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Approximation for Option Prices under Uncertain Volatility*

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Abstract. In this paper, we study the asymptotic behavior of the worst case scenario option prices as the volatility interval in an uncertain volatility model (UVM) degenerates to a single point and then provide an approximation procedure for the worst case scenario prices in a UVM with a small volatility interval. Numerical experiments show that this approximation procedure performs well even when the size of the volatility band is not so small.

Key words. uncertain volatility, nonlinear Black–Scholes–Barenblatt PDE, approximation

AMS subject classifications. 60H10, 91G80, 35Q93

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1. Introduction to UVM.

Since uncertain volatility models (UVMs) were initially introduced in [3], [14] and further developed in [8], for instance, the topic of model ambiguity has received intensive attention in mathematical finance, including literature on G-expectation, for instance, [16], [17], which studied fully nonlinear PDEs (like the BSB equation) through sublinear expectations. In a simplest UVM, it is assumed that the market has two assets: one riskless asset and one risky asset. Their price processes are denoted as (B_t) and (X_t). It is also assumed that the price process of the riskless asset (B_t) has dynamics

$$\mathrm{d}B_t = rB_t\mathrm{d}t,$$

where r is the constant risk-free rate.

The price process of the risky asset (X_t) solves the following stochastic differential equation (SDE):

$$\mathrm{d}X_t = rX_t\mathrm{d}t + \alpha_t X_t \mathrm{d}W_t,$$

where (W_t) is a Brownian motion on the probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{Q})$ and the volatility process $(\alpha_t) \in \mathcal{A}$, which is a family of progressively measurable and $[\underline{\sigma}, \overline{\sigma}]$-valued processes. For each stochastic volatility process $\alpha \in \mathcal{A}$, one has a general stochastic volatility model for (X_t). In a UVM, we know only that the true model lies in the above family of general stochastic volatility models. Note that we do not have a prior belief (a probability distribution) over the family of general stochastic volatility models. Therefore, we use "ambiguity" to distinguish this type of uncertainty. Intuitively, we can consider the size of the volatility interval as the degree of model ambiguity.

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Due to the presence of model ambiguity (or the absence of a prior distribution), the worst case scenario analysis is applied in derivatives pricing under a UVM. Suppose that χ is a European derivative written on the risky asset with maturity T and payoff $\varphi(X_T)$. It is known that its worst case scenario price at time $t < T$ is given by

$$V(t, X_t) := \exp(-r(T-t))\text{ess sup}_{\alpha\in\mathcal{A}}\mathbb{E}_t\left[\varphi(X_T)\right],$$

where $\mathbb{E}_t[\cdot]$ is the conditional expectation given $\mathscr{F}_t$ with respect to the measure $\mathbb{Q}$; see [3], [14]. It is proved in [19] that the seller of the derivative χ can superreplicate χ with initial wealth ess $\sup_{\alpha\in\mathcal{A}}\mathbb{E}\left[\varphi(X_T)\right]$ whatever the true volatility process is. The importance of the worst case scenario price is not only because of its superreplication property but also due to its relationship with coherent risk measures [2], [5]. Following the arguments in stochastic control theory, $V(t,x)$ satisfies the following Hamilton–Jacobi–Bellman (HJB) equation (in math finance it is called the Black–Scholes–Barenblatt (BSB) equation):

$$\partial_t V + r(x\partial_x V - V) + \sup_{\alpha\in[\underline{\sigma},\overline{\sigma}]}\left[\frac{1}{2}x^2\alpha^2\partial^2_{xx}V\right] = 0, \qquad V(T) = \varphi. \tag{1.1}$$

If φ is convex (like the payoff of a European call), it is known that the worst case scenario price of χ is equal to its Black–Scholes price with constant volatility $\overline{\sigma}$. For concave φ, we have a similar result; see [21] for details. However, the fully nonlinear PDE (1.1) does not have a closed form solution, like the Black–Scholes formula, for a general terminal payoff function φ. In order to evaluate the worst case scenario price, we have to resort to numerical methods [3], [18]. Similarly, the best case scenario price of χ at time $t = 0$ can be defined as e^{-rT}ess $\inf_{\alpha\in\mathcal{A}}\mathbb{E}\left[\chi\right]$. Moreover, it is shown in [21] that given any price between e^{-rT}ess $\inf_{\alpha\in\mathcal{A}}\mathbb{E}\left[\chi\right]$ and e^{-rT}ess $\sup_{\alpha\in\mathcal{A}}\mathbb{E}\left[\chi\right]$, the market is arbitrage-free.

It is clear that the worst case scenario price is larger than any Black–Scholes price with a constant volatility $\sigma\in[\underline{\sigma},\overline{\sigma}]$. In this paper, we shall consider how the worst case scenario price behaves as the volatility interval $[\underline{\sigma},\overline{\sigma}]$ degenerates to a single point $\sigma\in[\underline{\sigma},\overline{\sigma}]$. Intuitively, if the model ambiguity is reduced, then the extra price (which is included in the worst case scenario price) paid for that should be less. As is shown in what follows, the worst case scenario price of χ will converge to its Black–Scholes price with constant volatility σ. A measure of impact of model uncertainty on the worst case scenario price for any derivative χ is suggested in [5]. It is of the form

$$\mu(\chi) = \exp(-rT)\left\{\text{ess sup}_{\alpha\in\mathcal{A}}\mathbb{E}\left[\chi\right] - \text{ess inf}_{\alpha\in\mathcal{A}}\mathbb{E}\left[\chi\right]\right\},$$

which vanishes as the volatility interval shrinks to a single point.

In addition, our study partially answers the sensitivity problem of the worst case scenario to the degree of model ambiguity which is proposed in [5]. In fact, we obtain the rate of convergence of the worst case scenario prices as the volatility interval shrinks to a single point. Therefore, this result gives us an approximation of the worst case scenario price when the interval is sufficiently small. Throughout the paper we denote the Black–Scholes price as V_0 and the rate of convergence as V_1, which are the solutions to linear PDEs. Consequently, the first order approximation $V_0 + (\overline{\sigma} - \underline{\sigma})V_1$ of the worst case scenario price is achieved. Of course, the approximated price $V_0 + (\overline{\sigma} - \underline{\sigma})V_1$ does not have the property of superreplication. What did we gain in the approximation procedure? First, the problem of solving a fully nonlinear BSB equation is reduced to solving two Black–Scholes-like PDEs. The numerical examples also show that the approximation procedure is stable even with a reasonably large volatility interval. Second, we are able to see how a linear expectation turns into a sublinear expectation.

In order to study the asymptotic behavior of worst case scenario prices, we reparameterize our uncertain volatility model and assume that the risky asset price process $(X_t^{\alpha,\varepsilon})$ has a dynamic

$$dX_t^{\alpha,\varepsilon} = rX_t^{\alpha,\varepsilon}dt + \alpha_t X_t^{\alpha,\varepsilon}dW_t, \tag{1.2}$$

where $\alpha := (\alpha_t) \in \mathcal{A}^\varepsilon$, the family of progressively measurable, $[\sigma, \sigma+\varepsilon]$-valued processes, and (W_t) is a Brownian motion on the probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{Q})$. If $\varepsilon = 0$ and there is no danger of confusion, we shall use (X_t) to denote $(X^{\alpha,0})$, which is indeed a geometric Brownian motion with constant volatility σ,

$$dX_t = rX_t dt + \sigma X_t dW_t.$$

We define the worst case scenario price as a value function of a stochastic control problem

$$J^\varepsilon(t,x,\alpha) := \exp(-r(T-t))\mathbb{E}_{tx}[\varphi(X_T^{\alpha,\varepsilon})],$$
$$V^\varepsilon(t,x) = \text{ess sup}_{\alpha\in\mathcal{A}^\varepsilon}\left[J^\varepsilon(t,x,\alpha)\right],$$

where the conditional expectation $\mathbb{E}_{tx}[\cdot]$ is taken with respect to the law of $X_T^{\alpha,\varepsilon}$ given $X_t^{\alpha,\varepsilon} = x$. The worst case scenario option price when $\varepsilon = 0$ is the Black–Scholes price at volatility σ. We also represent it as a value function of a trivial stochastic control problem

$$V_0(t,x) = J(t,x,\sigma) := \exp(-r(T-t))\mathbb{E}_{tx}\left[\varphi(X_T)\right],$$

where the subscripts in $\mathbb{E}_{tx}[\cdot]$ also mean that $X_t = x$.

The paper is structured as follows. In section 2, we explain the continuity of the worst case scenario price with respect to the degree of model uncertainty: parameter ε. After that, we introduce the main result, differentiability of V^ε with respect to ε, through heuristic derivation. In section 3, we recall preliminary results from stochastic control and discuss properties of the worst case scenario price process. The proof of the main result is presented in section 4 by imposing technical conditions on the payoff function. A numerical experiment is presented in section 5, and we conclude in section 6.

2. Main result. In this section, we first introduce the Lipschitz continuity of the worst case scenario price V^ε with respect to the parameter ε. For the case where ε is small enough, we also heuristically derive a PDE that the first order correction term of V^ε should follow. At the end, we state the main result of this paper with more conditions imposed on the terminal payoff function. Its proof will be completed after a short review of preliminary results from stochastic control theory.

2.1. Continuity of V^ε.

Theorem 2.1. *Given φ, which is Lipschitz continuous with Lipschitz constant K_1, and for any $\varepsilon_0 \in [0,1)$,*

$$\lim_{\varepsilon \to \varepsilon_0} V^\varepsilon(t,x) = V^{\varepsilon_0}(t,x)$$

for all $(t,x) \in [0,T] \times \mathbb{R}$.

Proof. First, recall that $\mathcal{A}^\varepsilon$ is the family of progressively measurable and $[\sigma, \sigma+\varepsilon]$-valued processes. If $0 \le \varepsilon_0 < \varepsilon < 1$, we have that

$$e^{-r(T-t)} V^{\varepsilon_0}(t,x) = \text{ess sup}_{\alpha \in \mathcal{A}^{\varepsilon_0}} \mathbb{E}_{tx}\left[\varphi(X_T^{\alpha,\varepsilon_0})\right] = \text{ess sup}_{\alpha \in \mathcal{A}^{\varepsilon}} \mathbb{E}_{tx}\left[\varphi(X_T^{\alpha\wedge(\sigma+\varepsilon_0),\varepsilon})\right].$$

Therefore, by the estimate (3.3) presented in section 3.1 we obtain

$$\begin{aligned} e^{-r(T-t)}\left|V^\varepsilon(t,x) - V^{\varepsilon_0}(t,x)\right| &\le \text{ess sup}_{\alpha \in \mathcal{A}^\varepsilon}\left|\mathbb{E}_{tx}[\varphi(X_T^{\alpha,\varepsilon})] - \mathbb{E}_{tx}[\varphi(X_T^{\alpha\wedge(\sigma+\varepsilon_0),\varepsilon})]\right| \\ &\le K_1 \text{ess sup}_{\alpha \in \mathcal{A}^\varepsilon}\left(\mathbb{E}_{tx}\left|X_T^{\alpha,\varepsilon} - X_T^{\alpha\wedge(\sigma+\varepsilon_0),\varepsilon}\right|^2\right)^{1/2} \\ &\le K_1\left(N(T-t)e^{N(T-t)}(1+x^2)\right)^{1/2}\left|\varepsilon - \varepsilon_0\right|. \end{aligned}$$

It can be seen that for any fixed point $(t,x) \in [0,T] \times \mathbb{R}$, $|V^\varepsilon(t,x) - V^{\varepsilon_0}(t,x)| \to 0$ as ε approaches ε_0 from above.

It can be proved similarly that $\lim_{\varepsilon \uparrow \varepsilon_0} |V^\varepsilon(t,x) - V^{\varepsilon_0}(t,x)| = 0$ for $\varepsilon_0 > 0$. ■

In particular, when $\varepsilon_0 = 0$ we have the one-sided convergence of $\{V^\varepsilon(t,x)\}_{\varepsilon>0}$, which is stated in the following corollary.

Corollary 2.2. *When the conditions in Theorem 2.1 are satisfied,*

$$\lim_{\varepsilon \downarrow 0} V^\varepsilon(t,x) = V_0(t,x)$$

for all $(t,x) \in [0,T] \times \mathbb{R}$.

As the volatility interval $[\sigma, \sigma+\varepsilon]$ becomes smaller, the above corollary tells us that the Black–Scholes price V_0 of χ with constant volatility σ is the main contributor to its worst case scenario price relative to the extra price paid for the model ambiguity, i.e., V_0 would be the leading term in the approximation of V^ε. We move forward to investigate the first order correction term in this approximation, which will help us understand the sensitivity of the worst case scenario price to the degree of model ambiguity.

2.2. Main theorem and its heuristic derivation.

Recall that V^ε solves the following BSB equation:

$$\begin{aligned} \partial_t V^\varepsilon + r(x\partial_x V^\varepsilon - V^\varepsilon) + \sup_{\alpha \in [\sigma,\sigma+\varepsilon]} \frac{1}{2}\alpha^2 x^2 \partial_{xx}^2 V^\varepsilon = 0, \\ V^\varepsilon(T) = \varphi. \end{aligned}$$

To study the asymptotic behavior of V^ε, we also reparameterize the BSB equation as follows:

$$\partial_t V^\varepsilon + r(x\partial_x V^\varepsilon - V^\varepsilon) + \sup_{g\in[0,1]} \frac{1}{2}(\sigma + \varepsilon g)^2 x^2 \partial^2_{xx} V^\varepsilon = 0, \quad V^\varepsilon(T) = \varphi. \tag{2.1}$$

Note that V_0 is the solution of the following Black–Scholes equation:

$$\partial_t V_0 + r(x\partial_x V_0 - V_0) + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} V_0 = 0, \quad V_0(T) = \varphi. \tag{2.2}$$

In this heuristic derivation, we assume the differentiability of V^ε with respect to ε and the interchangeability of partial differential operators ∂_ε, ∂_t, ∂_x, and ∂^2_{xx}. Differentiating (2.1) with respect to ε,

$$\partial_\varepsilon \left\{ \partial_t V^\varepsilon + r(x\partial_x V^\varepsilon - V^\varepsilon) + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} V^\varepsilon + \varepsilon \sup_{g\in[0,1]} \left(g\sigma + \frac{1}{2}\varepsilon g^2 \right) x^2 \partial^2_{xx} V^\varepsilon \right\} = 0,$$

and noting that $V^\varepsilon|_{\varepsilon=0} = V_0$ by Corollary 2.2, it is seen that the derivative $V_1 := \partial_\varepsilon V^\varepsilon|_{\varepsilon=0}$ is the unique solution of the following linear Black–Scholes-type PDE with source and zero terminal condition:

$$\partial_t V_1 + r(x\partial_x V_1 - V_1) + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} V_1 + \sup_{g\in[0,1]} g\sigma x^2 \partial^2_{xx} V_0 = 0, \quad V_1(T) = 0. \tag{2.3}$$

Given V_0 as the solution to the Black–Scholes equation, the source term in the above equation is known. Indeed, it is equal to $\sigma x^2 \partial^2_{xx} V_0 \mathbb{I}_{\{\partial^2_{xx} V_0 > 0\}}$, which is nonlinear in V_0. This can be seen as the first manifestation of the nonlinearity of the full problem (2.1).

In the above heuristic argument, we obtained the equation characterizing V_1. It will be verified that V_1 which solves (2.3) is the first order derivative of V^ε with respect to ε at $\varepsilon = 0$. That is, we shall prove in our main result, under additional technical conditions imposed on the payoff φ, that the error term $V^\varepsilon - (V_0 + \varepsilon V_1)$ is of order $o(\varepsilon)$.

Theorem 2.3. *Assume that the payoff function $\varphi \in C^4_p(\mathbb{R}^+)$ (p for polynomial growth), φ is Lipschitz, and its derivatives up to order 4 have polynomial growth. Moreover, we also assume that the second derivative of φ has a finite number of zero points. Then, pointwise,*

$$\lim_{\varepsilon\downarrow 0} \frac{V^\varepsilon - (V_0 + \varepsilon V_1)}{\varepsilon} = 0.$$

To simplify the notation, we assume $r = 0$ in what follows, but all the results still hold when $r \neq 0$.

Remark 2.1. The conditions in the theorem are sufficient. Relaxing the regularity assumption on the payoff function is a complicated and technical issue. It has been done in the linear case in the context of option pricing under stochastic volatility [10]. The technique would consist in regularizing the payoff introducing a small parameter for the regularization and then showing that it can be removed as $\varepsilon \to 0$ without changing the accuracy estimate (choosing an appropriate regularization using the Black–Scholes equation with constant volatility σ would be a good choice, as in the linear case). This technically involved argument is beyond the scope of this paper.

3. Preliminary stochastic control results and application to the worst case scenario process. In order to prove the main result of this paper, Theorem 2.3, we briefly review necessary results from stochastic control theory. Based on the preliminary results, we will give corresponding estimates for our worst case scenario price process.

3.1. Preliminary results from stochastic control. In this brief review, we retain the notation in [13].

Given the SDE

$$\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}W_t, \tag{3.1}$$

if there exists a constant $K > 0$ such that its coefficients satisfy

$$|b_t(x) - b_t(y)| \leq K\,|x-y|\,, \quad |\sigma_t(x) - \sigma_t(y)| \leq K\,|x-y|$$

for all $t \in [0,T]$, $\omega \in \Omega$, and $x,\ y\ \in \mathbb{R}$, then we say that condition $(\mathscr{L})$ is satisfied. If for all $t \in [0,T]$, $\omega \in \Omega$, and $x\ \in \mathbb{R}$ there exists some $K > 0$ such that

$$|b_t(x)| \leq K\,|x| + h_t, \quad |\sigma_t(x)|^2 \leq K\,|x|^2 + r_t^2$$

for some stochastic processes (h_t) and (r_t), we say that condition (R) is satisfied. It can be seen that the condition $(\mathscr{L})$ implies condition (R) by noticing that

$$\begin{aligned}|b_t(x)| &\leq |b_t(x) - b_t(0)| + |b_t(0)|\,,\\ |\sigma_t(x)| &\leq |\sigma_t(x) - \sigma_t(0)| + |\sigma_t(0)|\,.\end{aligned}$$

Because we are interested only in the case where ε is close to 0, it is legitimate to assume that $\varepsilon \leq 1$. In particular, $|\alpha_t| \leq \sigma + 1$. By noting

$$|rx - ry| \leq r\,|x-y|\,, \quad |\alpha_t x - \alpha_t y| \leq |\alpha_t| \cdot |x-y| \leq (\sigma+1)\,|x-y|\,,$$

it is clear that the SDE (1.2) for $(X_t^{\alpha,\varepsilon})$ satisfies the condition $(\mathscr{L})$. According to Corollary 12 in section 2.5 of [13], we have the following universal estimates of the moments of $(X_t^{\alpha,\varepsilon})$:

$$\mathbb{E}\left[\sup_{s\in[0,t]} |X_s^{\alpha,\varepsilon}|^q\right] \leq Ne^{Nt}\,(1+|x_0|)^q \tag{3.2}$$

for all $\alpha \in \mathcal{A}^\varepsilon$, $t \in [0,T]$, and $q > 0$, where $N = N(q,\sigma,r)$ (we assumed that $\varepsilon < 1$) and $X_0^{\alpha,\varepsilon} = x_0$.

Given another $\varepsilon' \in (0,1]$, which is assumed to satisfy $\varepsilon' < \varepsilon$ without losing generality, we consider the process $(X_t^{\alpha\wedge(\sigma+\varepsilon'),\varepsilon})$ which satisfies the SDE (1.2) with volatility process $(\alpha_t \wedge (\sigma+\varepsilon'))$ for some $(\alpha_t) \in \mathcal{A}^\varepsilon$ and initial condition $X_0^{\alpha,\varepsilon'} = x_0$. By Theorem 9 in section 2.9 of [13] and the estimates of the moments (3.2), we can conclude that

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t]}\left|X_s^{\alpha,\varepsilon} - X_s^{\alpha\wedge(\sigma+\varepsilon'),\varepsilon}\right|^{2q}\right] \\
&\le N' t^{q-1} e^{N't}\mathbb{E}\left[\int_0^t |X^{\alpha,\varepsilon}|^{2q}\cdot\left|\alpha_s - \alpha_s\wedge(\sigma+\varepsilon')\right|^{2q}\mathrm{d}s\right] \\
&\le N' t^{q-1} e^{N't} N'' e^{N''t}(1+|x_0|^{2q})t\left|\varepsilon-\varepsilon'\right|^{2q} \\
&= N' t^{q} e^{N't} N'' e^{N''t}(1+|x_0|^{2q})\left|\varepsilon-\varepsilon'\right|^{2q} \\
&= N t^q e^{Nt}(1+|x_0|^{2q})\left|\varepsilon-\varepsilon'\right|^{2q}
\end{aligned}
\tag{3.3}
$$

for any $q \ge 1$, where $N' = N'(q,\sigma,r)$, $N'' = N''(q,\sigma,r)$, and $N = \max\{N'N'', N'+N''\}$.

3.2. Asset price process in the worst case scenario and estimates of its moments. It is known from [22], [23], [6] that if φ is locally Lipschitz continuous and φ and φ' have polynomial growth, then the viscosity solution V^ε of (2.1) belongs to $C_p^{1,2}([0,T)\times\mathbb{R})$ and there exists $\kappa \in (0,1]$ such that $\partial_{xx}^2 V^\varepsilon$ is Hölder-κ continuous.

3.2.1. Existence and uniqueness of $(X_t^{*,\varepsilon})$. Equation (2.1) would produce the worst case scenario volatility process $\alpha^{*,\varepsilon} = \sigma + \varepsilon g^{*,\varepsilon}$ for the claim χ, where

$$
g^{*,\varepsilon}(t,x) = \begin{cases} 1, & \partial_{xx}^2 V^\varepsilon(t,x) \ge 0, \\ 0, & \partial_{xx}^2 V^\varepsilon(t,x) < 0. \end{cases}
$$

By solving (2.3) of V_1, we would have another choice of the volatility process for the claim χ: $\bar\alpha = \sigma + \varepsilon\bar g$, where

$$
\bar g(t,x) = \begin{cases} 1, & \partial_{xx}^2 V_0(t,x) \ge 0, \\ 0, & \partial_{xx}^2 V_0(t,x) < 0. \end{cases}
$$

Therefore, the asset price process in the worst case scenario for the claim χ is a stochastic process which satisfies the SDE (1.2) with $(\alpha_t) = \left(\alpha^{*,\varepsilon}(t, x_t^{*,\varepsilon})\right)$ and $r = 0$, i.e.,

$$
\mathrm{d}X_t^{*,\varepsilon} = \alpha^{*,\varepsilon}(t, X_t^{*,\varepsilon})X_t^{*,\varepsilon}\mathrm{d}W_t. \tag{3.4}
$$

For simplicity, we use $\alpha_t^{*,\varepsilon}$ to denote the generic function $\alpha^{*,\varepsilon}(t, X_t^{*,\varepsilon})$ without confusion, and similarly we use the short notation $g_t^{*,\varepsilon}$ and $\bar g_t$ for $g^{*,\varepsilon}(t, X_t^{*,\varepsilon})$ and $\bar g(t, X_t^{*,\varepsilon})$, respectively. For the existence and uniqueness of the worst case scenario price process, we consider the transformation

$$
Y_t^{*,\varepsilon} := \log X_t^{*,\varepsilon},
$$

which is well defined for any $t < \tau^\delta$, where

$$
\begin{aligned}
\tau^\delta &= \inf\left\{t > 0 \mid X_t^{*,\varepsilon} = \delta \text{ or } X_t^{*,\varepsilon} = 1/\delta\right\} \\
&= \inf\left\{t > 0 \mid Y_t^{*,\varepsilon} = \log\delta \text{ or } Y_t^{*,\varepsilon} = -\log\delta\right\}
\end{aligned}
$$

for any $\delta > 0$. By Itô's formula, the process $(Y_t^{*,\varepsilon})$ satisfies the following SDE:

$$dY_t^{*,\varepsilon} = -\frac{1}{2}(\alpha_t^{*,\varepsilon})^2 dt + \alpha_t^{*,\varepsilon} dW_t. \tag{3.5}$$

It is noted that the coefficients in (3.5) are bounded and progressively measurable. Moreover, the diffusion coefficient is bounded away from zero: $\alpha_t^{*,\varepsilon} \geq \sigma > 0$. Therefore, thanks to Theorem 1 in section 2.6 of [13] or the result 7.3.3 of [20], the SDE (3.5) has a unique weak solution. That is, we have a unique solution to the SDE (3.4) until τ^δ for any $\delta > 0$. In order to prove that the SDE (3.4) has a unique solution for all $t \in (0, \infty)$, it suffices to show that for any $T > 0$

$$\lim_{\delta \downarrow 0} \mathbb{Q}\left(\tau^\delta < T\right) = 0. \tag{3.6}$$

Indeed, by the Chebyshev inequality, it holds that

$$\begin{aligned}
\lim_{\delta \downarrow 0} \mathbb{Q}\left(\tau^\delta < T\right) &\leq \lim_{\delta \downarrow 0} \mathbb{Q}\left(\sup_{t\in[0,T]} \left|Y_t^{*,\varepsilon}\right| > |\log \delta|\right) \\
&\leq \lim_{\delta \downarrow 0} \frac{\mathbb{E}\left[\sup_{t\in[0,T]} \left|Y_t^{*,\varepsilon}\right|\right]}{|\log \delta|} = 0,
\end{aligned}$$

which implies (3.6).

3.2.2. Estimates of moments and exit probability for $(X_t^{*,\varepsilon})$. As a special case of (3.2), we have that

$$\mathbb{E}_{tx}[\sup_{s\in[t,T]} |X_s^{*,\varepsilon}|^q] \leq N e^{N(T-t)}(1 + |x|^q) \tag{3.7}$$

for all $q > 0$, $N = N(q, \sigma)$, and $X_t^{*,\varepsilon} = x$.

Given $\rho > 0$, define a stopping time

$$\tau_\rho := \inf\left\{s \in [t, T] \,\text{such that}\, |X_s^{*,\varepsilon}| \geq \rho\right\}.$$

Conventionally, $\inf \emptyset = \infty$.

By using the estimates of moments and the Chebyshev inequality,

$$\mathbb{Q}_{tx}(\tau_\rho < T) \leq \mathbb{Q}_{tx}\left(\sup_{s\in[t,T]} |X_s^{*,\varepsilon}| \geq \rho\right) \leq \frac{N e^{N(T-t)}(1 + |x|)}{\rho}, \tag{3.8}$$

where $N = N(\sigma)$. This control on the exit probability enables us to use localization arguments in what follows.

Remark 3.1. It is important to notice that the estimates presented above are independent of ε, assuming $\varepsilon \leq 1$, for instance.

4. Analysis of the error term. Define the error term of the suggested approximation

$$Z^\varepsilon = V^\varepsilon - (V_0 + \varepsilon V_1)\,.$$

Let

$$\mathcal{L}(\sigma) := \partial_t + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx}, \tag{4.1}$$

and the operator $\mathcal{L}(\alpha_t^{*,\varepsilon})$ is defined by replacing the constant σ with the worst case scenario volatility $\alpha_t^{*,\varepsilon}$ at each point $(t,x) \in [0,T] \times \mathbb{R}^+$. Applying the operators to the error term, it follows that

$$\begin{aligned}
\mathcal{L}(\alpha_t^{*,\varepsilon})Z^\varepsilon &= -\mathcal{L}(\alpha_t^{*,\varepsilon})(V_0 + \varepsilon V_1) \\
&= -\left(\mathcal{L}(\alpha_t^{*,\varepsilon}) - \mathcal{L}(\sigma)\right) V_0 - \varepsilon \left(\mathcal{L}(\alpha_t^{*,\varepsilon}) - \mathcal{L}(\sigma)\right) V_1 - \varepsilon \mathcal{L}(\sigma) V_1 \\
&= \frac{1}{2}\left(\sigma^2 - \left(\sigma + \varepsilon g_t^{*,\varepsilon}\right)^2\right) x^2 \partial^2_{xx} V_0 + \varepsilon \frac{1}{2}\left(\sigma^2 - \left(\sigma + \varepsilon g_t^{*,\varepsilon}\right)^2\right) x^2 \partial^2_{xx} V_1 \\
&\quad + \varepsilon \bar{g}_t \sigma x^2 \partial^2_{xx} V_0 \\
&= -\varepsilon \left(g_t^{*,\varepsilon} - \bar{g}_t\right) \sigma x^2 \partial^2_{xx} V_0 - \varepsilon^2 \left(\frac{1}{2}\left(g_t^{*,\varepsilon}\right)^2 x^2 \partial^2_{xx} V_0 + g_t^{*,\varepsilon} \sigma x^2 \partial^2_{xx} V_1\right) \\
&\quad - \varepsilon^3 \left(\frac{1}{2}\left(g_t^{*,\varepsilon}\right)^2 x^2 \partial^2_{xx} V_1\right).
\end{aligned} \tag{4.2}$$

Note that the terminal condition of Z^ε is

$$Z^\varepsilon(T) = V^\varepsilon(T) - V_0(T) - \varepsilon V_1(T) = 0.$$

From now on, we impose more regularity conditions on the terminal data φ, i.e., polynomial growth conditions on the first four derivatives of $\varphi(x)$:

$$\begin{cases}
\varphi'(x) \le K_1, \\
\varphi''(x) \le K_2(1 + |x|^m), \\
\varphi'''(x) \le K_3(1 + |x|^n), \\
\varphi^{(4)}(x) \le K_4(1 + |x|^l),
\end{cases} \tag{4.3}$$

where K_i for $i \in \{1,2,3,4\}$, and m, n, and l are positive constants. Please note that the polynomial growth condition on $\varphi^{(4)}$ implies the rest. We specify all of them because they are used in the sequent arguments.

4.1. Feynman–Kac representation of the error term Z^ε. Given (4.2) for Z^ε together with the existence and uniqueness of $(X_t^{*,\varepsilon})$, we have the following probabilistic representation of Z^ε, where we use the notation $g_t^{*,\varepsilon}$ and $\bar{g}_t$ for $g^{*,\varepsilon}(t, X_t^{*,\varepsilon})$ and $\bar{g}(t, X_t^{*,\varepsilon})$, respectively, introduced after (3.4):

$$
\begin{aligned}
Z^{\varepsilon} &= -\varepsilon \mathbb{E}_{tx}\left[\int_t^T \left(g_s^{*,\varepsilon} - \bar{g}_s\right)\sigma \left(X_s^{*,\varepsilon}\right)^2 \partial_{xx}^2 V_0\left(s, X_s^{*,\varepsilon}\right) ds\right] \\
&\quad -\varepsilon^2 \mathbb{E}_{tx}\left[\int_t^T \left\{\frac{1}{2}\left(g_s^{*,\varepsilon}\right)^2 \left(X_s^{*,\varepsilon}\right)^2 \partial_{xx}^2 V_0\left(s, X_s^{*,\varepsilon}\right) + g_s^{*,\varepsilon}\sigma \left(X_s^{*,\varepsilon}\right)^2 \partial_{xx}^2 V_1\left(s, X_s^{*,\varepsilon}\right)\right\} ds\right] \\
&\quad -\varepsilon^3 \mathbb{E}_{tx}\left[\int_t^T \frac{1}{2}\left(g_s^{*,\varepsilon}\right)^2 \left(X_s^{*,\varepsilon}\right)^2 \partial_{xx}^2 V_1\left(s, X_s^{*,\varepsilon}\right) ds\right] \\
&= -\varepsilon I_1 - \varepsilon^2 I_2 - \varepsilon^3 I_3, \qquad (4.4)
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \mathbb{E}_{tx}\left[\int_t^T \left(g_s^{*,\varepsilon} - \bar{g}_s\right)\sigma \left(X_s^{*,\varepsilon}\right)^2 \partial_{xx}^2 V_0\left(s, X_s^{*,\varepsilon}\right) ds\right], \\
I_2 &= \mathbb{E}_{tx}\left[\int_t^T \left\{\frac{1}{2}\left(g_s^{*,\varepsilon}\right)^2 \left(X_s^{*,\varepsilon}\right)^2 \partial_{xx}^2 V_0\left(s, X_s^{*,\varepsilon}\right) + g_s^{*,\varepsilon}\sigma \left(X_s^{*,\varepsilon}\right)^2 \partial_{xx}^2 V_1\left(s, X_s^{*,\varepsilon}\right)\right\} ds\right], \\
I_3 &= \mathbb{E}_{tx}\left[\int_t^T \frac{1}{2}\left(g_s^{*,\varepsilon}\right)^2 \left(X_s^{*,\varepsilon}\right)^2 \partial_{xx}^2 V_1\left(s, X_s^{*,\varepsilon}\right) ds\right].
\end{aligned}
$$

We shall derive the bounds of I_2 and I_3 in the next section. The term I_1 will be dealt with in section 4.3.

4.2. Controls of the terms I_2 and I_3. All three terms in (4.4) involve partial derivatives of V_0 and V_1. To control the error of our approximation, it is necessary to derive estimates of their partial derivatives. Because both V_0 and V_1 satisfy linear PDEs, in particular since V_0 is the solution to the standard Black–Scholes equation, we have the following results.

Lemma 4.1. *Given $\varphi(x)$, which satisfies condition (4.3), there exist constants M_1, M_2, and M_3, which depend only on σ, T, m, n, and l, such that $\left|\partial_{xx}^2 V_0\right| \le M_1\left(1 + |x|^m\right)$, $\left|\partial_x^3 V_0\right| \le M_2\left(1 + |x|^n\right)$, and $\left|\partial_t \partial_{xx}^2 V_0\right| \le M_3\left(1 + |x|^{p_1}\right)$, where $p_1 = \max\{m, n+1, l+2\}$.*

Proof. This is a classical result for solutions of linear PDEs with appropriate assumptions on the coefficients and the terminal condition. It can also be easily derived by using the probabilistic representation

$$V_0(t,x) = \mathbb{E}_{tx}\left[\varphi\left(x e^{-\frac{\sigma^2 (T-t)}{2} + \sigma (W_T - W_t)}\right)\right], \qquad (4.5)$$

where (W_t) is a Brownian motion. ■

Given V_0, which solves (2.2), we define $f(t,x) := \frac{1}{2}\sigma x^2 \partial_{xx}^2 V_0(t,x)$, which has the first order partial derivatives

$$
\begin{aligned}
f_t(t,x) &= \frac{1}{2}\sigma x^2 \partial_t \partial_{xx}^2 V_0(t,x), \\
f_x(t,x) &= \frac{1}{2}\sigma\left[2x \partial_{xx}^2 V_0(t,x) + x^2 \partial_x^3 V_0(t,x)\right].
\end{aligned}
$$

Due to Lemma 4.1, it can be observed that

$$|f(t,x)| \le \frac{1}{2}\sigma x^2 M_1 (1 + |x|^m) \le K_5 (1 + |x|^{m+2}) \qquad (4.6)$$

for a constant $K_5 = K_5(M_1, \sigma)$ and that

$$|f_t(t,x)| + |f_x(t,x)| \le \frac{\sigma}{2}\left(x^2 M_3(1+|x|^{p_1}) + 2|x|M_1(1+|x|^m) + x^2 M_2(1+|x|^n)\right) \le K_6(1+|x|^{p_2}), \tag{4.7}$$

where $K_6 = K_6(M_1, M_2, M_3, p_1, m, n, \sigma)$ and $p_2 = \max\{p_1+2, m+1, n+2\}$.

Then, the source term in (2.3) for V_1 is given by

$$f^+(t,x) := -\sup_{g\in[0,1]}\left\{\frac{1}{2}g\sigma x^2\partial^2_{xx}V_0\right\} = -\max\left\{\frac{1}{2}\sigma x^2\partial^2_{xx}V_0, 0\right\} = -\max\{f, 0\}.$$

Therefore, it is clear that

$$\begin{aligned} \left|f^+(t,x)\right| &\le |f(t,x)|, \\ \left|f^+(t,x) - f^+(s,y)\right| &\le |f(t,x) - f(s,y)| \end{aligned} \tag{4.8}$$

for any (t,x), $(s,y) \in [0,T]\times\mathbb{R}$.

Proposition 4.2. *Given $\varphi(x)$, which satisfies condition (4.3), there exist two constants $p = \max\{p_2, m+2\}$ and M_5, which depend on K_5, K_6, T, p_2, m, and σ, such that $\left|x^2\partial^2_{xx}V_1\right| \le M_5(1+|x|^p)$.*

Proof. By noticing that

$$\partial_t V_1 + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} V_1 = f^+,$$

we have

$$\left|x^2\partial^2_{xx}V_1\right| \le \frac{2}{\sigma^2}\left(|\partial_t V_1| + \left|f^+\right|\right).$$

According to classical results in linear PDEs or arguments presented in section IV.8 of [9], it can be derived from (4.7) that $|\partial_t V_1| \le M_4(1+|x|^p)$ for some $M_4 = M_4(K_5, K_6, T, p_2, m, \sigma)$. Therefore, considering this fact together with (4.6) and (4.8), the proposition is obtained. ∎

Theorem 4.3. *Given $\varphi(x)$, which satisfies condition (4.3), there exists a constant D_1, which depends on M_1, M_5, $T-t$, m, p, and σ, such that I_2 and I_3 in (4.4) satisfy*

$$|I_2| + |I_3| \le D_1(1+|x|^p).$$

Proof. Due to Proposition 4.2 and (3.7), it follows that

$$\begin{aligned} |I_3| &= \mathbb{E}_{tx}\left[\int_t^T \frac{1}{2}\left(g_s^{*,\varepsilon}\right)^2\left(X_s^{*,\varepsilon}\right)^2\left|\partial^2_{xx}V_1\left(s, X_s^{*,\varepsilon}\right)\right| ds\right] \\ &\le \frac{M_5}{2}\mathbb{E}_{tx}\int_t^T \left(1+|X_s^{*,\varepsilon}|^p\right) ds \\ &\le \frac{M_5}{2}\left[\int_t^T \left[1 + N(\sigma,p)e^{N(\sigma,p)(T-t)}(1+|x|^p)\right] ds\right] \\ &= \frac{M_5}{2}\left[T - t + N(\sigma,p)e^{N(\sigma,p)(T-t)}(1+|x|^p)(T-t)\right]. \end{aligned}$$

Similarly, for I_2, we have

$$
\begin{aligned}
|I_2| &\le \mathbb{E}_{tx}\left[\int_t^T \frac{1}{2}\left(X_s^{*,\varepsilon}\right)^2 \left|\partial_{xx}^2 V_0\left(s, X_s^{*,\varepsilon}\right)\right| ds\right] \\
&\quad + \mathbb{E}_{tx}\left[\int_t^T \sigma\left(X_s^{*,\varepsilon}\right)^2 \left|\partial_{xx}^2 V_1\left(s, X_s^{*,\varepsilon}\right)\right| ds\right] \\
&\le \frac{M_1}{2}\mathbb{E}_{tx}\int_t^T |X_s^{*,\varepsilon}|^2\left(1+|X_s^{*,\varepsilon}|^m\right) ds + M_7\sigma\mathbb{E}_{tx}\int_t^T \left(1+|X_s^{*,\varepsilon}|^{p'}\right) ds \\
&\le \frac{M_1}{2}(T-t)N(\sigma)e^{N(\sigma)(T-t)}\left(1+|x|^2\right) \\
&\quad + \frac{M_1}{2}(T-t)N(m+2,\sigma)e^{N(m+2,\sigma)(T-t)}\left(1+|x|^{m+2}\right) \\
&\quad + M_5\sigma(T-t)N(p,\sigma)e^{N(p,\sigma)(T-t)}\left(1+|x|^p\right).
\end{aligned}
$$

By summarizing the above controls on I_2 and I_3, there exists $D_1 = D_1(M_1, M_5, T-t, m, p, \sigma)$ such that $|I_2| + |I_3| < D_1(1+|x|^p)$. ■

4.3. Convergence of the term I_1. Given the controls of I_2 and I_3 in Theorem 4.3, in order to prove

$$Z^\varepsilon(t,x) \sim o(\varepsilon)$$

for a fixed point $(t,x) \in [0,T]\times\mathbb{R}$, it suffices to prove that

$$\lim_{\varepsilon\downarrow 0} I_1 = 0.$$

Let $K_\rho := [-\rho,\rho]$ and $g^{d,\varepsilon} := g^{*,\varepsilon} - \bar{g}$, which depends on ε through $g^{*,\varepsilon}$ as follows:

$$\left|g^{d,\varepsilon}(t,x)\right| = \begin{cases} 1, & \partial_{xx}^2 V^\varepsilon(t,x)\partial_{xx}^2 V_0(t,x) < 0, \\ 0, & \partial_{xx}^2 V^\varepsilon(t,x)\partial_{xx}^2 V_0(t,x) \ge 0. \end{cases}$$

Intuitively, as ε approaches 0, V^ε and its derivatives will get closer to V_0 and its corresponding derivatives. In order to lay out the analysis of this intuition, we decompose the range of $X_s^{*,\varepsilon}$ for each $s \in [t,T]$ into two parts: a compact set K_ρ and its complement. Therefore, I_1 can be written as the expectation of a sum of two parts:

i. the compact part (when $X_s^{*,\varepsilon}$ lies in the compact set);
ii. the tail part (when $X_s^{*,\varepsilon}$ lies outside of the compact set).

That is,

$$
\begin{aligned}
I_1 &= \mathbb{E}_{tx}\left[\int_t^T g_s^{d,\varepsilon}\sigma\left(X_s^{*,\varepsilon}\right)^2 \partial_{xx}^2 V_0\left(s, X_s^{*,\varepsilon}\right)\mathbb{I}_{K_\rho}\left(X_s^{*,\varepsilon}\right) ds\right] \\
&\quad + \mathbb{E}_{tx}\left[\int_t^T g_s^{d,\varepsilon}\sigma\left(X_s^{*,\varepsilon}\right)^2 \partial_{xx}^2 V_0\left(s, X_s^{*,\varepsilon}\right)\mathbb{I}_{K_\rho^c}\left(X_s^{*,\varepsilon}\right) ds\right] \\
&= \mathbb{E}_{tx}[\text{i}] + \mathbb{E}_{tx}[\text{ii}]. \qquad (4.9)
\end{aligned}
$$

To achieve $\lim_{\varepsilon\downarrow 0} I_1 = 0$, we shall use localization arguments to deal with $\mathbb{E}_{tx}[\text{ii}]$ and the problem is reduced onto a compact set. On the compact set, it will be proved that V_ε and its partial derivatives converge to V_0 and its corresponding derivatives. Then, it is followed by the convergence of $\mathbb{E}_{tx}[\text{i}]$ to 0.

4.3.1. Control of the tail part. Recall that $\tau_\rho := \inf\left\{s \in [t,T]\,\text{such that}\,|X_s^{*,\varepsilon}| \geq \rho\right\}$, and in subsection 3.2.2 we have control of exit probability for the worst case scenario price process $(X_t^{*,\varepsilon})$. By applying Hölder's inequality to the second term of (4.9), it follows that

$$\mathbb{E}_{tx}[|\text{ii}|] \leq \sigma M_1 \left[\mathbb{E}_{tx}\int_t^T (X_s^{*,\varepsilon})^4 \left(1+|X_s^{*,\varepsilon}|^m\right)^2 ds\right]^{1/2} \left[\mathbb{Q}_{tx}\left(\tau_\rho < T\right)\right]^{1/2},$$

where

$$\begin{aligned}
&\mathbb{E}_{tx}\int_t^T (X_s^{*,\varepsilon})^4 \left(1+|X_s^{*,\varepsilon}|^m\right)^2 ds \\
&\leq \int_t^T \Big[N(4,\sigma)e^{N(4,\sigma)(T-t)}(1+|x|^4) + N(m+4,\sigma)e^{N(m+4,\sigma)(T-t)}(1+|x|^{m+4}) \\
&\quad + N(2m+4,\sigma)e^{N(2m+4,\sigma)(T-t)}(1+|x|^{2m+4})\Big]\, ds \\
&\leq N(T-t,m,\sigma)(1+|x|^{2m+4}). \qquad (4.10)
\end{aligned}$$

From (4.10) and the exit probability estimate of the process $(X_t^{*,\varepsilon})$ in (3.8), it is concluded that

$$\begin{aligned}
\mathbb{E}_{tx}[|\text{ii}|] &\leq \sigma M_1 \sqrt{N(T-t,m,\sigma)(1+|x|^{2m+4})}\sqrt{\frac{N(\sigma)e^{N(\sigma)(T-t)}(1+|x|)(T-t)}{\rho}} \\
&\leq D_2 \frac{\left(1+|x|^{m+3/2}\right)}{\sqrt{\rho}}
\end{aligned}$$

for some constant $D_2 = D_2(M_1, T-t, m, \sigma)$.

As ρ increases, i.e., the compact set K_ρ becomes larger, the process will be less likely to deviate outside of the set K_ρ. Then, we would expect the tail part to be small enough for sufficiently large ρ. This result is summarized in the following proposition.

Proposition 4.4. *Given $\varphi(x)$, which satisfies condition (4.3), $\mathbb{E}_{tx}[|\text{ii}|] \to 0$ as $\rho \to \infty$.*

4.3.2. Control of the compact part. In order to prove that the compact term is negligible when ε is sufficiently small, we need the convergence of $\partial_{xx}^2 V^\varepsilon$ to $\partial_{xx}^2 V_0$, which would imply that $g^{d,\varepsilon}$ gradually vanishes as ε tends to 0.

Recall the result regarding the regularity of the solution of the BSB equation in [22], [23], [6].

Theorem 4.5. *If φ is locally Lipschitz continuous and φ and φ' have polynomial growth, then the solution V^ε of (2.1) belongs to $C_p^{1,2}\left([0,T)\times\mathbb{R}\right)$. Moreover, $\partial_{xx}^2 V^\varepsilon(t,x)$ is Hölder continuous in x with an exponent $\kappa \in (0,1]$ for any $t \in [0,T)$.*

Remark 4.1. All the constants in the polynomial controls and Hölder continuity depend only on the bounds of the volatility interval $[\sigma, \sigma+\varepsilon]$. Since we are interested only in the cases where ε is small, we can choose these constants including the Hölder exponent to be universal, i.e., independent of ε; see [22], [23], [6], or [12].

We recapitulate the results as follows:

$$\begin{cases} |V^\varepsilon(t,x)| \le B_0(1+|x|^{b_0}), \\ |\partial_x V^\varepsilon(t,x)| \le B_1(1+|x|^{b_1}), \\ \left|\partial^2_{xx} V^\varepsilon(t,x)\right| \le B_2(1+|x|^{b_2}), \\ \left|\partial^2_{xx} V^\varepsilon(t,x) - \partial^2_{xx} V^\varepsilon(t,y)\right| \le B_3|x-y|^\kappa \end{cases} \tag{4.11}$$

for any $t \in [0,T)$, where all constants B_0, B_1, B_2, B_3, b_0, b_1, b_2, and κ are universal, i.e., independent of ε.

Lemma 4.6. *Given $\varphi(x)$, which satisfies condition (4.3), $\left\{\partial^2_{xx} V^\varepsilon(t,\cdot)\right\}$, as a family of functions of x indexed by ε, uniformly converges to $\partial^2_{xx} V_0(t,\cdot)$ on the compact set K_ρ as ε tends to 0 for any fixed $t \in [0,T)$.*

Proof. The lemma can be obtained by following the arguments in Theorem 5.2.5 of [11] or by applying the Stone–Weierstrass theorem. ■

Let

$$S_t^\varepsilon = \left\{ x \middle| \partial^2_{xx} V^{\varepsilon'}(t,x) \partial^2_{xx} V_0(t,x) \le 0, \quad \exists \varepsilon' \le \varepsilon \right\} \cap K_\rho.$$

Note that $\{S_t^\varepsilon\}_\varepsilon$ as a family of sets indexed by ε is nonincreasing as ε decreases to 0. Define

$$S_t^0 := \lim_{\varepsilon \downarrow 0} S_t^\varepsilon.$$

Lemma 4.7. *Given $\varphi(x)$, which satisfies condition (4.3), for any fixed $t \in [0,T)$,*

$$S_t^0 = \left\{ x \in K_\rho \middle| \partial^2_{xx} V_0(t,x) = 0 \right\}.$$

Proof. Notice that if $x \in K_\rho$ such that $\partial^2_{xx} V_0(t,x) = 0$, then $x \in S_t^\varepsilon$ for all $\varepsilon > 0$. It implies that

$$S_t^0 \supseteq \left\{ x \in K_\rho \middle| \partial^2_{xx} V_0(t,x) = 0 \right\}.$$

On the other hand, if $\partial^2_{xx} V_0(t,x) > 0$ for any $x \in K_\rho$, then due to the uniform convergence of $\left\{\partial^2_{xx} V^\varepsilon(t,\cdot)\right\}$ there exists $\varepsilon_0 > 0$ such that $\partial^2_{xx} V^\varepsilon(t,x) > 0$ for all $\varepsilon < \varepsilon_0$. Hence, $\partial^2_{xx} V^\varepsilon(t,x) \partial^2_{xx} V_0(t,x) > 0$ for all $\varepsilon < \varepsilon_0$, i.e., $x \notin S_t^\varepsilon$ for all $\varepsilon < \varepsilon_0$. It is followed by $x \notin S_t^0$.

Similarly, we can prove that any $x \in K_\rho$ such that $\partial^2_{xx} V_0(t,x) < 0$ does not lie in S_t^0 either. Therefore, we can claim that

$$S_t^0 \subseteq \left\{ x \in K_\rho \middle| \partial^2_{xx} V_0(t,x) = 0 \right\}. \quad ■$$

For any fixed $t \in [0,T)$, we denote by $\bar{S}_t^\varepsilon$ the closure of S_t^ε for each ε. For this sequence of closed, bounded, and nonincreasing sets, we define its limit

$$\bar{S}_t^0 := \lim_{\varepsilon \downarrow 0} \bar{S}_t^\varepsilon.$$

Due to the continuity of $\partial^2_{xx} V_0(t,x)$ in x, it is true that $\bar{S}_t^0 = S_t^0$ for any fixed $t \in [0,T)$. The following lemma tells us that the same relationship holds between $\bar{S}_t^\varepsilon$ and S_t^ε.

Lemma 4.8. *Given $\varphi(x)$, which satisfies condition (4.3), and any fixed $t \in [0,T)$, it holds that*

$$\bar{S}_t^\varepsilon = S_t^\varepsilon.$$

Proof. To prove the lemma, it suffices to show that for all $x_0 \in \bar{S}_t^\varepsilon$, either $x_0 \in S_t^\varepsilon$ or $x_0 \in S_t^0$. Indeed, if it is true, then together with the fact that $S_t^\varepsilon \supseteq S_t^0$ (by the monotonicity of $\{S_t^\varepsilon\}$) we can see that $\bar{S}_t^\varepsilon = S_t^\varepsilon$, i.e., S_t^ε is a closed set.

For any $x_0 \in \bar{S}_t^\varepsilon$, according to the definition of closure, there exists a sequence $\{x_n\} \subset S_t^\varepsilon$ such that $\lim_{n\to\infty} x_n = x_0$. For each x_n, there is a $\varepsilon_n \le \varepsilon$ such that

$$\partial_{xx}^2 V^{\varepsilon_n}(t,x_n)\partial_{xx}^2 V_0(t,x_n) \le 0.$$

Since $0 < \varepsilon_n \le \varepsilon$ for all $n > 0$, there exist a $\varepsilon_0 \in [0,\varepsilon]$ and a subsequence $\{\varepsilon_{n'}\}$ such that $\lim_{n\to\infty} \varepsilon_{n'} = \varepsilon_0$.

We take the corresponding subsequence $\{x_{n'}\}$ which converges to x_0 by assumption. Then,

$$\lim_{n'\to\infty} \partial_{xx}^2 V^{\varepsilon_{n'}}(t,x_{n'})\partial_{xx}^2 V_0(t,x_{n'}) = \partial_{xx}^2 V^{\varepsilon_0}(t,x_0)\partial_{xx}^2 V_0(t,x_0) \le 0. \tag{4.12}$$

Indeed,

$$\begin{aligned}
&\left|\partial_{xx}^2 V^{\varepsilon_{n'}}(t,x_{n'})\partial_{xx}^2 V_0(t,x_{n'}) - \partial_{xx}^2 V^{\varepsilon_0}(t,x_0)\partial_{xx}^2 V_0(t,x_0)\right| \\
&\le \left|\partial_{xx}^2 V^{\varepsilon_{n'}}(t,x_{n'})\partial_{xx}^2 V_0(t,x_{n'}) - \partial_{xx}^2 V^{\varepsilon_{n'}}(t,x_0)\partial_{xx}^2 V_0(t,x_{n'})\right| \\
&\quad + \left|\partial_{xx}^2 V^{\varepsilon_{n'}}(t,x_0)\partial_{xx}^2 V_0(t,x_{n'}) - \partial_{xx}^2 V^{\varepsilon_{n'}}(t,x_0)\partial_{xx}^2 V_0(t,x_0)\right| \\
&\quad + \left|\partial_{xx}^2 V^{\varepsilon_{n'}}(t,x_0)\partial_{xx}^2 V_0(t,x_0) - \partial_{xx}^2 V^{\varepsilon_0}(t,x_0)\partial_{xx}^2 V_0(t,x_0)\right| \\
&\le \partial_{xx}^2 V_0(t,x_{n'})B_3\left|x_{n'} - x_0\right|^\kappa + \partial_{xx}^2 V^{\varepsilon_{n'}}(t,x_0)B_3\left|x_{n'} - x_0\right|^\kappa \\
&\quad + \partial_{xx}^2 V_0(t,x_0)\left|\partial_{xx}^2 V^{\varepsilon_{n'}}(t,x_0) - \partial_{xx}^2 V^{\varepsilon_0}(t,x_0)\right|,
\end{aligned} \tag{4.13}$$

where the last inequality is due to the Hölder continuity of $\partial_{xx}^2 V^\varepsilon$ for all ε. Then, due to the continuity of $\partial_{xx}^2 V_0(t,\cdot)$, $\partial_{xx}^2 V_0(t,x_{n'})$ converges to $\partial_{xx}^2 V_0(t,x_0)$ as n' tends to ∞. Based on the fact of the uniform boundedness of $\left\{\partial_{xx}^2 V^\varepsilon(t,\cdot)\right\}$ on K_ρ and the convergence of $\left\{\partial_{xx}^2 V^{\varepsilon_{n'}}(t,x_0)\right\}$ to $\partial_{xx}^2 V^{\varepsilon_0}(t,x_0)$, it is clear that all three terms in (4.13) converge to 0. Therefore, (4.12) follows. We discuss two possible cases for the value of ε_0.

1. Case $\varepsilon_0 > 0$: We also know that $\varepsilon_0 \le \varepsilon$, since $\varepsilon_{n'} \le \varepsilon$. According to the definition of S_t^ε, $x_0 \in S_t^\varepsilon$.
2. Case $\varepsilon_0 = 0$: From (4.12), $\left(\partial_{xx}^2 V_0(t,x_0)\right)^2 \le 0$. Therefore, $\partial_{xx}^2 V_0(t,x_0) = 0$, i.e., $x \in S_t^0$ due to Lemma 4.8.

Therefore, the lemma is proved. ∎

Note that any $x \in \bar{S}_t^0$ is a zero point of $\partial_{xx}^2 V_0$ at time t. Let $U(t,x) := \partial_{xx}^2 V_0(t,x)$. We shall consider the zero set of $U(t,x)$ for any fixed $t \in [0,T)$. With the assumption (4.3) on φ, V_0 has up to fourth derivative with respect to x. Therefore, we can derive the equation for $U(t,x)$ from (2.2) as follows:

$$\partial_t U + \sigma^2 U + 2\sigma x \partial_x U + \frac{1}{2}\sigma^2 x^2 \partial^2_{xx} U = 0,$$
$$U(T) = \varphi''.$$

Let $x := \log y$, $\tilde{U}(t,y) := U(t,x)$. Then, $\tilde{U}(t,x)$ solves the following PDE:

$$\partial_t \tilde{U} + \frac{3}{2}\sigma^2 \partial_y \tilde{U} + \frac{1}{2}\sigma^2 \partial^2_y \tilde{U} + \sigma^2 \tilde{U} = 0.$$

Notice that all coefficients in the above equation are constant. Therefore, Theorem B in [1] and the remark below it are applicable to $\tilde{U}$. They directly imply that the size of the zero set of $\tilde{U}$,

$$Z_t = \left\{ y \middle| \tilde{U}(t,y) = 0 \right\},$$

is nonincreasing as variable t goes from T to 0. Noting that the change of variables $x = \log y$ is a one-to-one mapping, we can conclude the following proposition.

Proposition 4.9. *If φ satisfies (4.3), then $\partial^2_{xx} V_0$ has at most the same number of zero points as φ'' does for any fixed t.*

At this point, we can conclude that if φ'' has a finite number of zero points, then the $g^{d,\varepsilon}$ will be vanishing as ε decreases to 0. However, to achieve the goal that the expectation of compact part goes to 0, we still need to show that the law of variable $X^{*,\varepsilon}_s$ does not give a positive probability to any single point for any fixed $s \in [t,T]$.

Recall the main theorem (Theorem 1.1) in [15] that for every $t \in (0,T)$, the marginal law of

$$M_t = \int_0^t \alpha_s dW_s$$

does not weight points, where (α_t) is any progressively measurable process such that

$$0 < \sigma \le \alpha_t \le \sigma + \varepsilon.$$

The following lemma is a simple extension to the above result.

Lemma 4.10. *Let $(X^{*,\varepsilon}_t)$ solve the SDE (3.4). For any $t \in (0,T]$, $X^{*,\varepsilon}_t$ does not weight points.*

Proof. Due to the transformation applied in section 3.2.1, we need only prove the claim for the process $(Y^{*,\varepsilon}_t)$ which solves

$$dY^{*,\varepsilon}_t = -\frac{1}{2}\left(\alpha^{*,\varepsilon}_t\right)^2 dt + \alpha^{*,\varepsilon}_t dW_t$$

on the probability space $(\Omega, \mathbb{F}, \mathbb{Q})$.

Let

$$\xi_t = \exp\left(-\int_0^t \frac{\left(\alpha^{*,\varepsilon}_s\right)^2}{8} \mathrm{d}s + \int_0^t \frac{\alpha^{*,\varepsilon}_s}{2} \mathrm{d}W_s \right) \quad \text{for } t \le T.$$

Define a measure $\tilde{\mathbb{Q}}$ on $\mathscr{F}_T$ by

$$\mathrm{d}\tilde{\mathbb{Q}} = \xi_T \mathrm{d}\mathbb{Q}.$$

According to Girsanov's theorem, under the measure $\tilde{\mathbb{Q}}$,

$$\tilde{W}_t = -\int_0^t \frac{\alpha_s^{*,\varepsilon}}{2} \mathrm{d}s + W_t$$

is a Brownian motion and $(Y_t^{*,\varepsilon})$ has the following dynamic:

$$dY_t^{*,\varepsilon} = \alpha_t^{*,\varepsilon} d\tilde{W}_t.$$

Note that the worst case scenario volatility process $(\alpha_t^{*,\varepsilon})$ for χ is an adapted and bounded process. According to the Novikov condition, (ξ_t) is a martingale, and therefore $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$ are two equivalent measures.

From Theorem 1.1 in [15], we learn that the law of $Y_t^{*,\varepsilon}$ does not weight points under the measure $\tilde{\mathbb{Q}}$ for any $t \in [0, T)$. Due to equivalence between $\mathbb{Q}$ and $\tilde{\mathbb{Q}}$, we can claim that $Y_t^{*,\varepsilon}$ does not weight points under the measure $\mathbb{Q}$. Therefore, the lemma follows. ∎

For given $s \in [t, T]$, we cannot directly use the continuity of a probability measure to claim that $\lim_{\varepsilon \downarrow 0} \mathbb{Q}_{tx} \left[X_s^{*,\varepsilon} \in \bar{S}_s^\varepsilon \right] = \mathbb{Q}_{tx}[X_s^{*,0} \in \bar{S}_s^0]$, because both the process $(X_t^{*,\varepsilon})$ and the set $\bar{S}_t^\varepsilon$ depend on ε. Therefore, we define a capacity from the laws of a family of random variables $\left\{ X_s^{*,\varepsilon} \right\}_\varepsilon$ as follows:

$$c(A) := \sup_{\varepsilon \in [0,1]} \mathbb{Q}_{tx} \left(X_s^{*,\varepsilon} \in A \right)$$

for any $A \in \mathscr{B}(\mathbb{R})$.

Proposition 4.11. *$X_s^{*,\varepsilon}$ converges weakly to X_s for any $s > 0$.*

Proof. It is a direct implication of Theorem 2.1. ∎

If φ'' has a finite number of zero points, then due to Lemma 4.10, $\mathbb{Q}_{tx} \left(X_s^{*,\varepsilon} \in \bar{S}_s^0 \right) = 0$ for any $\varepsilon > 0$. This fact directly leads to $c(\bar{S}_s^0) = 0$. Due to the weak convergence of $\left\{ X_s^{*,\varepsilon} \right\}$, the family of laws of $\left\{ X_s^{*,\varepsilon} \right\}$ is weakly compact.

Lemma 4.12. *If φ satisfies condition (4.3) and φ'' has a finite number of zero points, then*

$$\lim_{\varepsilon \downarrow 0} \mathbb{Q}_{tx} \left[X_s^{*,\varepsilon} \in \bar{S}_s^\varepsilon \right] = 0$$

for any $s \in (t, T)$.

Proof. First, we observe that

$$0 \leq \mathbb{Q}_{tx} \left(X_s^{*,\varepsilon} \in \bar{S}_s^\varepsilon \right) \leq c(\bar{S}_s^\varepsilon).$$

Notice that $\left\{ \bar{S}_s^\varepsilon \right\}$ is a sequence of decreasing closed sets and converges to $\bar{S}_s^0$ as ε goes to 0. Because of the weak compactness of the laws of $\left\{ X_s^{*,\varepsilon} \right\}$ and Lemma 8 in [7], it can be seen that

$$c(\bar{S}_s^\varepsilon) \downarrow c(\bar{S}_s^0) = 0.$$

Then, the lemma is true. ∎

Theorem 4.13. *If φ satisfies (4.3) and φ'' has a finite number of zero points, then there exists $\varepsilon_0 > 0$ such that $\mathbb{E}_{tx}[|\text{i}|] < \delta$ for any fixed $\rho > 0$, for any fixed point $(t, x) \in (0, T] \times \mathbb{R}$, and for all $\delta > 0$ as long as $\varepsilon < \varepsilon_0$.*

Proof. Recall from (4.9) that

$$\mathbb{E}_{tx}[\text{i}] = \mathbb{E}_{tx}\left[\int_t^T g^{d,\varepsilon}(t, X_s^{*,\varepsilon})\mathbb{I}_{K_\rho}(X_s^{*,\varepsilon})\sigma\left(X_s^{*,\varepsilon}\right)^2 \partial_{xx}^2 V_0(s, X_s^{*,\varepsilon})ds\right].$$

Note that $g^{d,\varepsilon}(t, x)$ can take only three possible values: $\{-1, 0, 1\}$. Indeed,

$$\left|g^{d,\varepsilon}(t, x)\right| = \begin{cases} 1 & \text{if } \partial_{xx}^2 V^\varepsilon(t, x)\partial_{xx}^2 V_0(t, x) < 0, \\ 0 & \text{if } \partial_{xx}^2 V^\varepsilon(t, x)\partial_{xx}^2 V_0(t, x) \geq 0. \end{cases}$$

Therefore, due to (4.10) it follows that

$$\begin{aligned} \mathbb{E}_{tx}[|i|] &= \mathbb{E}_{tx}\left[\int_t^T \left|g^{d,\varepsilon}(s, X_s^{*,\varepsilon})\right| \mathbb{I}_{K_{\rho(\delta)}}(X_s^{*,\varepsilon})\sigma\left(X_s^{*,\varepsilon}\right)^2 \left|\partial_{xx}^2 V_0(s, X_s^{*,\varepsilon})\right| ds\right] \\ &\leq \sigma M_1 \left[\mathbb{E}_{tx}\int_t^T \left(X_s^{*,\varepsilon}\right)^4 \left(1 + |X_s^{*,\varepsilon}|^m\right)^2 ds\right]^{1/2} \left[\int_t^T \mathbb{Q}_{tx}(X_s^{*,\varepsilon} \in \bar{S}_s^\varepsilon)ds\right]^{1/2} \\ &\leq \sigma M_1 \sqrt{N(T-t, m, \sigma)(1 + |x|^{2m+4})}\left[\int_t^T \mathbb{Q}_{tx}(X_s^{*,\varepsilon} \in \bar{S}_s^\varepsilon)ds\right]^{1/2}. \end{aligned}$$

Due to Lemma 4.12, it follows that

$$\int_t^T \mathbb{Q}_{tx}(X_s^{*,\varepsilon} \in \bar{S}_s^\varepsilon)ds \to 0$$

as $\varepsilon \downarrow 0$. Therefore, the lemma follows. ■

Now, we are ready to claim the main result of this section.

Theorem 4.14. *If φ satisfies (4.3) and φ'' has a finite number of zero points, then $\lim_{\varepsilon\downarrow 0} I_1 = 0$ for any fixed $(t, x) \in [0, T) \times \mathbb{R}$.*

Proof. Note that

$$|I_1| \leq \mathbb{E}_{tx}[|\text{i}|] + \mathbb{E}_{tx}[|\text{ii}|].$$

For any $\delta > 0$, due to Proposition 4.4 there exists $\rho_0(t, x, \delta) > 0$ such that $\mathbb{E}_{tx}[|\text{ii}|] < \delta$ for all $\rho > \rho_0(t, x, \delta)$.

By Theorem 4.13, for the given $\rho_0(t, x, \delta)$ and δ there exist $\varepsilon_0(t, x, \rho_0(t, x, \delta))$ such that $\mathbb{E}_{tx}[|\text{i}|] < \delta$ for any $\varepsilon < \varepsilon_0(t, x, \rho_0(t, x, \delta))$. Therefore, the theorem follows. ■

From the arguments in this section, we essentially proved that $\bar{\alpha} = \sigma + \varepsilon\bar{g}$ is a good approximation of the worst case scenario volatility $\alpha^{*,\varepsilon}$; see their definitions in section 3.2.1. Together with the properties of the law of the asset price process in the worst case scenario, we proved the main theorem that $Z^\varepsilon(t,x)$ goes to 0 as $\varepsilon \downarrow 0$ for any $(t,x) \in [0,T) \times \mathbb{R}$. Therefore, we can construct an approximation of V^ε of order $o(\varepsilon)$: $V_0 + \varepsilon V_1$. The performance of this approximation procedure is studied numerically in the next section.

5. Numerical results. In this section, we will work on a nontrivial example: a symmetric European butterfly with the payoff function

$$\varphi(x) = (x-90)^+ - 2(100-x)^+ + (x-110)^+ \tag{5.1}$$

represented in Figure 1.

Even though the payoff function (5.1) does not satisfy the conditions of the main theorem (Theorem 2.3), we could consider a regularization $\bar{\varphi}$ of φ, which would satisfy the conditions of Theorem 2.3. As explained in Remark 2.1, the justification of this regularization step is beyond the scope of this paper. Here, we simply apply our approximation to the payoff (5.1). In the numerical computation presented below we directly discretize the payoff function (5.1) using discretization steps $\Delta x = 0.05$ and $\Delta t = 0.005$.

We numerically compute the worst case scenario price $V^\varepsilon(\varphi)$ by the scheme provided in [18]. It is proved by Barles [4] that the numerical solution from that scheme is locally uniformly convergent to V^ε, the unique viscosity solution of (2.1), as the scheme becomes finer. We also compare the numerically computed worst case scenario price with its approximation, $V_0(\varphi) + \varepsilon V_1(\varphi)$, where $V_0(\varphi)$ is given by the Black–Scholes formula and $V_1(\varphi)$ is numerically computed by a simple difference scheme according to (2.3). Because the scheme for computing $V^\varepsilon(\varphi)$ uses the Newton iteration in each time step to deal with the nonlinearity, our approximation is computed a lot more efficiently. For visual comparison of the worst case scenario prices with corresponding approximations, we show complete numerical results for a very small $\varepsilon = 0.006$, a small $\varepsilon = 0.01$, and a $\varepsilon = 0.05$ which is not so small. Throughout all the experiments, we set $\sigma = 0.15$, $T = 0.25$, and $r = 0$. When $\varepsilon = 0.05$, the upper bound of the volatility interval is $\sigma + \varepsilon = 0.20$, which is 1/3 larger than the base volatility level $\sigma = 0.15$. In other words, even if $\varepsilon = 0.05$ is small, 5% volatility is significant. From Figure 2, we note that the worst case scenario prices are higher than the Black–Scholes prices. That is, we need extra cash to superreplicate the option when facing the model ambiguity. It also can be noted that the first order corrected prices capture the main feature of the worst case scenario prices V^ε for different values of ε.

To see the trend of the error of our approximation as ε increases, we choose eight equally spaced values from 0 to 0.05 for ε. For each ε, we compute the error of the approximation, which is defined by

$$\mathrm{error}(\varepsilon) = \sup_x |V^\varepsilon(0,x) - V_0(0,x) - \varepsilon V_1(0,x)|. \tag{5.2}$$

As shown in Figure 3 the error increases as ε becomes larger, so the second order approximation will be needed to improve accuracy for large values of ε. However, there is not an abrupt change in the range of ε we choose.

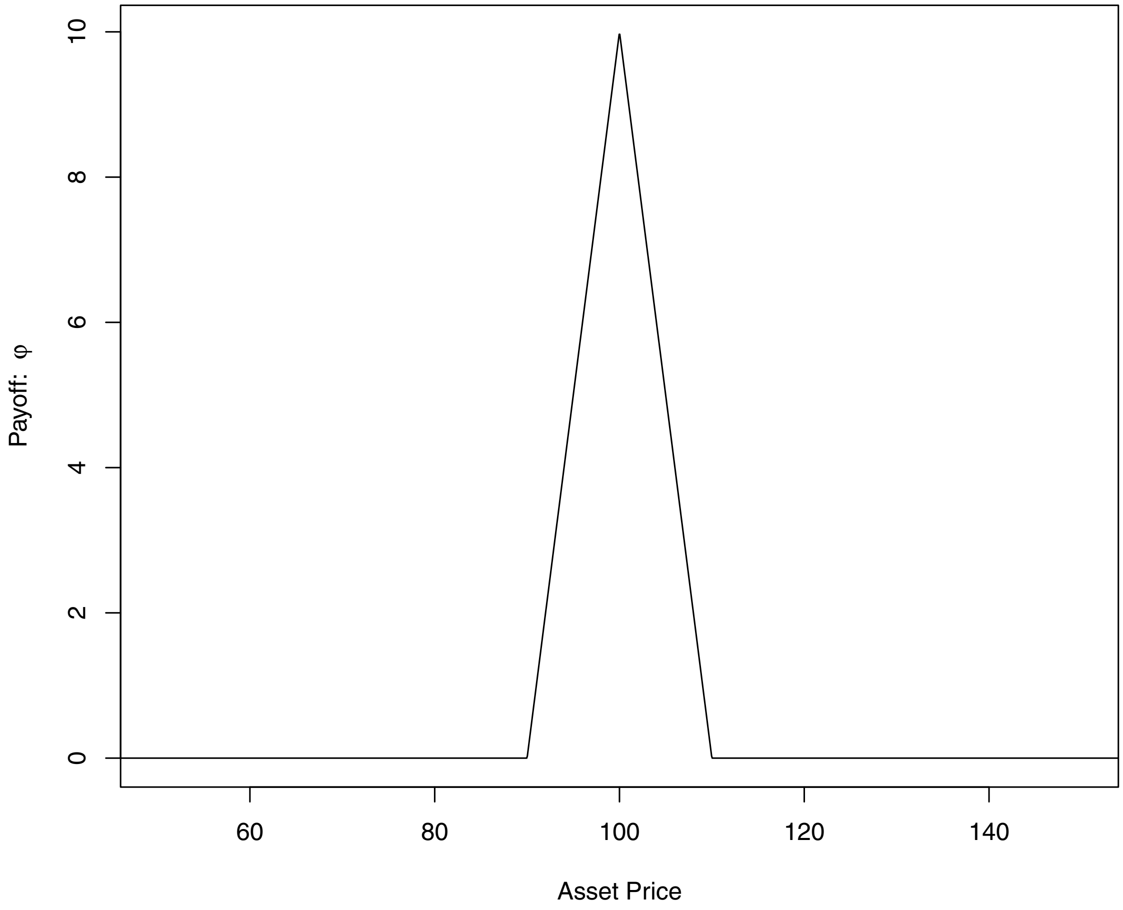

Figure 1. *The payoff function of a symmetric European butterfly.*

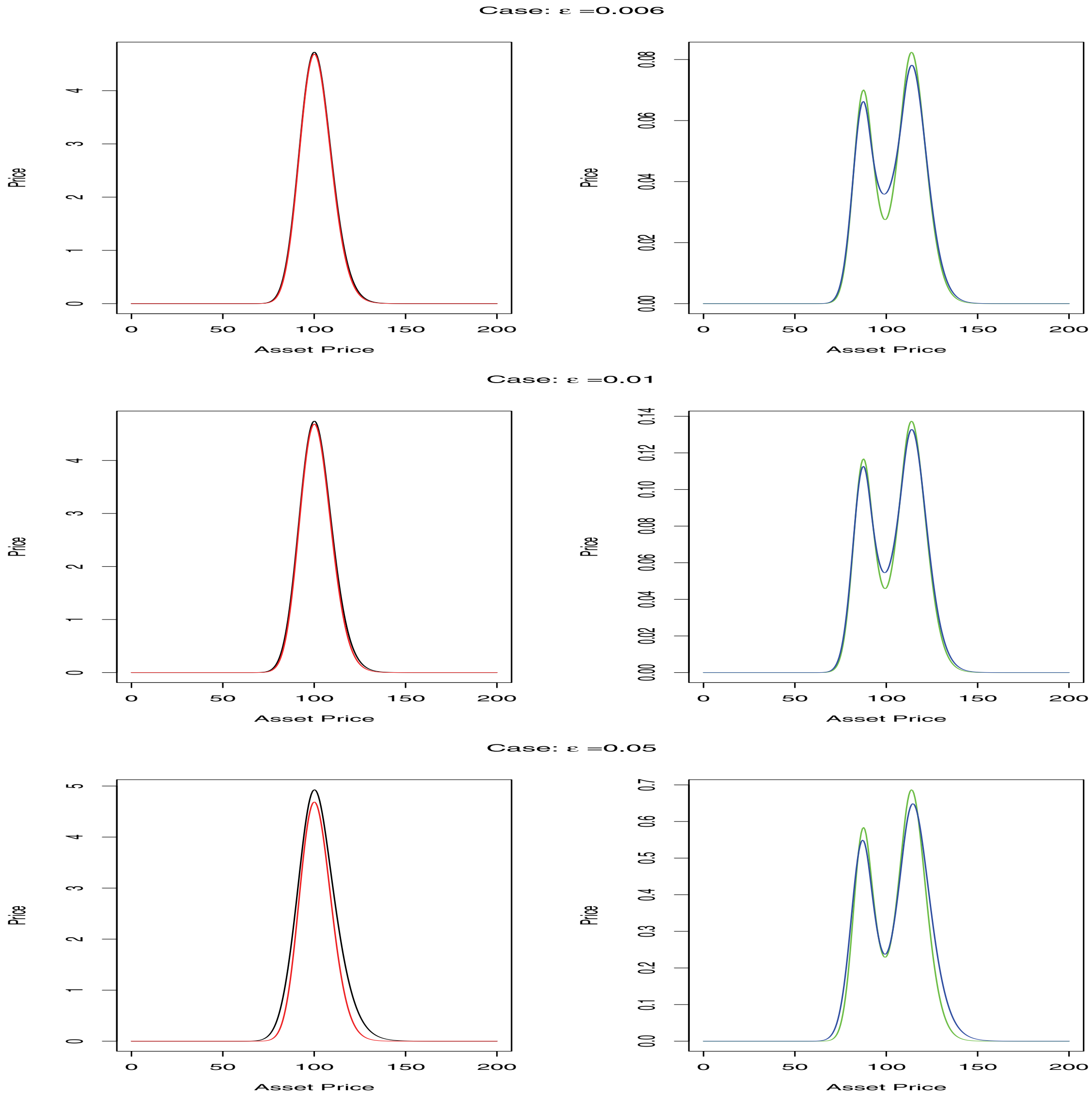

Figure 2. *The black curve represents the worst case scenario prices, and the red curve represents the Black–Scholes prices; the blue curve represents the difference between the worst case scenario prices and their Black–Scholes prices, and the green curve is εV_1; all curves are plotted against asset prices.*

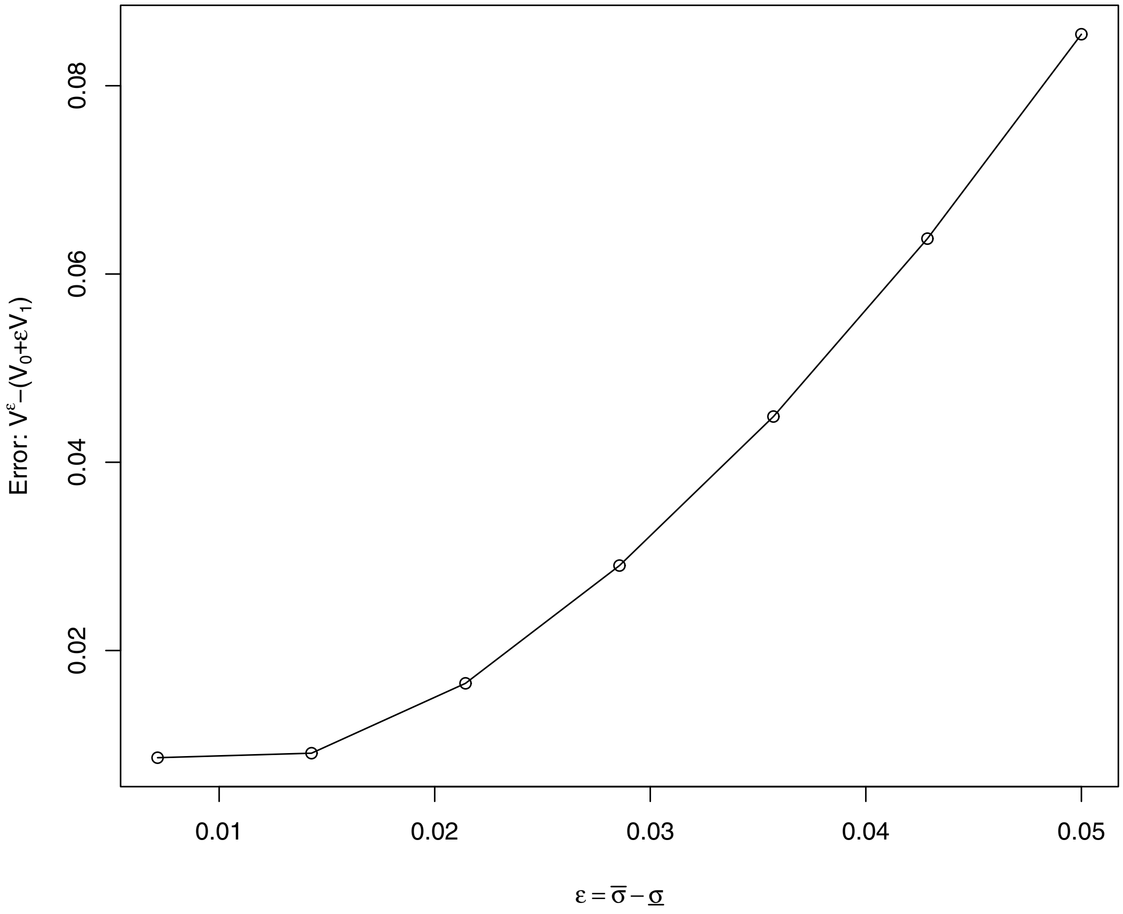

Figure 3. *Error for different values of ε.*

6. Conclusion. In this paper, we have studied the asymptotic behavior of the worst case scenario option prices as the degree of model ambiguity vanishes. This study not only helps us understand how a linear expectation turns into a sublinear expectation but also gives us an approximation procedure of worst case scenario option prices when ε is small. From the numerical results, we see that the approximation procedure works well even when the upper bound volatility is $1/3$ larger than the lower bound.

Note that the worst case scenario price is often computed to evaluate the risk in a portfolio. Our approximation procedure improves the efficiency of this evaluation, because it avoids the Newton iteration which is employed in the scheme for V^{ε}. Moreover, the worst case scenario price V^{ε} has to be recomputed for a new value of ε. However, (2.2) and (2.3) for V_0 and V_1 are independent of ε, so the approximation requires us only to compute V_0 and V_1 once for all small values of ε.

REFERENCES

[1] S. Angenent, *The zero set of a solution of a parabolic equation*, J. Reine Angew. Math., 390 (1988), pp. 79–96.

[2] P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath, *Coherent measures of risk*, Math. Finance, 9 (1999), pp. 203–228.

[3] M. Avellaneda, A. Levy, and A. Paras, *Pricing and hedging derivative securities in markets with uncertain volatilities*, Appl. Math. Finance, 2 (1995), pp. 73–88.

[4] G. Barles, *Convergence of numerical schemes for degenerate parabolic equations arising in finance theory*, in Numerical Methods in Finance, Cambridge University Press, Cambridge, UK, 1997, pp. 1–21.

[5] R. Cont, *Model uncertainty and its impact on the pricing of derivative instruments*, Math. Finance, 16 (2006), pp. 519–547.

[6] M. G. Crandall and A. Święch, *L^p-theory for fully nonlinear uniformly parabolic equations*, Comm. Partial Differential Equations, 25 (2000), pp. 1997–2053.

[7] L. Denis, M. Hu, and S. Peng, *Function spaces and capacity related to a sublinear expectation: Application to G-Brownian motion paths*, Potential Anal., 34 (2011), pp. 139–161.

[8] L. Denis and C. Martini, *A theoretical framework for the pricing of contingent claims in the presence of model uncertainty*, Ann. Appl. Probab., 16 (2006), pp. 827–852.

[9] W. H. Fleming and H. M. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer-Verlag, New York, 1992.

[10] J.-P. Fouque, G. Papanicolaou, R. Sircar, and K. Solna, *Singular perturbations in option pricing*, SIAM J. Appl. Math., 63 (2003), pp. 1648–1665.

[11] M.-H. Giga, Y. Giga, and J. Saal, *Nonlinear Partial Differential Equations: Asymptotic Behavior of Solutions and Self-Similar Solutions*, Birkhäuser Boston, Boston, MA, 2010.

[12] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin, 2001.

[13] N. V. Krylov, *Controlled Diffusion Processes*, Springer, New York, Berlin, 1980.

[14] T. J. Lyons, *Uncertain volatility and the risk-free synthesis of derivatives*, Appl. Math. Finance, 2 (1995), pp. 117–133.

[15] C. Martini, *On the marginal laws of one-dimensional stochastic integrals with uniformly elliptic integrand*, Ann. Inst. H. Poincaré Probab. Statist., 36 (2000), pp. 35–43.

[16] S. Peng, *G-Brownian Motion and Dynamic Risk Measure under Volatility Uncertainty*, preprint, arXiv: 0711.2834, 2007.

[17] S. Peng, *G-expectation, G-Brownian motion and related stochastic calculus of Itô type*, in Stochastic Analysis and Applications, Springer, Berlin, 2007, pp. 541–567.

[18] D. M. Pooley, P. A. Forsyth, and K. R. Vetzal, *Numerical convergence properties of option pricing PDEs with uncertain volatility*, IMA J. Numer. Anal., 23 (2003), pp. 241–267.

[19] H. M. Soner, N. Touzi, and J. Zhang, *Dual formulation of second order target problems*, Ann. Appl. Probab., 23 (2013), pp. 308–347.
[20] D. W. Stroock and S. R. S. Varadhan, *Multidimensional Diffusion Processes*, Springer, Berlin, 2005.
[21] J. Vorbrink, *Financial Markets with Volatility Uncertainty*, preprint, arXiv:1012.1535, 2010.
[22] L. Wang, *On the regularity theory of fully nonlinear parabolic equations*: I, Comm. Pure Appl. Math., 45 (1992), pp. 27–76.
[23] L. Wang, *On the regularity theory of fully nonlinear parabolic equations*: II, Comm. Pure Appl. Math. 45 (1992), pp. 141–178.