

Option Pricing under Hybrid Stochastic and Local Volatility

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Abstract This paper deals with an option pricing model which can be thought of as a hybrid stochastic and local volatility model. This model is built on the local volatility term of the well-known constant elasticity of variance (CEV) model multiplied by a stochastic volatility term driven by a fast mean-reverting Ornstein-Uhlenbeck process. An asymptotic formula for European option price is derived to complement the existing CEV pricing formula. Subsequently, empirical experiments are presented showing a better agreement of the proposed model with market data than the CEV model in terms of dynamics and geometric nature of the implied volatilities.

Keywords: stochastic volatility, constant elasticity of variance, asymptotic analysis, option pricing, implied volatility

1. INTRODUCTION

The geometric Brownian motion assumption for underlying asset price in the standard Black-Scholes model (1973) is well-known not to capture the accumulated empirical evidence in financial industry. A main draw back in the assumption of the Black-Scholes model is in flat implied volatilities, which is contradictory to empirical results showing that the implied volatilities of the equity options exhibit the smile or skew curve. For example, Rubinstein (1985) before the 1987 crash and by Jackwerth and Rubinstein (1996) after the crash belong to those representative results. Before the 1987 crash, geometry of implied volatilities against the strike price was often observed to be U-shaped with minimum at or near the money but, after the crash, the shape of smile curve becomes more typical.

Among those several ways of overcoming the above draw back and extending the geometric Brownian motion to incorporate the smile effect, there is a way that makes the volatility depend on underlying asset price as well as time. One renowned model in this category is the well-known constant elasticity of variance diffusion model. This model was initially studied by Cox (1975), and Cox and Ross (1976) and was designed and developed to incorporate the negative correlation between underlying asset price change and volatility change. The CEV diffusion has been applied to exotic options as well as standard options by many authors. For example, see Boyle and Tian (1999), Boyle et al. (1999), Davydov and Linetsky (2001, 2003), Lipton and McGhee (2002), and Linetsky (2004) for studies of exotic options. See Beckers (1980), Emanuel and MacBeth (1982), Schroder (1989), Delbaen and Shirakawa (2002), and Carr and Linetsky (2006) for general option pricing studies. The book by Jeanblanc, Yor, and Chesney (2006) may serve as a general reference.

One disadvantage in local volatility models is, however, that volatility and underlying risky asset price changes are perfectly correlated either positively or negatively depending on the chosen elasticity parameter, whereas empirical studies show that prices may decrease when volatility increases or vice versa but it seems that there is no definite correlation all the time. For example, Harvey

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(2001) and Ghysels et al. (1996) show that it is time varying. Some empirical works may better motivate the need for an extension of the CEV model. It is found in Ballestra and Pacelli (2011) that “the CEV model does not offer a correct description of equity prices”, whereas Emanuel and Macbeth (1982) states that the CEV model with stationary parameters does not appear to be able to explain the mispricing of call options by the Black-Scholes model.

Based on these observations together with the renowned contribution of stochastic volatility formulation to option pricing, it seems to be required to have a hybrid structure of local and stochastic volatility. In fact, models like this hybrid type have been used by practitioners and academics. Jex et al. (1999), Lipton (2002) and Hagan et al. (2002) are important references belonging to this class. The volatility of the first paper is modelled by the square root function of two components, that is, a stochastic mean reverting process and a deterministic local component. The second paper considers the jump nature of the spot in addition to the local and stochastic features of its volatility. The third paper deals with a model (called the SABR model) of stochastic volatility multiplied by a CEV term. Our paper formulates a volatility combining a parametric CEV local volatility, which is lacking in the first and second papers, and a mean reverting stochastic volatility, which is absent from the third paper. Particularly, we take a mean-reversion rate sufficiently high enough to apply the averaging theory developed by Fouque et al. (2011) and references therein. If the mean reversion is fast but still finite, then the CEV model needs to be corrected to account for stochastic volatility. In general, it is very difficult to find explicitly solvable derivative’s prices for the mixed stochastic and local volatilities. The third paper among those mentioned above has an explicit formula in this respect but their model does not contain a drift term so that it has the limitation of application to option pricing. Our paper exploits asymptotic technique of Fouque et al. (2011) to extend analytically the well-known explicit formula for the CEV model and demonstrate some improvement over the traditional CEV model.

This paper is organized as follows. In Section 2 we review the well-known CEV option price formula for the completeness of this paper. Then in Section 3 we formulate an option pricing problem as a stochastic volatility extension of the CEV model in terms of a partial differential equation (PDE). In Section 4, we use multiscale asymptotic technique on the option pricing PDE to derive a correction to the well-known CEV option price formula. Section 5 is devoted to give an error estimate to verify mathematical rigor for the approximation. In Section 6, we study geometry and dynamics of the implied volatilities and some calibration experiment for our model. Some concluding remarks are given in Section 7.

2. THE CEV FORMULA - REVIEW

In this section we review briefly the well-known CEV option price formula.

The underlying asset price of the renowned CEV model is governed by the SDE

$$(1) \quad dX_t = \mu X_t dt + \sigma X_t^{\frac{\theta}{2}} dW_t,$$

where μ , σ (positive) and θ are constants. In this model, volatility is a function of underlying asset price which is given by $\sigma X_t^{\frac{\theta}{2}-1}$. Depending upon the parameter $\frac{\theta}{2}$, this model can reduce to the well-known popular option pricing models. For example, If $\theta = 2$, it reduces to the Black-Scholes model (1973) with constant volatility σ . If $\theta = 1$, it becomes the square root model or Cox-Ross model (1976). If $\theta = 0$, then it is the absolute model. If $\theta < 2$, the volatility and the stock price

move inversely. If $\theta > 2$, the volatility and the stock price move in the same direction. We note that the formal definition of elasticity is given by $\eta := \theta - 2$.

There is an analytic form of European option price formula for the CEV diffusion solving (1). It was first given by Cox (1975) using the transition probability density function for the CEV diffusion. He did not show the derivation of the transition probability density function but it is not difficult to obtain it. Basically, one transforms X_t into Y_t via $Y_t = X_t^{2-\theta}$, $\theta \neq 2$ and then apply the Ito lemma and the Feynman-Kac formula. Then using the well-known result on the parabolic PDE by Feller (1951) one can obtain the transition probability density function as follows:

$$(2) \quad D(\tilde{x}, T; x, t) = (2 - \theta)k^{\frac{1}{2-\theta}}(uv^{1-\theta})^{\frac{1}{2(2-\theta)}}e^{-u-v}I_{\frac{1}{2-\theta}}(2(uv)^{\frac{1}{2}}), \quad 0 \leq \theta < 2,$$

where

$$\begin{aligned} k &= \frac{2r}{\sigma^2(2-\theta)(e^{r(2-\theta)(T-t)} - 1)}, \\ u &= kx^{2-\theta}e^{r(2-\theta)(T-t)}, \\ v &= k\tilde{x}^{2-\theta} \end{aligned}$$

and $I_q(x)$ is the modified Bessel function of the first kind of order q defined by

$$I_q(x) = \sum_{n=0}^{\infty} \frac{(\frac{x}{2})^{2n+q}}{r! \Gamma(n+1+q)}$$

Once the transition probability density function is obtained, the call option pricing formula for the CEV model follows as sum of two integrals:

$$\begin{aligned} C_{CEV}(t, x) &= \int_{-\infty}^{\infty} e^{-r(T-t)}(\tilde{x} - K)^+ D(\tilde{x}, T; x, t) d\tilde{x} \\ &= e^{-r(T-t)} \int_K^{\infty} \tilde{x} D(\tilde{x}, T; x, t) d\tilde{x} - e^{-r(T-t)} K \int_K^{\infty} D(\tilde{x}, T; x, t) d\tilde{x}. \end{aligned}$$

This Cox's result is further extended to the case of $\theta > 2$ by Emanuel and MacBeth (1982). The computation of the integrals in the Cox formula was done by Schroder (1989) and Chen and Lee (1993) in terms of an infinite series. We write here the simpler result of Davydov and Linetsky (2001) which includes both cases of $\theta < 2$ and $\theta > 2$. Of course, the put option price can be given by the put-call parity.

Theorem 2.1: The call option price $C_{CEV}(t, x)$ is given by

$$(3) \quad C_{CEV}(t, x) = \begin{cases} xQ(\zeta; n-2, \delta_0) - e^{-rT}K(1 - Q(\delta_0; n, \zeta)), & \beta > 0 \\ xQ(\delta_0; n, \zeta) - e^{-rT}K(1 - Q(\zeta; n-2, \delta_0)), & \beta < 0 \end{cases}$$

where

$$(4) \quad \begin{aligned} \beta &= \frac{\theta - 2}{2}, \\ n &= 2 + \frac{1}{|\beta|}, \\ \zeta &= \frac{2\mu x^{-2\beta}}{\sigma^2 \beta (e^{2\mu\beta T} - 1)}, \\ \delta_0 &= \frac{2\mu K^{-2\beta}}{\sigma^2 \beta (1 - e^{-2\mu\beta T})} \end{aligned}$$

and $Q(x; u, v)$ is the complementary noncentral chi-square distribution function with u degrees of freedom and the noncentrality parameter v .

Proof: Refer to Davydov and Linetsky (2001).

Alternatively, the call option price can be obtained by solving the pricing PDE

$$(5) \quad \partial_t C + \frac{1}{2} \sigma^2 x^\theta \partial_{xx}^2 C + r(x \partial_x C - C) = 0,$$

$$(6) \quad C(T, x) = (x - K)^+.$$

This PDE can be solved, for example, by the Green's function method after being reduced to a simple form. See Lipton (2001) for details.

Sometimes it is more convenient to rewrite the above option formula in the form that directly generalizes the standard Black-Scholes formula (see, for example, Lipton (2001)):

$$(7) \quad C_{CEV}(t, x) = e^{-r(T-t)} x \int_{\check{K}}^{\infty} \left(\frac{\check{x}}{y} \right)^{\frac{1}{2(2-\theta)}} e^{-(\check{x}+y)} I_{\frac{1}{2-\theta}}(2\sqrt{\check{x}y}) dy \\ + e^{-r(T-t)} K \int_{\check{K}}^{\infty} \left(\frac{y}{\check{x}} \right)^{\frac{1}{2(2-\theta)}} e^{-(\check{x}+y)} I_{\frac{1}{2-\theta}}(2\sqrt{\check{x}y}) dy,$$

where

$$\check{x} = \frac{2xe^{r(2-\theta)(T-t)}}{(2-\theta)^2 \xi}, \\ \xi = \frac{\sigma^2}{(2-\theta)r} (e^{r(2-\theta)T} - e^{r(2-\theta)t}), \\ \check{K} = \frac{2K^{2-\theta}}{(2-\theta)^2 \xi}.$$

Note that by using the asymptotic formula

$$I_{\frac{1}{2\gamma}} \left(\frac{x}{\gamma^2} \right) \sim \frac{\gamma e^{x/\gamma^2 - 1/8x}}{\sqrt{2\pi x}}, \quad \gamma \rightarrow 0$$

for the Bessel function, one can show that as θ goes to 2 the CEV call price C_{CEV} goes to the usual Black-Scholes call option price C_{BS} given by

$$(8) \quad C_{BS}(t, x) = xN(d_1) - Ke^{-r(T-t)}N(d_2),$$

where

$$d_1 = \frac{\log(x/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 = d_1 - \sigma\sqrt{T-t},$$

and

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-y^2/2} dy.$$

3. PROBLEM FORMULATION

In this section, we establish a hybrid option pricing model that extends the local volatility of the CEV model to incorporate stochastic volatility.

First, the local volatility term $X_t^{\frac{\theta}{2}}$ of the CEV model is changed into such a form as a product of a function of external process Y_t and the internal term $X_t^{\frac{\theta}{2}}$ so that the volatility of the asset price depends on both internal and external noise sources. In this form, it would be natural to choose Y_t as a mean-reverting diffusion process since the newly formed volatility gets back to the mean level that corresponds to the CEV case. The mean-reverting Ornstein-Uhlenbeck process is an example of such process.

As an underlying asset price model in this paper, therefore, we take the SDEs

$$(9) \quad dX_t = \mu X_t dt + f(Y_t) X_t^{\frac{\theta}{2}} dW_t$$

$$(10) \quad dY_t = \alpha(m - Y_t) dt + \beta d\hat{Z}_t,$$

where $\alpha > 0$, $\beta > 0$ and W_t and \hat{Z}_t are correlated Brownian motions such that $d\langle W, \hat{Z} \rangle_t = \rho dt$. If $\rho > 0$, then X_t may fail to be a true martingale because Y_t may explode to infinite under the risk neutral measure. Refer to Andersen and Piterbarg (2007). So, $\rho \leq 0$ is imposed here. We write $\hat{Z}_t = \rho W_t + \sqrt{1 - \rho^2} Z_t$ for some standard Brownian motion Z_t independent of W_t . We do not specify the concrete form of $f(y)$ since it will not play an essential role in the asymptotic analysis performed in this paper. However, $f(y)$ has to satisfy a sufficient growth condition to avoid some kind of bad behavior such as the non-existence of moments of X_t . We assume that $0 < c_1 \leq f \leq c_2 < \infty$ for some constants c_1 and c_2 . It is well-known from the Ito formula that the solution of (10) is a Gaussian process given by

$$Y_t = m + (Y_0 - m)e^{-\alpha t} + \beta \int_0^t e^{-\alpha(t-s)} d\hat{Z}_s$$

and thus $Y_t \sim N(m + (Y_0 - m)e^{-\alpha t}, \frac{\beta^2}{2\alpha}(1 - e^{-2\alpha t}))$ leading to invariant distribution given by $N(m, \frac{\beta^2}{2\alpha})$. Later, we will use notation $\langle \cdot \rangle$ for the average with respect to the invariant distribution, i.e.,

$$\langle g \rangle = \frac{1}{\sqrt{2\pi\nu^2}} \int_{-\infty}^{+\infty} g(y) e^{-\frac{(y-m)^2}{2\nu^2}} dy, \quad \nu^2 \equiv \frac{\beta^2}{2\alpha}$$

for arbitrary function g . Notice that ν^2 denotes the variance of the invariant distribution of Y_t .

Once an equivalent martingale measure Q , under which the discounted asset price $e^{-rt}X_t$ is a martingale, is provided, the option price is given by the formula

$$(11) \quad P(t, x, y) = E^Q[e^{-r(T-t)}h(X_T)|X_t = x, Y_t = y]$$

under Q , where h is the payoff function for the European option. The existence of the martingale measure Q is guaranteed by the well-known Girsanov theorem which further provides that processes defined by

$$W_t^* = W_t + \int_0^t \frac{\mu - r}{f(Y_s)X_s^{\frac{\theta-2}{2}}} ds,$$

$$Z_t^* = Z_t + \int_0^t \gamma_s ds$$

are independent standard Brownian motion under Q , where γ_t is arbitrary adapted process to be determined. We will use notation Q^γ in stead of Q to emphasize the dependence on γ . It may raise a question whether Novikov's condition is satisfied on W^* since the integrand of W^* may blow up depending upon the behavior of X_t or Y_t . This issue is usually addressed by assuming that μ depends on (X_t, Y_t) in such a way that the ratio $(\mu(X_s, Y_s) - r)/(f(Y_s)X_s^{\frac{\theta-2}{2}})$ is bounded (see Chapter 2 in Fouque et al. (2011) for instance).

We first change the subjective SDEs (9)-(10) into the risk-neutral version as follows. Under an equivalent martingale measure Q^γ , we have

$$(12) \quad dX_t = rX_t dt + f(Y_t)X_t^{\frac{\theta}{2}} dW_t^*,$$

$$(13) \quad dY_t = (\alpha(m - Y_t) - \beta\Lambda(t, y)) dt + \beta d\hat{Z}_t^*,$$

where $\hat{Z}_t^* = \rho W_t^* + \sqrt{1 - \rho^2} Z_t^*$ and Λ (the market price of volatility risk), a function of γ , is assumed to be independent of x . Then the Feynman-Kac formula leads to the following pricing PDE for the option price $P(t, x, y)$:

$$(14) \quad \begin{aligned} P_t + \frac{1}{2}f^2(y)x^\theta P_{xx} + \rho\beta f(y)x^{\frac{\theta}{2}}P_{xy} + \frac{1}{2}\beta^2 P_{yy} \\ + rxP_x + (\alpha(m - y) - \beta\Lambda(t, y))P_y - rP = 0. \end{aligned}$$

4. ASYMPTOTIC OPTION PRICING

Volatility is known to be fast mean-reverting when looked at over the time scale of an option. Refer to Fouque et al. (2000) for some empirical analysis. To obtain an asymptotic pricing result on the fast mean-reversion, we need a small parameter, say ϵ , to denote the inverse of the rate of mean reversion α :

$$(15) \quad \epsilon = \frac{1}{\alpha}$$

which is also the typical correlation time of the OU process Y_t . We assume that ν , which was introduced in Section 3 as $\nu = \frac{\beta}{\sqrt{2\alpha}}$ (the standard deviation of the invariant distribution of Y_t), remains fixed in scale as ϵ becomes zero. Thus we have $\alpha \sim O(\epsilon^{-1})$, $\beta \sim O(\epsilon^{-1/2})$, and $\nu \sim O(1)$.

The methodology to be used here is asymptotic analysis outlined by Fouque et al. (2011) which is based upon a mean-reverting Ornstein-Uhlenbeck (OU) diffusion which decorrelates rapidly while fluctuating on a fast time-scale. The choice of fast mean-reverting OU process provides us analytic advantage. This is generically related to averaging principle and ergodic theorem or, more directly, asymptotic diffusion limit theory of stochastic differential equations with a small parameter which has been initiated by Khasminskii (1966) and developed by Papanicolaou (1978), Asch et al. (1991), Kim (2004), Cerrai (2009) and etc.

After rewritten in terms of ϵ , the PDE (14) becomes

$$(16) \quad \left(\frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2 \right) P^\epsilon(t, x, y) = 0,$$

where

$$(17) \quad \mathcal{L}_0 = \nu^2 \partial_{yy}^2 + (m - y) \partial_y,$$

$$(18) \quad \mathcal{L}_1 = \sqrt{2\rho\nu} f(y) x^{\frac{\theta}{2}} \partial_{xy}^2 - \sqrt{2\nu} \Lambda(t, y) \partial_y,$$

$$(19) \quad \mathcal{L}_2 = \partial_t + \frac{1}{2} f(y)^2 x^\theta \partial_{xx}^2 + r(x \partial_x - \cdot).$$

Here, the operator $\alpha \mathcal{L}_0$ which acts on the y variable is the infinitesimal generator of the OU process Y_t under the physical measure. The operator \mathcal{L}_1 is the sum of two terms; the first one is the mixed partial differential operator due to the correlation of the two Brownian motions W^* and \hat{Z}^* and the second one is the first-order differential operator with respect to y due to the market price of elasticity risk. Finally, the third operator \mathcal{L}_2 is a generalized version of the classical Black-Scholes operator at the volatility level $\sigma(x, y) = f(y) x^{\frac{\theta}{2}-1}$ in stead of constant volatility.

Before we solve the problem (16)-(19), we write a useful lemma about the solvability (centering) condition on the Poisson equation related to the above operator \mathcal{L}_0 as follows:

Lemma 4.1: If solution to the Poisson equation

$$(20) \quad \mathcal{L}_0 \chi(y) + \psi(y) = 0$$

exists, then the condition $\langle \psi \rangle = 0$ must be satisfied, where $\langle \cdot \rangle$ is the expectation with respect to the invariant distribution of Y_t . If then, solutions of (20) are given by the form

$$(21) \quad \chi(y) = \int_0^{+\infty} E^y[\psi(Y_t) | Y_0 = y] dt.$$

Proof: Refer to Fouque et al. (2011). \square

Although the problem (16)-(19) is a singular perturbation problem, we are able to obtain a limit of P^ϵ as ϵ goes to zero and also characterize the first correction for small but nonzero ϵ . In order to do it, we first expand P^ϵ in powers of $\sqrt{\epsilon}$:

$$(22) \quad P^\epsilon = P_0 + \sqrt{\epsilon} P_1 + \epsilon P_2 + \dots$$

Here, the choice of the power unit $\sqrt{\epsilon}$ in the power series expansion was determined by the method of matching coefficient.

Substituting (22) into (16) yields

$$(23) \quad \begin{aligned} & \frac{1}{\epsilon} \mathcal{L}_0 P_0 + \frac{1}{\sqrt{\epsilon}} (\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0) \\ & + (\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0) + \sqrt{\epsilon} (\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1) + \dots = 0 \end{aligned}$$

which holds for arbitrary $\epsilon > 0$.

Under some reasonable growth condition, one can show that the first corrected price $P_0 + \sqrt{\epsilon} P_1$ is independent of the unobserved variable y as follows.

Theorem 4.1 Assume that P_0 and P_1 do not grow as much as

$$\frac{\partial P_i}{\partial y} \sim e^{y^2}, \quad y \rightarrow \infty, \quad i = 0, 1.$$

Then P_0 and P_1 do not depend on the variable y .

Proof: From the asymptotic expansion (23), we first have

$$(24) \quad \mathcal{L}_0 P_0 = 0.$$

Solving this equation yields

$$P_0(t, x, y) = c_1(t, x) \int_0^y e^{-\frac{(\theta-z)^2}{2\nu^2}} dz + c_2(t, x)$$

for some functions c_1 and c_2 independent of y . From the imposed growth condition on P_0 , $c_1 = 0$ must hold. This leads that $P_0(t, x, y)$ becomes a function of only t and x . We denote it by

$$(25) \quad P_0 = P_0(t, x).$$

Since each term of the operator \mathcal{L}_1 contains y -derivative, the y -independence of P_0 yields $\mathcal{L}_1 P_0 = 0$. On the other hand, from the expansion (23), $\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0 = 0$ holds. These facts are put together and yield $\mathcal{L}_0 P_1 = 0$ so that P_1 also is a function of only t and x ;

$$(26) \quad P_1 = P_1(t, x).$$

□

We next derive a PDE that the leading order price P_0 satisfies based upon the observation (Theorem 4.1) that P_0 and P_1 do not depend on the current level y of the process Y_t .

Theorem 4.2: Under the growth condition on P_i ($i = 0, 1$) expressed in Theorem 4.1, the leading term $P_0(t, x)$ is given by the solution of the PDE

$$(27) \quad \partial_t P_0 + \frac{1}{2} \bar{\sigma}^2 x^\theta \partial_{xx}^2 P_0 + r(x \partial_x P_0 - P_0) = 0$$

with the terminal condition $P_0(T, x) = h(x)$, where

$$\bar{\sigma} = \sqrt{\langle f^2 \rangle}.$$

So, $P_0(t, x)$ (call) is the same as the price in the CEV formula in Theorem 2.1, where only σ is replaced by the effective coefficient $\bar{\sigma}$.

Proof: From the expansion (23), the PDE

$$(28) \quad \mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0 = 0$$

holds. Since each term of the operator \mathcal{L}_1 contains y -derivative, $\mathcal{L}_1 P_1 = 0$ holds so that (28) leads to

$$(29) \quad \mathcal{L}_0 P_2 + \mathcal{L}_2 P_0 = 0$$

which is a Poisson equation. Then the centering condition must be satisfied. From Lemma 4.1 with $\psi = \mathcal{L}_2 P_0$, the leading order $P_0(t, x)$ has to satisfy the PDE

$$(30) \quad \langle \mathcal{L}_2 \rangle P_0 = 0$$

with the terminal condition $P_0(T, x) = h(x)$, where

$$\langle \mathcal{L}_2 \rangle = \partial_t + \frac{1}{2} \langle f^2 \rangle x^\theta \partial_{xx}^2 + r(x \partial_x - \cdot).$$

Thus P_0 solves the PDE (27). \square

Next, we derive the first correction term $P_1(t, x)$. Later in Section 5, we will estimate the error of the approximation $P_0 + \sqrt{\epsilon}P_1$. For convenience, we need notation

$$(31) \quad V_3 = \frac{\rho\nu}{\sqrt{2}}\langle f\psi' \rangle,$$

$$(32) \quad V_2 = -\frac{\nu}{\sqrt{2}}\langle \Lambda\psi' \rangle,$$

where $\psi(y)$ is solution of the Poisson equation

$$(33) \quad \mathcal{L}_0\psi = \nu^2\psi'' + (m - y)\psi' = f^2 - \langle f^2 \rangle.$$

Note that V_3 and V_2 are constants.

Theorem 4.3: The correction term $P_1(t, x)$ satisfies the PDE

$$(34) \quad \begin{aligned} \partial_t P_1 + \frac{1}{2}\bar{\sigma}^2 x^\theta \partial_{xx}^2 P_1 + r(x\partial_x P_1 - P_1) &= v(t, x), \\ v(t, x) &:= V_3 x^{\frac{\theta}{2}} \frac{\partial}{\partial x} \left(x^\theta \frac{\partial^2 P_0}{\partial x^2} \right) + V_2 x^\theta \frac{\partial^2 P_0}{\partial x^2} \end{aligned}$$

with the final condition $P_1(T, x) = 0$, where V_3 and V_2 are given by (31) and (32), respectively. Further, the solution $P_1(t, x)$ ($\theta \neq 2$) of (34) is given by

$$(35) \quad P_1(t, x) = - \int_{-\infty}^{\infty} \int_t^T e^{-r(s-t)} v(s, \tilde{x}) D(\tilde{x}, s; x, t) ds d\tilde{x},$$

where $D(\tilde{x}, s; x, t)$ is the same as (2) except that σ is now replaced by the effective one $\bar{\sigma} = \sqrt{\langle f^2 \rangle}$.

Proof: From (19) and (30)

$$\begin{aligned} \mathcal{L}_2 P_0 &= \mathcal{L}_2 P_0 - \langle \mathcal{L}_2 P_0 \rangle \\ &= \frac{1}{2} x^\theta (f^2 - \langle f^2 \rangle) \partial_{xx}^2 P_0 \end{aligned}$$

holds. Then (29) and (33) lead to

$$(36) \quad \begin{aligned} P_2 &= -\mathcal{L}_0^{-1}(\mathcal{L}_2 P_0) \\ &= -\frac{1}{2} x^\theta \mathcal{L}_0^{-1} (f^2 - \langle f^2 \rangle) \partial_{xx}^2 P_0 \\ &= -\frac{1}{2} (\psi(y) + c(t, x)) x^\theta \partial_{xx}^2 P_0 \end{aligned}$$

for arbitrary function $c(t, x)$ independent of y .

On the other hand, from the expansion (23)

$$\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 = 0$$

holds. This is a Poisson PDE for P_3 . Thus the centering condition $\langle \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1 \rangle = 0$ has to be satisfied. Then using the result (36) we obtain

$$\begin{aligned} \langle \mathcal{L}_2 \rangle P_1 &= \langle \mathcal{L}_2 P_1 \rangle \\ &= -\langle \mathcal{L}_1 P_2 \rangle \\ &= \frac{1}{2} \langle \mathcal{L}_1 \psi \rangle x^\theta \partial_{xx}^2 P_0. \end{aligned}$$

Here, the term $c(t, x)$ disappears since the operator \mathcal{L}_1 given by (18) takes derivatives with respect to y . Applying (18), (19), (31) and (32) to this result, we obtain the PDE (34).

Now, by applying the Feynman-Kac formula to the PDE problem (34), we obtain

$$P_1(t, x) = E^{Q^\gamma} \left[- \int_t^T e^{-r(s-t)} \left(V_3 X_s^\theta \frac{\partial}{\partial \tilde{x}} (X_s^\theta \frac{\partial^2 P_0}{\partial \tilde{x}^2}) + V_2 X_s^\theta \frac{\partial^2 P_0}{\partial \tilde{x}^2} \right) ds \mid X_t = x \right]$$

which leads to the integral (35) from the transition probability density function (2). \square

5. ACCURACY - ERROR ESTIMATE

Combining the leading order P_0 and the first correction term $\tilde{P}_1 = \sqrt{\epsilon} P_1$, we have the approximation $P_0 + \tilde{P}_1$ to the original price P . Naturally, the next concern is about the accuracy of the approximation $P_0 + \tilde{P}_1$ to the price P .

First, we write a well-known property of solutions to the Poisson equation (20) about their boundedness or growth as follows:

Lemma 5.1 If there are constant C and integer $n \neq 0$ such that

$$(37) \quad |\psi(y)| \leq C(1 + |y|^n),$$

then solutions of the Poisson equation (20), i.e. $\mathcal{L}_0 \chi(y) + \psi(y) = 0$, satisfy

$$(38) \quad |\chi(y)| \leq C_1(1 + |y|^n)$$

for some constant C_1 . If $n = 0$, then $|\chi(y)| \leq C_1(1 + \log(1 + |y|))$.

Proof: Refer to Fouque et al. (2011). \square

Now, we are ready to estimate the error of the approximation $P_0 + \tilde{P}_1$ to the price P .

Theorem 5.1 If each of f and Λ is bounded by a constant independent of ϵ , then we have

$$(39) \quad P^\epsilon - (P_0 + \tilde{P}_1) = O(\epsilon).$$

Proof: We first define $Z^\epsilon(t, x, y)$ by

$$Z^\epsilon = \epsilon P_2 + \epsilon \sqrt{\epsilon} P_3 - [P^\epsilon - (P_0 + \tilde{P}_1)].$$

Once we show that $Z^\epsilon = O(\epsilon)$, the estimate (39) will follow immediately.

Using the operators (17)-(19), we define

$$\mathcal{L}^\epsilon = \frac{1}{\epsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathcal{L}_1 + \mathcal{L}_2$$

and apply the operator \mathcal{L}^ϵ to Z^ϵ to obtain

$$\begin{aligned}
(40) \quad \mathcal{L}^\epsilon Z^\epsilon &= \mathcal{L}^\epsilon(P_0 + \sqrt{\epsilon}P_1 + \epsilon P_2 + \epsilon\sqrt{\epsilon}P_3 - P^\epsilon) \\
&= \mathcal{L}^\epsilon(P_0 + \sqrt{\epsilon}P_1 + \epsilon P_2 + \epsilon\sqrt{\epsilon}P_3) \\
&= \frac{1}{\epsilon}\mathcal{L}_0 P_0 + \frac{1}{\sqrt{\epsilon}}(\mathcal{L}_0 P_1 + \mathcal{L}_1 P_0) + (\mathcal{L}_0 P_2 + \mathcal{L}_1 P_1 + \mathcal{L}_2 P_0) \\
&\quad + \sqrt{\epsilon}(\mathcal{L}_0 P_3 + \mathcal{L}_1 P_2 + \mathcal{L}_2 P_1) + \epsilon(\mathcal{L}_1 P_3 + \mathcal{L}_2 P_2) + \epsilon\sqrt{\epsilon}\mathcal{L}_2 P_3 \\
&= \epsilon(\mathcal{L}_1 P_3 + \mathcal{L}_2 P_2) + \epsilon\sqrt{\epsilon}\mathcal{L}_2 P_3.
\end{aligned}$$

Here, we have used $\mathcal{L}^\epsilon P^\epsilon = 0$ and the fact that the expansion (23) holds for arbitrary ϵ . The result (40) yields $\mathcal{L}^\epsilon Z^\epsilon = O(\epsilon)$.

Now, let F^ϵ and G^ϵ denote

$$(41) \quad F^\epsilon(t, x, y) = \mathcal{L}_1 P_3 + \mathcal{L}_2 P_2 + \sqrt{\epsilon}\mathcal{L}_2 P_3,$$

$$(42) \quad G^\epsilon(x, y) = P_2(T, x, y) + \sqrt{\epsilon}P_3(T, x, y),$$

respectively. Then from (40)-(42) we have a parabolic PDE for Z^ϵ of the form

$$\mathcal{L}^\epsilon Z^\epsilon - \epsilon F^\epsilon = 0$$

with the final condition $Z^\epsilon(T, x, y) = \epsilon G^\epsilon(x, y)$. Applying the Feynman-Kac formula to this PDE problem leads to

$$Z^\epsilon = \epsilon E^{Q^\gamma} \left[e^{-r(T-t)} G^\epsilon - \int_t^T e^{-r(s-t)} F^\epsilon ds \mid X_t^\epsilon = x, Y_t^\epsilon = y \right]$$

in the risk-neutral world. Assuming smoothness and boundedness of the payoff h , one can derive that F^ϵ and G^ϵ are bounded uniformly in x and at most linearly growing in $|y|$ from Lemma 5.1 (see Chapter 4 in Fouque et al. (2011) for details). Consequently, the desired error estimate $Z^\epsilon = O(\epsilon)$ follows.

In general, as for a call option for instance, the payoff h is not smooth as assumed above. However, generalization of the proof to the non-smooth case of European vanilla call or put option is possible following to the regularization argument in Fouque et al. (2003). The detailed proof is omitted here. \square

6. IMPLIED VOLATILITIES AND CALIBRATION

6.1. Asymptotics near $\theta = 2$. To obtain the correction term \tilde{P}_1 , whose PDE is given by Theorem 4.3, it is required to estimate the parameters V_3 and V_2 , which is not an easy task. Since the (non-homogeneous) Black-Scholes equation is easier to solve than the (non-homogeneous) CEV equation, it is desirable to exploit the known results on the Black-Scholes model as much as possible in order to estimate V_3 and V_2 . Also, one can observe that, practically, the value of the parameter θ is close to 2 as shown in Table 1 under some appropriate adjustment of the numerical boundary condition. So, in this section, we perform another perturbation for θ near 2.

For convenience, we use the implied volatility I defined by

$$C_{BS}(t, x; K, T; I) = C^{\text{observed}}(K, T),$$

where $C^{\text{observed}}(K, T)$ is the observed call option price in the market. After we choose $I_0(K, T)$ such that

$$C_{BS}(t, x; K, T; I_0) = P_0(t, x) = C_{CEV}(\bar{\sigma}),$$

if we use the expansion $I = I_0 + \sqrt{\epsilon}I_1 + \dots$ on the left-hand side and our approximation $P = P_0 + \tilde{P}_1 + \dots$ on the right-hand side, then we have

$$C_{BS}(t, x; K, T; I_0) + \sqrt{\epsilon}I_1 \frac{\partial C_{BS}}{\partial \sigma}(t, x; K, T; I_0) + \dots = P_0(t, x) + \tilde{P}_1(t, x) + \dots,$$

which yields that the implied volatility is given by

$$(43) \quad I = I_0 + \tilde{P}_1(t, x) \left(\frac{\partial C_{BS}}{\partial \sigma}(t, x; K, T; I_0) \right)^{-1} + O(1/\alpha),$$

where $\frac{\partial C_{BS}}{\partial \sigma}$ (Vega) is well-known to be given by

$$\begin{aligned} \frac{\partial C_{BS}}{\partial \sigma}(t, x; K, T; I_0) &= \frac{x e^{-d_1^2/2} \sqrt{T-t}}{\sqrt{2\pi}}, \\ d_1 &= \frac{\log(x/K) + (r + \frac{1}{2}I_0^2)(T-t)}{I_0 \sqrt{T-t}}. \end{aligned}$$

We observe that when $\theta = 2$, P_0 becomes the Black-Scholes call option price with volatility $\bar{\sigma}$, denoted by $C_{BS}(\bar{\sigma})$, and \tilde{P}_1 is given as follows by the formula obtained by Fouque et al. (2011) as a consequence of the fact that the Black-Scholes operator and the operator \mathcal{A} defined below commute:

$$(44) \quad \tilde{P}_1 = -(T-t)H_0,$$

where

$$(45) \quad H_0 = \mathcal{A}(V_3^*, V_2^*)C_{BS}(\bar{\sigma})$$

with

$$\begin{aligned} V_3^* &= \frac{\rho\nu}{\sqrt{2\alpha}} \langle f \psi' \rangle, \\ V_2^* &= -\frac{\nu}{\sqrt{2\alpha}} \langle \Lambda \psi' \rangle. \end{aligned}$$

Here, we used a differential operator $\mathcal{A}(\alpha, \beta)$ defined by

$$\mathcal{A}(\alpha, \beta) = \alpha x \frac{\partial}{\partial x} \left(x^2 \frac{\partial^2}{\partial x^2} \right) + \beta x^2 \frac{\partial^2}{\partial x^2},$$

and $\psi(y)$ as a solution of the Poisson equation (33). Note that to compute $\mathcal{A}(V_3^*, V_2^*)$ it is required to have Gamma $\left(= \frac{\partial^2}{\partial x^2} P_0 \right)$ and Speed $\left(= \frac{\partial^3}{\partial x^3} P_0 \right)$ of the Black-Scholes price. So, by fitting (43) to volatilities implied by the Black-Scholes model, one can obtain V_3^* and V_2^* .

From now on we approximate P_0 and \tilde{P}_1 near $\theta = 2$. For this, we use notation

$$\phi = 2 - \theta$$

in terms of which we expand P_0 as

$$P_0 = C_{CEV}(\bar{\sigma}) = C_{BS}(\bar{\sigma}) + \phi P_{0,1} + \phi^2 P_{0,2} + \dots$$

and H , defined by $H(t, x) = \mathcal{A}(V_3^*, V_2^*)P_0$, and \tilde{P}_1 , respectively, as

$$\begin{aligned} H &= H_0 + \phi H_1 + \phi^2 H_2 + \dots, \\ \tilde{P}_1 &= -(T-t)H_0 + \phi \tilde{P}_{1,1} + \phi^2 \tilde{P}_{1,2} + \dots. \end{aligned}$$

Then from Theorem 4.2 $P_{0,1}$ satisfies the Black-Scholes equation with a non-zero source but with a zero terminal condition. It is given by

$$(46) \quad \begin{aligned} \mathcal{L}_{BS}(\bar{\sigma})P_{0,1} &= \frac{1}{2}\bar{\sigma}^2 x^2 \ln x \frac{\partial^2 C_{BS}}{\partial x^2}, \\ P_{0,1}(T, x) &= 0. \end{aligned}$$

We have $\mathcal{L}_{BS}(\bar{\sigma})H_0 = 0$ as it should be. From Theorem 4.3 $\tilde{P}_{1,1}$ is given by

$$(47) \quad \begin{aligned} \mathcal{L}_{BS}(\bar{\sigma})\tilde{P}_{1,1} &= H_1 - \frac{1}{2}\bar{\sigma}^2(T-t)x^2 \ln x \frac{\partial^2 H_0}{\partial x^2}, \\ \tilde{P}_{1,1}(T, x) &= 0, \end{aligned}$$

where H_0 is (45) and H_1 is given by

$$(48) \quad H_1 = \mathcal{A}(V_3^*, V_2^*)P_{0,1}.$$

Up to order ϕ^2 , therefore, the leading order term P_0 is given by

$$P_0 = C_{BS}(\bar{\sigma}) + \phi P_{0,1},$$

where $P_{0,1}$ is the solution of (46), and the correction \tilde{P}_1 is given by

$$(49) \quad \tilde{P}_1 = -(T-t)\mathcal{A}(V_3^*, V_2^*)C_{BS}(\bar{\sigma}) + \phi\tilde{P}_{1,1},$$

where $\tilde{P}_{1,1}$ is the solution of (47).

Synthesizing the discussion above, pricing methodology can be implemented as follows in the case near $\theta = 2$. After estimating $\bar{\sigma}$ (the effective historical volatility) from the risky asset price returns and confirming from the historical asset price returns that volatility is fast mean-reverting as in Fouque et al. (2011), fit (43) to the implied volatility surface I to obtain V_i^* , $i = 2, 3$. Then the call price of a European option corrected for stochastic volatility is approximated by

$$(50) \quad P_0 + \tilde{P}_1 = C_{BS}(\bar{\sigma}) - (T-t)\mathcal{A}(V_3^*, V_2^*)C_{BS}(\bar{\sigma}) + (2-\theta)(P_{0,1} + \tilde{P}_{1,1}),$$

where $\tilde{P}_{0,1}$ and $\tilde{P}_{1,1}$ are given (in integral form) by

$$\begin{aligned} P_{0,1} &= \frac{1}{2}\bar{\sigma}^2 E^{Q^\gamma} \left[- \int_t^T X_s^2 \ln X_s \frac{\partial^2 C_{BS}(\bar{\sigma})}{\partial x^2} ds \right], \\ \tilde{P}_{1,1} &= E^{Q^\gamma} \left[- \frac{1}{\sqrt{\alpha}} \int_t^T \mathcal{A}(V_3^*, V_2^*) P_{0,1} ds \right. \\ &\quad \left. + \frac{1}{2}\bar{\sigma}^2 \int_t^T (T-s) X_s^2 \ln X_s \frac{\partial^2}{\partial x^2} \left(\mathcal{A}(V_3^*, V_2^*) C_{BS}(\bar{\sigma}) \right) ds \right] \end{aligned}$$

from the Feynman-Kac formula applied to (46) and (47), respectively.

6.2. Empirical Results. As mentioned above, practically, the parameter θ can be shown to be close to 2. We calibrate the θ by using market data from the call option prices on the S&P500 index (SPX) taken from December 10th, 2010. We limit our data set to options with maturities between 66 and 508 days. Table 1 shows some typical values of the θ . They are less than 2 and close to the value 2 (zero elasticity corresponding to the Black-Scholes case). We note that the values of θ might be greater than 2 depending upon the type of underlying assets.

Next, we employ a calibration procedure for both the CEV model and our model and compare the results in terms of implied volatility. The corrected price $P_0 + \tilde{P}_1$ has been constructed based

Time to Maturity (day)	66	128	191	253	378	508
θ	1.9910	1.9913	1.9920	1.9900	1.9928	1.9923

TABLE 1. The Typical Values of the Parameter θ for SPX.

upon a volatility given by a function of the OU process multiplied by a local CEV volatility function. So, let us call our model as ‘the SVCEV model’ for convenience.

In Figure 1 from (a) to (f), we plot the implied volatilities of call options as well as the calibrated implied volatility curves for the CEV and SVCEV models. Figure 1 shows a proof of the improvement of the SVCEV model over the CEV model in term of the geometric structure of implied volatilities. Two CEV lines of implied volatility corresponding to $\theta = 1.8$ (CEV line (1)) and $\theta = 1.9$ (CEV line (2)) are shown in Figure 1. This linear structure of the CEV lines seems to be typical and far away from the real data. As θ decreases from 2, the corresponding CEV line may have a higher angle to become close to the real data but it still has a linear structure which is not the case, particularly, for short time-to-maturity options. In this experiment, one can also observe that if θ is smaller enough than 2, then the volatility constant σ may have to be chosen to be larger than 1 in order to fit the data.

In Figure 2, we plot the implied volatility curves as a function of underlying SPX price using calibration results of $\bar{\sigma} = 0.1354$, $V_2 = -0.0863$, $V_3 = -0.3209$, $\theta = 1.9920$ for SPX (Days to Maturity 191). Figure 2 demonstrates that the implied volatilities of the SVCEV model move in the desirable direction with respect to the underlying asset price, which is contrary to the CEV model.

7. CONCLUSION

Our option pricing model based upon a hybrid structure of stochastic volatility and constant elasticity of variance has demonstrated some improvements over the traditional CEV model. First, the implied volatility curves move in accordance with the real market phenomena with respect to the underlying asset price. This is an important result in the sense that otherwise it may create a possible hedging instability problem as stated by Hagan et al (2002). Second, the geometric structure of the implied volatilities shows a smile fitting the market data better than the CEV model. Consequently, the underlying asset price model may serve as a sound alternative one replacing the CEV model for financial problems such as credit risk and portfolio optimization apart from European option pricing discussed in this paper. As one of possible future works, a large-time large deviation principle for our model will be considered to derive large-time asymptotics for European options and implied volatility like a recent work done by Forde (2011) on an uncorrelated OU stochastic volatility model.

Acknowledgments

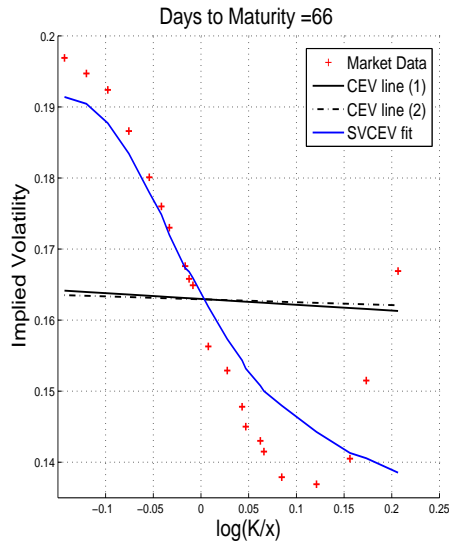
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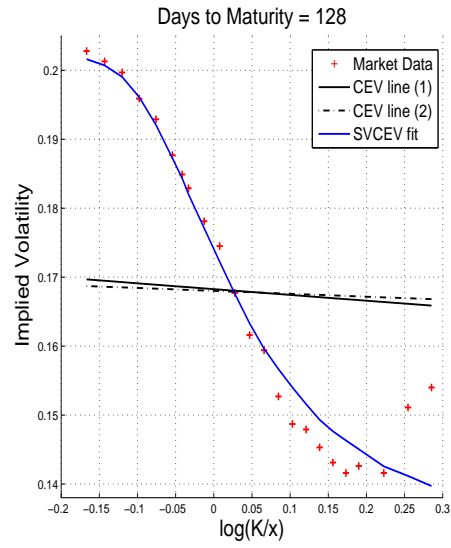
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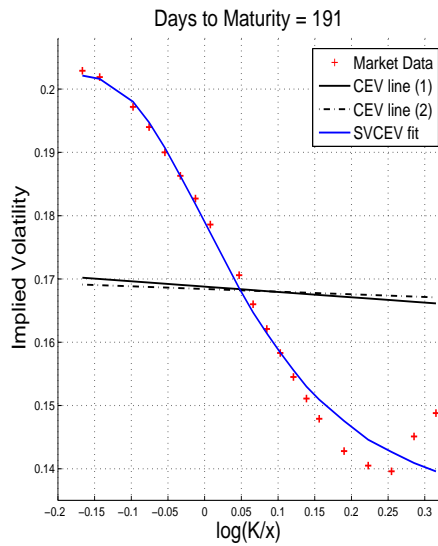
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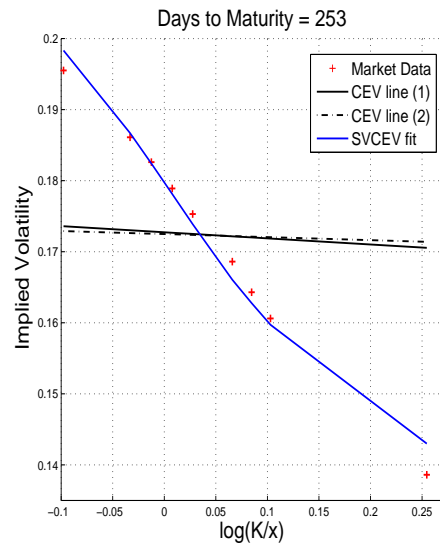
(a)



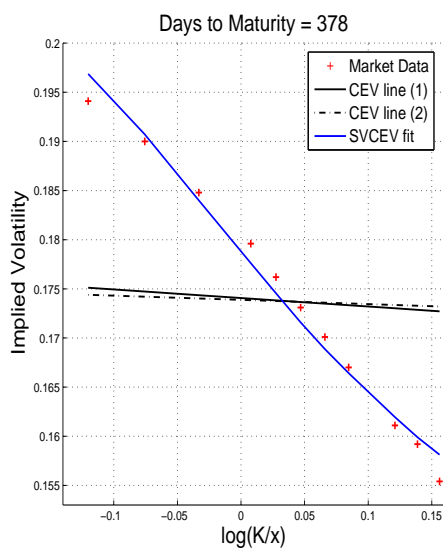
(b)



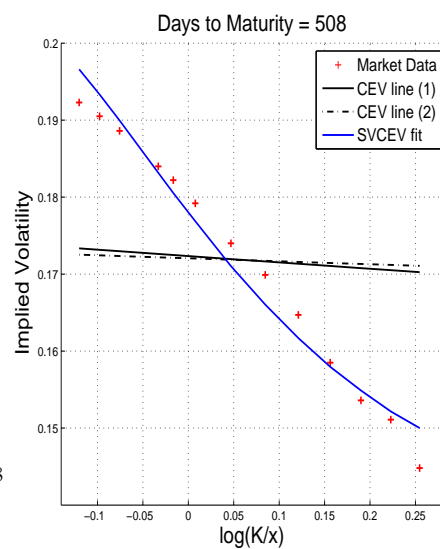
(c)



(d)



(e)



(f)

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FIGURE 1. SPX Implied Volatilities from December 10, 2010 from Days to Maturity 66 through 508

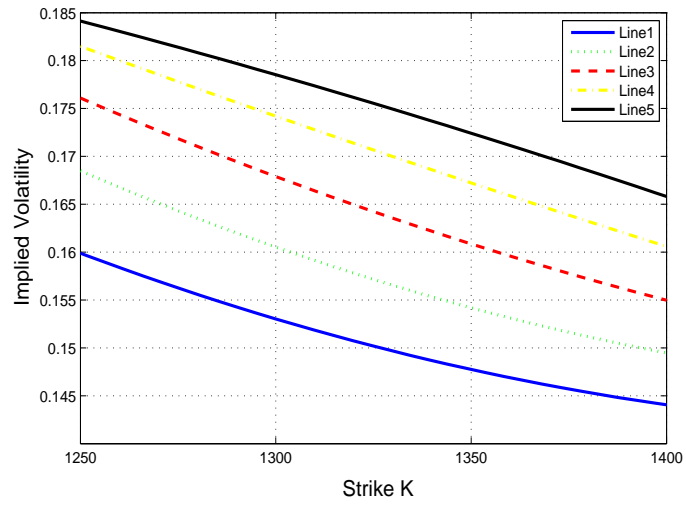


FIGURE 2. SPX, days to maturity 191, implied volatility curves are plotted as a function of SPX price; the Line 1: $x = 1116$, Line 2: $x = 1178$, Line 3: $x = 1240$, Line 4: $x = 1302$ and Line 5: $x = 1364$