MEAN FIELD GAMES AND SYSTEMIC RISK

RENÉ CARMONA*, JEAN-PIERRE FOUQUE†, AND LI-HSIEN SUN‡

Abstract.
We propose a simple model of inter-bank borrowing and lending where the evolution of the log-monetary reserves of N banks is described by a system of diffusion processes coupled through their drifts in such a way that stability of the system depends on the rate of inter-bank borrowing and lending. Systemic risk is characterized by a large number of banks reaching a default threshold by a given time horizon. Our model incorporates a game feature where each bank controls its rate of borrowing/lending to a central bank. The optimization reflects the desire of each bank to borrow from the central bank when its monetary reserve falls below a critical level or lend if it rises above this critical level which is chosen here as the average monetary reserve. Borrowing from or lending to the central bank is also subject to a quadratic cost at a rate which can be fixed by the regulator. We solve explicitly for Nash equilibria with finitely many players, and we show that in this model the central bank acts as a clearing house, adding liquidity to the system without affecting its systemic risk. We also study the corresponding Mean Field Game in the limit of large number of banks in the presence of a common noise.

Key words. Systemic risk, interbank borrowing and lending, stochastic games, Nash equilibrium, Mean Field Game.

Subject classifications. 60H30, 91A15, 91G20, 93E20

Dedicated to George Papanicolaou in honor of his 70th birthday

1. Introduction

Systemic risk is becoming a central research topic. We refer to the Handbook [8] for recent developments on systemic risk from many points of view (Statistics, Finance, Mathematical Finance, Behavioral Finance, Networks, Counterparty Risk, High Frequency Trading, ...). Here, we propose a simple model of inter-bank borrowing and lending where the evolution of the log-monetary reserves of N banks is described by a system of diffusion processes coupled through their drifts in such a way that stability of the system depends on the rate of inter-bank borrowing and lending. Systemic risk is characterized by a large number of banks reaching a default threshold by a given time horizon. This type of interaction and the relation *stability–systemic risk* has been recently studied in [9], [10], [11], and [12]. Here, we introduce a game feature where each bank controls its rate of borrowing/lending to a central bank. The control of each individual bank reflects the desire to borrow from the central bank when its monetary reserve falls below a critical level or lend if it rises above this critical level which is chosen here as the average monetary reserve. Borrowing from or lending to the central bank is also subject to a quadratic cost at a rate which can be fixed by the regulator. As written, our model is an example of Linear-Quadratic Mean Field Game with finitely many players which can be solved explicitly. We first solve for open-loop equilibria using the Pontryagin stochastic maximum principle. We also solve for closed-loop equilibria using the probabilistic approach based on the Pontryagin stochastic maximum principle leading to the solution of Forward-Backward Stochastic Differential Equations, and the dynamic programming principle leading to

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the solution of Hamilton-Jacobi-Bellman partial differential equations. We also study the corresponding Mean Field Game in the limit of a large number of banks. Adding a game component to the model has a non-trivial effect on the stability of the system which our simple model enables to analyze. Our conclusion is that inter-bank borrowing and lending creates stability, this stability is enhanced by the possibility of borrowing and lending from a central bank which in our model appears as a clearing house providing additional liquidity, and systemic risk is described as a rare event with probability quantified by large deviation theory.

In the model discussed below, the diffusion processes X_t^i for $i = 1, \ldots, N$ represent the log-monetary reserves of N banks lending to and borrowing from each other. The system is driven by N (possibly correlated) standard Brownian motions $\widetilde{W}_t^i, i = 1, \cdots, N$ written as $\widetilde{W}_t^i = \rho W_t^0 + \sqrt{1-\rho^2} W_t^i$ where $W_t^j, j = 0, 1, \cdots, N$ are independent standard Brownian motions, W_t^0 being the *common noise*, and $|\rho| \leq 1$. The system starts at time $t = 0$ from i.i.d. random variables $X_0^i = \xi^i$ independent of the Brownian motions and such that $\mathbb{E}(\xi^i) = 0$. We assume that the diffusion coefficients are constant and identical, denoted by $\sigma > 0$. Our model of lending and borrowing consists in introducing an interaction through drift terms representing the rate at which bank i borrows from or lends to bank j. In this case, the rates are proportional to the difference in log-monetary reserves, and our model is:

$$dX_t^i = \frac{a}{N} \sum_{j=1}^{N} (X_t^j - X_t^i)\, dt + \alpha_t^i dt + \sigma d\widetilde{W}_t^i\,, \quad i = 1, \ldots, N, \tag{1.1}$$

where the overall rate of "mean-reversion" a/N has been normalized by the number of banks with $a \geq 0$. Bank i controls its rate of borrowing/lending to a central bank through the control rate α_t^i. Using the notation

$$\overline{X}_t = \frac{1}{N} \sum_{i=1}^{N} X_t^i,$$

for the empirical mean, the dynamics can be rewritten in the mean field form:

$$dX_t^i = \left[a(\overline{X}_t - X_t^i) + \alpha_t^i\right] dt + \sigma d\widetilde{W}_t^i, \quad i = 1, \cdots, N. \tag{1.2}$$

Bank $i \in \{1, \cdots, N\}$ controls its rate of lending and borrowing at time t by choosing the control α_t^i in order to minimize

$$J^i(\alpha^1, \cdots, \alpha^N) = \mathbb{E}\left\{\int_0^T f_i(X_t, \alpha_t^i) dt + g_i(X_T^i)\right\}, \tag{1.3}$$

where the running cost function f_i is defined by

$$f_i(x, \alpha^i) = \left[\frac{1}{2}(\alpha^i)^2 - q\alpha^i(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2\right], \tag{1.4}$$

and the terminal cost function g_i by

$$g_i(x) = \frac{c}{2}\left(\overline{x} - x^i\right)^2. \tag{1.5}$$

Notice that the running quadratic cost $\frac{1}{2}(\alpha^i)^2$ has been normalized and that the effect of the parameter $q > 0$ is to control the *incentive* to borrowing or lending: the bank i

will want to borrow ($\alpha_t^i > 0$) if X_t^i is smaller than the empirical mean ($\overline{X}_t$) and lend ($\alpha_t^i < 0$) if X_t^i is larger than $\overline{X}_t$. Equivalently, after dividing by $q > 0$, this parameter can be thought as a control by the regulator of the cost of borrowing or lending (with q large meaning low fees).

The quadratic terms in $\left(\overline{x} - x^i\right)^2$ in the running cost ($\epsilon > 0$) and in the terminal cost ($c > 0$) penalize departure from the average. We assume that

$$q^2 \leq \epsilon, \tag{1.6}$$

so that $f_i(x, \alpha)$ is convex in (x, α).

In the spirit of structural models of defaults, we introduce a default level $D < 0$ and say that bank i defaults by time T if its log-monetary reserve reached the level D before time T. Note that in this simple model, even after reaching the default level, bank i stays in the system until time T and continues to participate in inter-bank and central bank borrowing and lending activities.

The paper is organized as follows. In Section 2 we recall the results of the descriptive model presented in [10] without control, that is $\alpha_t^i = 0$. We illustrate the fact that inter-bank borrowing and lending creates *stability*, and we define and quantify the *systemic risk*.

Section 3 is devoted to the analysis of the stochastic differential game (1.2-1.3). We derive exact Nash equilibria for the open-loop as well as for the closed-loop Markovian models, using both probabilist and analytic approaches.

The financial implications in terms of liquidity and role of a central bank are discussed in Section 4.

Stochastic differential games with a large number of players are usually not tractable. It is because of the very special nature of our model (linear dynamics, quadratic costs, and interactions through the empirical mean) that we are able to construct explicitly Nash equilibria, both open and closed loops. For generic models which are not amenable to explicit solutions, Lasry and Lions [18, 19, 20] have recently provided an elegant way to tackle the construction of approximate Nash equilibria for large games with mean field interactions. Their methodology, known as *Mean Field Game* (MFG), has been applied to a wide variety of problems (see [13, 17] for some examples). A similar research program was developed independently by Caines, Huang, and Malhamé with the name of Nash Certainty Equivalent. See for example [14] and [15]. The approach of Lasry and Lions (e.g. [20]) is based on the solution of a system of partial differential equations (PDEs): a Hamilton-Jacobi-Bellman equation evolving backward in time, and a Kolmogorov equation evolving forward in time, these two PDEs being strongly coupled. By its probabilistic nature, it is natural to recast the MFG strategy using appropriate forms of the Pontryagin stochastic maximum principle, leading to the solution of new models. See, for example, [6, 2, 5, 4] or [7] for a probabilistic approach based on the weak formulation of stochastic control. We consider the MFG problem and discuss the existence of approximate Nash equilibria in Section 5.

2. Stability and Systemic Risk

In this section, we consider a system of N banks without the possibility of bor-

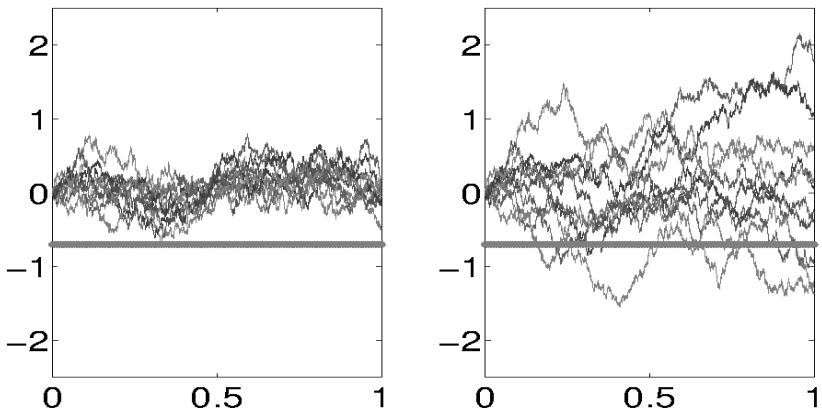

FIG. 2.1. *One realization of $N = 10$ trajectories of the coupled diffusions (2.1) (left plot) with $a = 10$ and trajectories of independent Brownian motions ($a = 0$) (right plot) using the same Gaussian increments. The solid horizontal line represents the "default" level $D = -0.7$.*

rowing or lending to a central bank, that is (1.1) or (1.2) with $\alpha_t^i = 0$:

$$\begin{aligned} dX_t^i &= \frac{a}{N}\sum_{j=1}^{N}(X_t^j - X_t^i)\,dt + \sigma d\widetilde{W}_t^i \\ &= a(\overline{X}_t - X_t^i)dt + \sigma d\widetilde{W}_t^i, \quad i = 1, \cdots, N. \end{aligned} \tag{2.1}$$

To start with, we assume that the Brownian motions $\widetilde{W}^i$ are independent, that is $\rho = 0$ and $\widetilde{W}^i = W^i$. For simplicity, we also assume that $X_0^i = \xi^i = 0$. In Section 2.4, we will consider the correlated (or *common noise*) case.

2.1. Simulations

In Figure 2.1 we show a typical realization of the N trajectories with a relatively large rate of borrowing/lending $a = 10$, the diffusion coefficient being set at $\sigma = 1$ (we used the Euler scheme with a time-step $\Delta = 10^{-4}$, up to time $T = 1$). We see that the trajectories generated by (2.1) are more grouped than the ones generated by independent Brownian motions corresponding to no inter-bank borrowing or lending ($a = 0$). This is the "swarming" or "flocking" effect more pronounced for a larger rate a. Consequently, less (or almost no) trajectories will reach the default level D, creating stability of the system.

Next, we compare the loss distribution (distribution of number of defaults) for the coupled and independent cases. We compute these loss distributions by Monte Carlo method using 10^4 simulations, and with the same parameters as previously.

In the independent case ($a = 0$), the loss distribution is Binomial(N, p) with parameter p given by

$$p = \mathbb{P}\left(\min_{0\le t\le T}(\sigma W_t) \le D\right) = 2\Phi\left(\frac{D}{\sigma\sqrt{T}}\right),$$

where Φ denotes the $\mathcal{N}(0,1)$-cdf, and we used the explicitly known distribution of the minimum of a Brownian motion. With our choice of parameters, we have $p \approx 0.5$ and

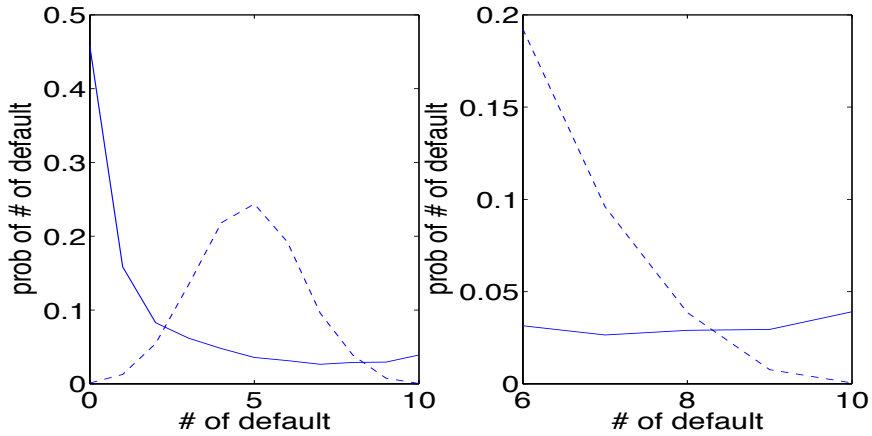

FIG. 2.2. *On the left, we show plots of the loss distribution for the coupled diffusions with* $a = 10$ *(solid line) and for the independent Brownian motions (dashed line). The plots on the right show the corresponding tail probabilities.*

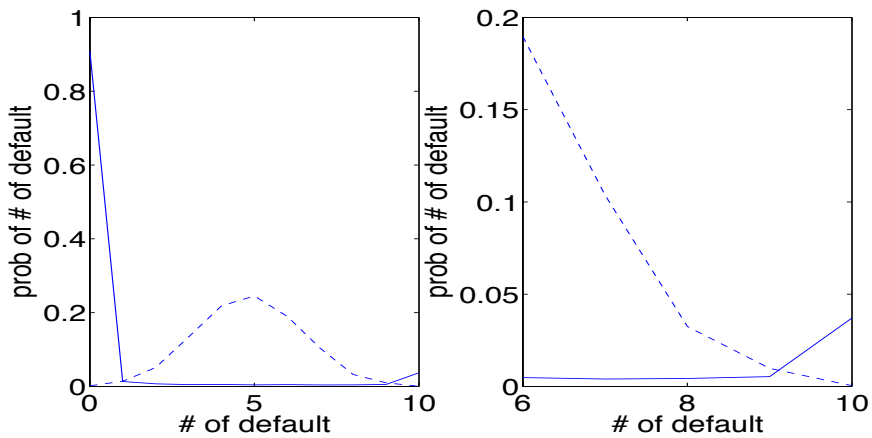

FIG. 2.3. *On the left, we show plots of the loss distribution for the coupled diffusions with* $a = 100$ *(solid line) and for the independent Brownian motions (dashed line). The plots on the right show the corresponding tail probabilities.*

therefore, the corresponding loss distribution is almost symmetric as can be seen on the left panels (dashed lines) in Figures 2.2 and 2.3. We see that increasing a, that is the rate of inter-bank borrowing and lending, pushes most of the mass to zero default, in other words, it improves the stability of the system by keeping the diffusions near zero (away from default) most of the time. However, we also see that there is a small but non-negligible probability, that almost all diffusions reach the default level. On the right panels of Figures 2.2 and 2.3 we zoom on this tail probability. In fact, this tail corresponds to the small probability of the ensemble average reaching the default level, and to almost all diffusions following this average due to "flocking" for large a.

2.2. Mean-field Limit

From (2.1) we see that the processes (X_t^i)'s are "OUs" mean-reverting to their *ensemble average* $(\overline{X}_t)$. Summing up the equations in (2.1) and using $X_0^i = 0$, $i = 1, \ldots, N$, one observes that this ensemble average is given by

$$\overline{X}_t = \frac{\sigma}{N} \sum_{i=1}^{N} W_t^i, \tag{2.2}$$

which is distributed as a Brownian motion with diffusion coefficient $\sigma/\sqrt{N}$ independently of a.

In the limit $N \to \infty$, the strong law of large numbers gives

$$\frac{1}{N} \sum_{j=1}^{N} W_t^j \to 0 \quad a.s.,$$

and, more generally, the processes X_t^i given by

$$X_t^i = \frac{\sigma}{N} \sum_{j=1}^{N} W_t^j + \sigma e^{-at} \int_0^t e^{as} dW_s^i - \frac{\sigma}{N} \sum_{j=1}^{N} \left(e^{-at} \int_0^t e^{as} dW_s^j \right),$$

converge to the independent OU processes $\sigma e^{-\alpha t} \int_0^t e^{\alpha s} dW_s^i$ with long-run mean zero. This is in fact a simple example of a mean-field limit and propagation of chaos studied in general in [21].

2.3. Large Deviations and Systemic Risk

In this section, we focus on the event where the ensemble average given by (2.2) reaches the default level. The probability of this event is small (when N becomes large), and is given by the theory of Large Deviations. In our simple example, this probability can be computed explicitly as follows:

$$\begin{aligned} \mathbb{P}\left(\min_{0 \le t \le T} \left(\frac{\sigma}{N} \sum_{i=1}^{N} W_t^i \right) \le D \right) &= \mathbb{P}\left(\min_{0 \le t \le T} \widetilde{W}_t \le \frac{D\sqrt{N}}{\sigma} \right) \\ &= 2\Phi\left(\frac{D\sqrt{N}}{\sigma\sqrt{T}} \right), \end{aligned} \tag{2.3}$$

where $\widetilde{W}$ is a standard Brownian motion and Φ is the $\mathcal{N}(0,1)$-cdf. Therefore, using classical equivalent for the Gaussian cumulative distribution function, we obtain

$$\lim_{N \to \infty} -\frac{1}{N} \log \mathbb{P}\left(\min_{0 \le t \le T} \left(\frac{\sigma}{N} \sum_{i=1}^{N} W_t^i \right) \le D \right) = \frac{D^2}{2\sigma^2 T}. \tag{2.4}$$

In other words, for a large number of banks, the probability that the ensemble average reaches the default barrier is of order $\exp(-D^2 N/(2\sigma^2 T))$. Recalling (2.2), we identify

$$\left\{ \min_{0 \le t \le T} \overline{X}_t \le D \right\} \tag{2.5}$$

as a *systemic event*. Observe that this event does not depend on $a > 0$, in other words, increasing stability by increasing the rate of borrowing and lending a does not

prevent a systemic event where a large number of banks default. In fact, once in this event, increasing a creates even more defaults by "flocking to default". This is illustrated in the Figure 2.3, where $a = 100$ and the probability of systemic risk is roughly 3% (obtained using formula (2.3)).

To summarize this section, our simple model with prescribed dynamics (no game) and independent noises shows that "lending and borrowing improves stability but also contributes to systemic risk". We have quantified this behavior and identified the crucial role played by the inter-bank rate of borrowing and lending.

2.4. Systemic Risk and Common Noise

In this section, we discuss the coupled diffusions driven by correlated Brownian motions $(\widetilde{W}_t^i)$ and without control ($\alpha^i = 0$). The dynamics (1.1) or (1.2) becomes:

$$dX_t^i = a\left(\frac{1}{N}\sum_{j=1}^N X_t^j - X_t^i\right)dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t^i\right), \quad i = 1, \cdots, N, \quad (2.6)$$

where $\left(W_t^0, W_t^i, i = 1, \cdots, N\right)$ are independent standard Brownian motions and W_t^0 is a common noise. As previously, we calculate the ensemble average

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^N X_t^i = \frac{\sigma}{N}\sum_{i=1}^N \widetilde{W}_t^i &= \sigma\left(\rho W_t^0 + \frac{\sqrt{1-\rho^2}}{N}\sum_{i=1}^N W_t^i\right)\\ &\overset{\mathcal{D}}{=} \sigma\sqrt{\rho^2 + \frac{(1-\rho^2)}{N}}B_t,\end{aligned}$$

where B_t is a standard Brownian motion. Moreover, the explicit solution for X_t^i is

$$X_t^i = \sigma\rho W_t^0 + \sigma\sqrt{1-\rho^2}\left(\frac{1}{N}\sum_{j=1}^N W_t^j + \int_0^t e^{a(s-t)}dW_s^i - \frac{1}{N}\sum_{j=1}^N\int_0^t e^{a(s-t)}dW_s^j\right).$$

The probability of the systemic event (2.5) becomes

$$\begin{aligned}\mathbb{P}\left(\min_{0\le s\le T}\frac{1}{N}\sum_{i=1}^N X_s^i < D\right) &= \mathbb{P}\left(\min_{0\le s\le T} B_s < \frac{D}{\sigma}\sqrt{\frac{N}{N\rho^2 + (1-\rho^2)}}\right)\\ &= 2\Phi\left(\frac{D}{\sigma\sqrt{T}}\sqrt{\frac{N}{N\rho^2+(1-\rho^2)}}\right).\end{aligned}$$

From the formula above, we see that in the correlated case ($\rho \neq 0$), the probability of systemic risk does not vanish as N becomes large, instead it converges to $2\Phi\left(\frac{D}{\sigma|\rho|\sqrt{T}}\right)$. This is in dramatic contrast with the independent case ($\rho = 0$) where the probability of systemic risk is exponentially small in N. We illustrate this instability created by the common noise in Figure 2.4.

3. Constructions of Exact Nash Equilibria

We now return to the model where each bank controls its rate of borrowing and lending and we search for Nash equilibria of the system (1.2-1.5). We first construct open-loop equilibria using the Pontryagin stochastic maximum principle leading naturally to the solution of Forward-Backward SDEs (FBSDEs). Next, we construct

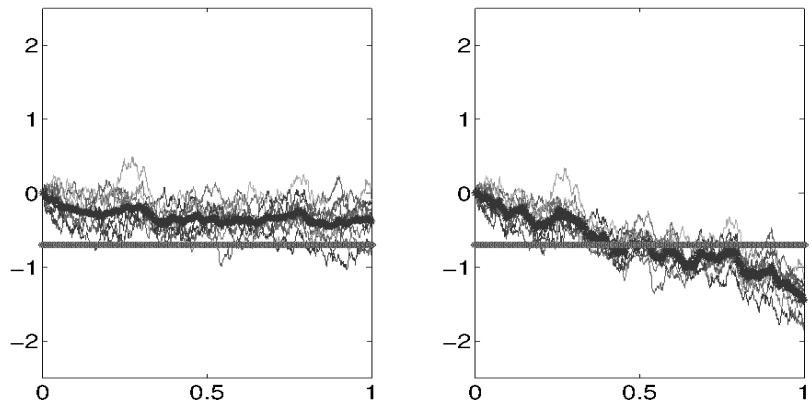

FIG. 2.4. *One realization of $N = 10$ trajectories of the coupled diffusions with independent Brownian motions (2.1) (left plot) and trajectories of the coupled diffusions with correlated Brownian motions with $\rho = 0.5$ (2.6) (right plot) using the common $a = 10$. The solid horizontal line represents the "default" level $D = -0.7$.*

closed-loop Markovian equilibria using two different approaches: the first one based on a modified version of Pontryagin stochastic maximum principle used in the open-loop case, and the other one based on the dynamic programming principle leading to the solution Hamilton-Jacobi-Bellman (HJB) PDEs. Recall that in all cases we try to find an equilibrium for the following borrowing-and-lending problem. The dynamics of the log-capitalizations X_t^i for $i = 1, \cdots, N$ are in the form:

$$dX_t^i = \left[a(\overline{X}_t - X_t^i) + \alpha_t^i\right] dt + \sigma\left(\sqrt{1-\rho^2}dW_t^i + \rho dW_t^0\right), \tag{3.1}$$

where $W_t^i, i = 0, 1, \ldots, N$ are independent Brownian motions, $\sigma > 0$ and $a \geq 0$. Bank $i \in \{1, \cdots, N\}$ controls its rate of lending and borrowing (to a central bank) at time t by choosing the control α_t^i in order to minimize

$$J^i(\alpha^1, \cdots, \alpha^N) = \mathbb{E}\left\{\int_0^T f_i(X_t, \alpha_t^i)dt + g_i(X_T^i)\right\}, \tag{3.2}$$

with

$$f_i(x, \alpha^i) = \left[\frac{1}{2}(\alpha^i)^2 - q\alpha^i(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2\right], \tag{3.3}$$

$$g_i(x) = \frac{c}{2}\left(\overline{x} - x^i\right)^2, \tag{3.4}$$

and where $f_i(x, \alpha)$ is convex in (x, α) under the assumption $q^2 \leq \epsilon$.

Our model falls in the class of *Linear-Quadratic* (LQ) Mean-Field games. What differentiates the various problems is the set of admissible strategies $\{\alpha_t^i, i = 1, \cdots, N\}$ searched for equilibria.

3.1. Open-Loop Equilibria

In the deterministic case ($\sigma = 0$), the open-loop problem corresponds to searching for an equilibrium among strategies which are (deterministic) functions $\{\alpha_t^i, i = 1, \cdots, N\}$ given at time $t = 0$. See for instance [1]. In the stochastic case ($\sigma > 0$), the open-loop problem consists in searching for an equilibrium among strategies $\{\alpha_t^i, i = 1, \cdots, N\}$ which are adapted processes satisfying some integrability property such as $\mathbb{E}\left(\int_0^T |\alpha_t^i| dt\right) < \infty$. See [3] for example.

The Hamiltonian for bank i is given by

$$\begin{aligned} &H^i(x^1, \cdots, x^N, y^{i,1}, \cdots, y^{i,N}, \alpha^1, \cdots, \alpha^N) \\ &= \sum_{k=1}^N \left[a(\overline{x} - x^k) + \alpha^k\right] y^{i,k} + \frac{1}{2}(\alpha^i)^2 - q\alpha^i(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2. \end{aligned} \tag{3.5}$$

For a given $\alpha = (\alpha^i)_{i=1,\cdots,n}$, the controlled forward dynamics of the states X_t^i are given by (3.1) with initial conditions $X_0^i = x^i$, and the adjoint processes $Y_t^i = (Y_t^{i,j}; j = 1, \cdots, N)$ and $Z_t^i = (Z_t^{i,j,k};\ j = 1, \cdots, N,\ k = 0, 1, \cdots, N)$ for $i = 1, \cdots, N$ are defined as the solutions of the backward stochastic differential equations (BSDEs):

$$dY_t^{i,j} = -\partial_{x^j} H^i(X_t, Y_t^i, \alpha_t)dt + \sum_{k=0}^N Z_t^{i,j,k} dW_t^k \tag{3.6}$$

with terminal conditions $Y_T^{i,j} = \partial_{x^j} g_i(X_T)$ for $i, j = 1, \cdots, N$. For each admissible (i.e. satisfying the above integrability condition) strategy profile $\alpha = (\alpha^i)_{i=1,\cdots,n}$, standard existence and uniqueness results for BSDEs apply and the existence of the adjoint processes is guaranteed. Note that from (3.5), we have

$$\partial_{x^j} H^i = \frac{a}{N}\sum_{k=1}^N (y^{i,k} - y^{i,j}) - q\alpha^i(\frac{1}{N} - \delta_{i,j}) + \epsilon(\overline{x} - x^i)(\frac{1}{N} - \delta_{i,j}).$$

The necessary condition of the Pontryagin stochastic maximum principle suggests that one minimizes the Hamiltonian H^i with respect to α^i. See for example the discussion of the Isaacs conditions in [3]. This leads to the choice:

$$\hat{\alpha}^i = -y^{i,i} + q(\overline{x} - x^i). \tag{3.7}$$

In order to prove that these candidates actually form a Nash equilibrium, we assume that all players are making that choice, and let player i finds his best response by solving the BSDE (3.6) to identify his own adjoint process. With this choice for the controls α^i, the forward equation (3.1) becomes coupled with the backward equation (3.6) and this forward-backward coupled system is in general extremely difficult to solve. In the present linear case, we make the ansatz

$$Y_t^{i,j} = (\frac{1}{N} - \delta_{i,j})(\overline{X}_t - X_t^i)\phi_t, \tag{3.8}$$

for some deterministic scalar function ϕ_t satisfying the terminal condition $\phi_T = c$. Using this ansatz, a careful computation shows that the backward equations (3.6) become

$$dY_t^{i,j} = (\frac{1}{N} - \delta_{i,j})(\overline{X}_t - X_t^i)\left[(a + (1 - \frac{1}{N})q)\phi_t - (\epsilon - q^2)\right]dt + \sum_{k=0}^N Z_t^{i,j,k} dW_t^k. \tag{3.9}$$

Using (3.7) and (3.8), the forward equation (3.1) becomes

$$dX_t^i = \left[a + q + (1 - \frac{1}{N})\phi_t\right](\overline{X}_t - X_t^i)dt + \sigma\left(\sqrt{1-\rho^2}dW_t^i + \rho dW_t^0\right), \tag{3.10}$$

which by summation gives

$$d\overline{X}_t = \sigma\rho dW_t^0 + \sigma\sqrt{1-\rho^2}\left(\frac{1}{N}\sum_{i=1}^N dW_t^i\right). \tag{3.11}$$

Consequently, one obtains

$$d(\overline{X}_t - X_t^i) = -\left[a + q + (1 - \frac{1}{N})\phi_t\right](\overline{X}_t - X_t^i)dt + \sigma\sqrt{1-\rho^2}\left(\frac{1}{N}\sum_{k=1}^N dW_t^k - dW_t^i\right). \tag{3.12}$$

Differentiating the ansatz (3.8) and using (3.12), we get

$$\begin{aligned} dY_t^{i,j} &= (\frac{1}{N} - \delta_{i,j})(\overline{X}_t - X_t^i)\left[\dot{\phi}_t - \phi_t\left(a + q + (1 - \frac{1}{N})\phi_t\right)\right]dt \\ &\quad + \phi_t\sigma\sqrt{1-\rho^2}(\frac{1}{N} - \delta_{i,j})\left(\frac{1}{N}\sum_{k=1}^N dW_t^k - dW_t^i\right), \end{aligned} \tag{3.13}$$

where $\dot{\phi}_t$ denotes the time-derivative of ϕ_t. Comparing the two Itô decompositions (3.9) and (3.13), the martingale terms give the processes $Z_t^{i,j,k}$

$$Z_t^{i,j,0} = 0, \quad Z_t^{i,j,k} = \phi_t\sigma\sqrt{1-\rho^2}(\frac{1}{N} - \delta_{i,j})(\frac{1}{N} - \delta_{i,k}) \text{ for } \ k = 1, \cdots, N,$$

which turn out to be determistic in our case and hence adapted. Identifying the drift terms shows that the function ϕ_t must satisfy the scalar Riccati equation

$$\dot{\phi}_t = 2(a + (1 - \frac{1}{2N})q)\phi_t + (1 - \frac{1}{N})\phi_t^2 - (\epsilon - q^2), \tag{3.14}$$

with the terminal condition $\phi_T = c$. This equation can be solved explicitly. We defer the solution to Section 3.4 where we will provide a comparison with the closed-loop case for which a similar equation appears. We also defer to Section 4 the discussion of the financial interpretation of this equilibrium which will be similar to the one for the closed-loop equilibrium.

Note that the form (3.7) of the control α_t^i, and the ansatz (3.8) combine to give:

$$\alpha_t^i = \left[q + (1 - \frac{1}{N})\phi_t\right](\overline{X}_t - X_t^i). \tag{3.15}$$

It is interesting to remark that these controls, while constructed to form an open-loop equilibrium, are in fact in closed-loop feedback form! In this equilibrium, each bank i can implement its strategy by knowing $\overline{X}_t - X_t^i$. Further implications will be discussed in Section 4. For the time being, we note here that since the optimal strategies are functions of $\overline{X}_t - X_t^i$, in equilibrium, the dynamics of the state $X_t = (X_t^i)_{1=1,\cdots,N}$ are Markovian . So it is natural to search for equilibria within closed-loop Markovian strategies as we discuss next.

3.2. Closed-Loop Equilibria, still via the FBSDE Approach

In this section we solve for an exact Nash equilibrium in closed-loop form when the players/banks at time t have complete information of the states of all the other players at time t, and use Markovian strategies.

In this context, when all the other players $k \neq i$ have chosen strategies in feedback form given by deterministic functions $\alpha^k(t,x)$ of time and state, player i needs to solve a Markovian control problem to find his best response to these choices, and the Hamiltonian of his control problem is given by (see [3]):

$$\begin{aligned} H^i(x, y^{i,1}, \cdots, y^{i,N}, \alpha^1(t,x), \cdots, \alpha^i_t, \cdots, \alpha^N(t,x)) \\ = \sum_{k\neq i} \left[a(\overline{x} - x^k) + \alpha^k(t,x)\right] y^{i,k} + \left[a(\overline{x} - x^i) + \alpha^i\right] y^{i,i} \\ + \frac{1}{2}(\alpha^i)^2 - q\alpha^i(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2. \end{aligned} \tag{3.16}$$

The dynamics of the state $X_t = (X^i_t)_{i=1,\cdots,N}$ are again given by (3.1) with initial conditions $X^i_0 = x^i$, and the adjoint processes $Y^i = (Y^{i,j})_{j=1,\cdots,N}$ and $Z^i = (Z^{i,j,k})_{j=1,\cdots,N,\, k=0,1,\cdots,N}$ of player i are the solutions of the same BSDE (3.6) with H^i given in (3.16). Consequently, $\partial_{x^j} H^i$ is computed from (3.16), giving:

$$\begin{aligned} \partial_{x^j} H^i = a\sum_{k=1}^N (\frac{1}{N} - \delta_{k,j}) y^{i,k} + \sum_{k\neq i} (\partial_{x_j}\alpha^k(t,x)) y^{i,k} \\ -q\alpha^i(\frac{1}{N} - \delta_{i,j}) + \epsilon(\overline{x} - x^i)(\frac{1}{N} - \delta_{i,j}). \end{aligned} \tag{3.17}$$

Again, the necessary part of the stochastic maximum principle suggests that one minimizes H^i with respect to α^i, leading again to the choice (3.7). Injecting these functions of X^i_t and $Y^{i,i}_t$ into the state equations and the adjoint equations gives, as before, a large system of forward backward stochastic differential equations which, if solved, provides a Markovian Nash equilibrium because of the sufficient part of the stochastic maximum principle. In order to solve this large FBSDE, we make an ansatz of the same form as in (3.8):

$$Y^{i,j}_t = \left(\frac{1}{N} - \delta_{i,j}\right)(\overline{X}_t - X^i_t)\eta_t, \tag{3.18}$$

for some deterministic scalar function η_t satisfying the terminal condition $\eta_T = c$. This choice ensures that the control α^i is a feedback control. With this choice of optimal controls and ansatz (3.18), we get

$$\begin{aligned} \alpha^k(t,x) &= \left[q + (1 - \frac{1}{N})\eta_t\right](\overline{x} - x^k), \\ \partial_{x^j}\alpha^k(t,x) &= \left[q + (1 - \frac{1}{N})\eta_t\right](\frac{1}{N} - \delta_{k,j}), \end{aligned}$$

and a straightforward computation using (3.17) reduces the backward equations to

$$\begin{aligned} dY_t^{i,j} &= -\partial_{x^j} H^i dt + \sum_{k=0}^{N} Z_t^{i,j,k} dW_t^k \\ &= (\frac{1}{N} - \delta_{i,j})(\overline{X}_t - X_t^i)\Big[(a+q)\eta_t - \frac{1}{N}(\frac{1}{N}-1)\eta_t^2 + q^2 - \epsilon\Big]dt \\ &\quad + \sum_{k=0}^{N} Z_t^{i,j,k} dW_t^k, \end{aligned} \tag{3.19}$$

with terminal conditions $Y_T^{i,j} = c(\frac{1}{N} - \delta_{i,j})(\overline{X}_T - X_T^i)$. Using (3.7) and (3.18), the forward equations become:

$$dX_t^i = \Big[a + q + (1 - \frac{1}{N})\eta_t\Big](\overline{X}_t - X_t^i)dt + \sigma\Big(\sqrt{1-\rho^2}dW_t^i + \rho dW_t^0\Big), \tag{3.20}$$

and by summation we deduce that $\overline{X}_t$ satisfies again (3.11). Differentiating the ansatz (3.18) and using (3.11) and (3.20) for the forward dynamics, we obtain that $Y_t^{i,j}$ satisfies the equation (3.13) with ϕ_t replaced with η_t:

$$\begin{aligned} dY_t^{i,j} &= \Big(\frac{1}{N} - \delta_{i,j}\Big)(\overline{X}_t - X_t^i)\Big[\dot{\eta}_t - \Big(a + q + (1 - \frac{1}{N})\eta_t\Big)\eta_t\Big]dt \\ &\quad + \eta_t \sigma\sqrt{1-\rho^2}(\frac{1}{N} - \delta_{i,j})\sum_{k=1}^{N}(\frac{1}{N} - \delta_{i,k})dW_t^k. \end{aligned} \tag{3.21}$$

Next, identifying term by term the two Itô decompositions (3.19) and (3.21), we obtain from the martingale terms

$$Z_t^{i,j,0} = 0, \qquad Z_t^{i,j,k} = \eta_t\sigma\sqrt{1-\rho^2}(\frac{1}{N} - \delta_{i,j})(\frac{1}{N} - \delta_{i,k}) \text{ for } \ k = 1, \cdots, N,$$

which are indeed adapted and square integrable, and from the drift terms:

$$\dot{\eta}_t - \Big(a + q + (1 - \frac{1}{N})\eta_t\Big)\eta_t = (a+q)\eta_t - \frac{1}{N}(\frac{1}{N} - 1)\eta_t^2 + q^2 - \epsilon.$$

Therefore, η_t must satisfy the scalar Riccati equation

$$\dot{\eta}_t = 2(a+q)\eta_t + (1 - \frac{1}{N^2})\eta_t^2 - (\epsilon - q^2), \tag{3.22}$$

with the terminal condition $\eta_T = c$. Equation (3.22) admits the solution

$$\eta_t = \frac{-(\epsilon - q^2)\left(e^{(\delta^+ - \delta^-)(T-t)} - 1\right) - c\left(\delta^+ e^{(\delta^+ - \delta^-)(T-t)} - \delta^-\right)}{\left(\delta^- e^{(\delta^+ - \delta^-)(T-t)} - \delta^+\right) - c(1 - \frac{1}{N^2})\left(e^{(\delta^+ - \delta^-)(T-t)} - 1\right)}, \tag{3.23}$$

where we used the notation

$$\delta^{\pm} = -(a+q) \pm \sqrt{R}, \tag{3.24}$$

with

$$R := (a+q)^2 + \Big(1 - \frac{1}{N^2}\Big)(\epsilon - q^2) > 0. \tag{3.25}$$

Observe that η_t is well defined for any $t \leq T$ since the denominator in (3.23) can be written as

$$-\left(e^{(\delta^+-\delta^-)(T-t)}+1\right)\sqrt{R}-\left(a+q+c\left(1-\frac{1}{N^2}\right)\right)\left(e^{(\delta^+-\delta^-)(T-t)}-1\right),$$

which stays negative because $\delta^+ - \delta^- = 2\sqrt{R} > 0$. In fact, using $q^2 \leq \epsilon$, we see that η_t is positive with $\eta_T = c$ as required and illustrated in Figure 3.1.

We delay the discussion of the implications of our analysis in terms of banking system to Section 4. We first briefly present the dynamic programming approach to the problem which produces the same equilibrium as the one obtained in this section.

3.3. Closed-Loop Equilibria via the HJB Approach

In the Markovian setting, the value function of player i is given by

$$V^i(t,x) = \inf_\alpha \mathbb{E}_{t,x}\left\{\int_t^T f_i(X_s,\alpha_s^i)ds + g_i(X_T^i)\right\},$$

with the cost functions f_i and g_i given in (3.3) and (3.4), and where the dynamics of X_t is given as before by:

$$dX_t^i = \left[a(\overline{X}_t - X_t^i) + \alpha_t^i\right]dt + \sigma\left(\sqrt{1-\rho^2}dW_t^i + \rho dW_t^0\right),\ i = 1,\cdots,N.$$

Using the dynamic programming principle in search for a closed-loop equilibrium, the corresponding HJB equations read

$$\begin{aligned}\partial_t V^i + \inf_\alpha \Bigg\{ &\sum_{j=1}^N \left[a\left(\overline{x}-x^j\right)+\alpha^j\right]\partial_{x^j}V^i \\ &+ \frac{\sigma^2}{2}\sum_{j=1}^N\sum_{k=1}^N\left(\rho^2+\delta_{j,k}(1-\rho^2)\right)\partial_{x^jx^k}V^i \\ &+ \frac{(\alpha^i)^2}{2} - q\alpha^i\left(\overline{x}-x^i\right)+\frac{\epsilon}{2}(\overline{x}-x^i)^2\Bigg\} = 0,\end{aligned} \tag{3.26}$$

with terminal conditions $V^i(T,x) = \frac{c}{2}(\overline{x}-x^i)^2$. Assuming that all controls α^j for $j \neq i$ are chosen, player i will choose the optimal strategy $\hat{\alpha}^i = q(\overline{x}-x^i) - \partial_{x^i}V^i$ where V^i is still unknown. Next, assuming that all players are following the strategies $\hat{\alpha}^i = q(\overline{x}-x^i) - \partial_{x^i}V^i$, the HJB equations (3.26) become

$$\begin{aligned}\partial_t V^i + &\sum_{j=1}^N\left[(a+q)\left(\overline{x}-x^j\right)-\partial_{x^j}V^j\right]\partial_{x^j}V^i \\ &+ \frac{\sigma^2}{2}\sum_{j=1}^N\sum_{k=1}^N\left(\rho^2+\delta_{j,k}(1-\rho^2)\right)\partial_{x^jx^k}V^i \\ &+ \frac{1}{2}(\epsilon-q^2)\left(\overline{x}-x^i\right)^2+\frac{1}{2}(\partial_{x^i}V^i)^2 = 0.\end{aligned} \tag{3.27}$$

We then make the ansatz

$$V^i(t,x) = \frac{\tilde{\eta}_t}{2}(\overline{x}-x^i)^2 + \mu_t, \tag{3.28}$$

where $\tilde{\eta}_t$ and μ_t are deterministic functions satisfying $\tilde{\eta}_T = c$ and $\mu_T = 0$ in order to match the terminal conditions for V^i. Note that the adjoint variables $y^{i,j}$ introduced in the FBSDE approach correspond to $\partial_{x^j} V^i$ and the ansatz (3.28) corresponds to the ansatz (3.18). The optimal strategies will be

$$\hat{\alpha}^i = q(\overline{X}_t - X_t^i) - \partial_{x^i} V^i = \left(q + (1 - \frac{1}{N})\tilde{\eta}_t\right)(\overline{X}_t - X_t^i), \tag{3.29}$$

and the controlled dynamics will become

$$dX_t^i = \left(a + q + (1 - \frac{1}{N})\tilde{\eta}_t\right)(\overline{X}_t - X_t^i)dt + \sigma\left(\sqrt{1-\rho^2}dW_t^i + \rho dW_t^0\right). \tag{3.30}$$

Using

$$\partial_{x^j} V^i = (\frac{1}{N} - \delta_{i,j})\left(\overline{x} - x^i\right)\tilde{\eta}_t, \quad \partial_{x^j x^k} V^i = (\frac{1}{N} - \delta_{i,j})(\frac{1}{N} - \delta_{i,k})\tilde{\eta}_t,$$

plugging into (3.27), and canceling terms in $\left(\overline{x} - x^i\right)^2$ and state-independent terms, we obtain

$$\dot{\tilde{\eta}}_t = 2(a+q)\tilde{\eta}_t + (1 - \frac{1}{N^2})\tilde{\eta}_t^2 - (\epsilon - q^2), \tag{3.31}$$

$$\dot{\mu}_t = -\frac{1}{2}\sigma^2(1-\rho^2)\left(1 - \frac{1}{N}\right)\tilde{\eta}_t, \tag{3.32}$$

with the terminal conditions $\tilde{\eta}_T = c$ and $\mu_T = 0$. Therefore, $\tilde{\eta}_t$ must satisfy the same Riccati equation as the one satisfied by η_t (3.22). They have the same terminal conditions and by unicity of the solution for this equation, we deduce that $\tilde{\eta}_t = \eta_t$ for all $t \leq T$ and, consequently, the two closed-loop approaches (FBSDE and HJB) produce the same equilibrium. The explicit solution for η_t is given by (3.23) and, furthermore, the solution μ_t of (3.32) with the terminal condition $\mu_T = 0$ is given by

$$\mu_t = \frac{1}{2}\sigma^2(1-\rho^2)\left(1 - \frac{1}{N}\right)\int_t^T \tilde{\eta}_s\, ds, \tag{3.33}$$

and the value functions V^i under this exact Nash equilibrium are given by (3.28). Note that the correlation, quantified by the parameter ρ, affects the controls $\hat{\alpha}_t^i$ given by (3.29) only through the dynamics of $\overline{X}_t - X_t^i$ since η_t does not depend on ρ. However, it affects the value function V^i given by (3.28) also through the state-independent term μ_t.

3.4. Comparison of the Open- and Closed-Loop Equilibria

Our analysis in Sections 3.1 and 3.2 shows that the two equilibria we obtained are very similar. In fact, the only difference is that in the open-loop case we obtained the Riccati equation (3.14) for the function ϕ_t (with a factor $(1 - \frac{1}{N})$ in front of ϕ_t^2 and a factor $(1 - \frac{1}{2N})$ in front of q), and in the closed-loop case we obtained the Riccati equation (3.22) for the function η_t (with a factor $(1 - \frac{1}{N^2})$ in front of η_t^2 and a factor 1 in front of q).

In the closed-loop case we saw that the optimal strategy is given by

$$\alpha_t^i = \left[q + (1 - \frac{1}{N})\eta_t\right](\overline{X}_t - X_t^i),$$

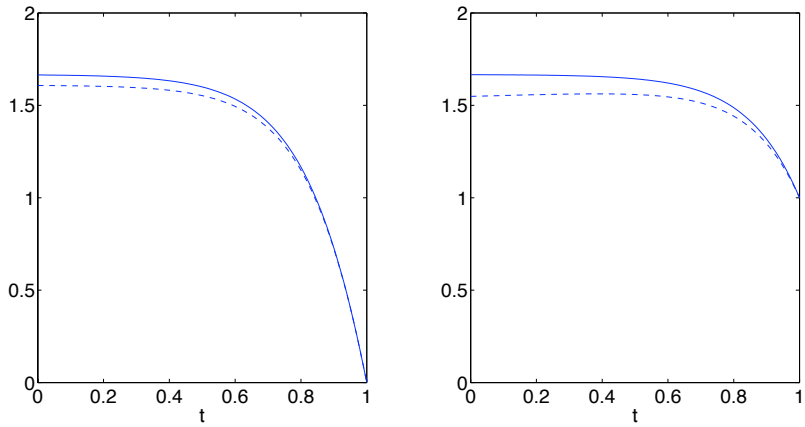

FIG. 3.1. *Plots of ϕ_t (solid line) and η_t (dashed line) with $N = 10$, $a = 1$, $q = 1$, $\epsilon = 10$, $T = 1$, and $c = 0$ on the left, $c = 1$ on the right.*

and the forward dynamics are:

$$dX_t^i = \left[a + q + (1 - \frac{1}{N})\eta_t\right](\overline{X}_t - X_t^i)dt + \sigma\left(\sqrt{1-\rho^2}dW_t^i + \rho dW_t^0\right),$$

with

$$d\overline{X}_t = \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}\frac{1}{N}\sum_{k=1}^{N} dW_t^k\right).$$

These equations are identical in the open-loop case with η_t replaced by ϕ_t. Note that η_t is given explicitly by formula (3.23) and ϕ_t can be obtained similarly by replacing the factor $(1 - \frac{1}{N^2})$ with $(1 - \frac{1}{N})$ and using

$$\delta^{\pm} = -(a + (1 - \frac{1}{2N})q) \pm \sqrt{R},$$

with

$$R := (a + (1 - \frac{1}{2N})q)^2 + \left(1 - \frac{1}{N}\right)(\epsilon - q^2) > 0.$$

In Figure 3.1, we show the functions ϕ_t and η_t involved respectively in the open-loop and closed-loop strategies. As expected, the difference is relatively small for $N = 10$. However, it is enhanced by our choice of $\epsilon = 10$ giving a rather large factor $\epsilon - q^2$ in front of $(1 - \frac{1}{N})$ in the open-loop case or in front of $(1 - \frac{1}{N^2})$ in the closed-loop case. Note that the presence of a terminal cost $c = 1$ in the right panel produces a significant difference.

The individual value functions can be calculated as follows. Considering for instance the closed-loop case, we want to calculate

$$V^i(x) = \mathbb{E}\left\{\int_0^T \left[\frac{1}{2}(\alpha^i)^2 - q\alpha^i(\overline{X}_t - X_t^i) + \frac{\epsilon}{2}(\overline{X}_t - X_t^i)^2\right]dt + \frac{c}{2}(\overline{X}_T - X_T^i)^2\right\},$$

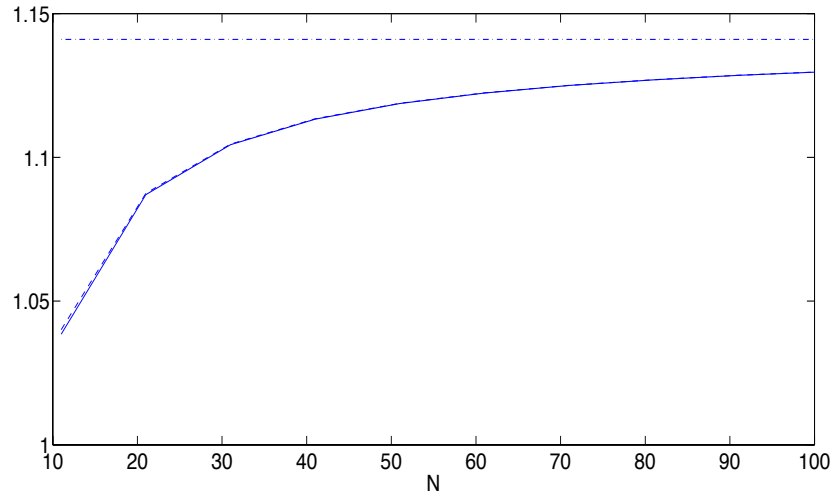

FIG. 3.2. *Plots of the value function V^i at $t = 0$ and $x^i = 0$ for $i = 1, \cdots, N$, as N increases: open loop (solid line), closed loop (dashed line), and common limit as $N \to \infty$ (dotted line) with $a = 1$, $q = 1$, $\epsilon = 10$, $\rho = 0.2$, $T = 1$, and $c = 10$.*

where x is the initial position of the system and $(\alpha_t^i, X_t^i, \overline{X}_t)$ are given by the equations above. Then, one easily obtains by direct computation

$$V^i(x) = \frac{1}{2}\int_0^T \left[\epsilon - q^2 + (1 - \frac{1}{N})^2\eta_t^2\right] \mathbb{E}\left\{(\overline{X}_t - X_t^i)^2\right\} dt + \frac{c}{2}\mathbb{E}\left\{(\overline{X}_T - X_T^i)^2\right\},$$

with

$$\begin{aligned}\mathbb{E}\left\{(\overline{X}_t - X_t^i)^2\right\} &= (\overline{x} - x^i)^2 e^{-2\int_0^t (a+q+(1-\frac{1}{N})\eta_s)ds} \\ &\quad + (1 - \frac{1}{N})\sigma^2(1-\rho^2)\int_0^t e^{-2\int_s^t (a+q+(1-\frac{1}{N})\eta_u)du} ds.\end{aligned}$$

The formula in the open-loop case is simply obtained by replacing η_t by ϕ_t.

In Figure 3.2, we compare the value functions V^i in the open-loop and closed-loop equilibria. For the choice of parameters and as $N \to \infty$, these two functions are very close.

4. Financial Implications

The comments below made in the case of the closed-loop equilibrium with the function η_t would be identical in the case of the open-loop equilibrium with η_t replaced by ϕ_t.

1. Once the function η_t has been obtained in (3.23), bank i implements its strategy by using its control $\hat{\alpha}^i$ given by (3.29). It requires its own log-reserve X_t^i but also the average reserve $\overline{X}_t$ which may or may not be known to the individual bank i. Here, we assume that the central bank/regulator observes the monetary reserves of all banks and therefore $\overline{X}_t$, and that it gives this aggregated information to each individual banks. Consequently, even though $\overline{X}_t$ is given by (3.11), the banks do not need to know the two parameters σ and

ρ in order to implement their optimal strategies. Observe also that the average $\overline{X}_t$ is identical to the average found in Section 2.4. Therefore, systemic risk occurs in the same manner as in the case of uncontrolled dynamics with or without common noise as presented respectively in Sections 2.4 and 2.3.

2. However, (3.30) shows that the control affects the rate of borrowing and lending by adding the time-varying component $q + (1 - \frac{1}{N})\eta_t$ to the uncontrolled rate a.
3. In fact, from (3.30) rewritten as

$$dX_t^i = \left(a + q + (1 - \frac{1}{N})\eta_t\right) \frac{1}{N} \sum_{j=1}^{N} (X_t^j - X_t^i)dt \tag{4.1}$$
$$+\sigma\left(\sqrt{1-\rho^2}dW_t^i + \rho dW_t^0\right),$$

we see that the effect of the banks using their optimal strategies corresponds to inter-bank borrowing and lending at the increased **effective rate**

$$A_t := a + q + (1 - \frac{1}{N})\eta_t$$

with no central bank (or a central bank acting as an instantaneous **clearing house**). As a consequence, under this equilibrium, the system is operating as if banks were borrowing from and lending to each other at the rate A_t, and the net effect is **additional liquidity** quantified by the rate of lending/borrowing.
Note that the comment above is valid not only if $a > 0$, in which case the effect of the game is to increase the rate of interbank lending and borrowing, but also if $a = 0$, in which case the effect of the game is to "create" an interbank lending and borrowing activity. In both cases, the central bank acts as a clearing house but needs to provide the information $\overline{X}_t$ so that individual banks can implement their strategies.
4. Observe that the presence of a common noise (quantified by ρ) does not affect the form of the optimal strategies (the function η_t does not depend on ρ). However it affects the value function $V^i(t,x)$ and the dynamics X_t^i, and, as we have seen in Section 2.4, it has a drastic effect on systemic risk.
5. It is also interesting to note that for T large, most of the time ($T - t$ large), η_t is mainly constant. For instance, with $c = 0$,

$$\lim_{T\to\infty} \eta_t = \frac{\epsilon - q^2}{-\delta^-} := \overline{\eta},$$

as illustrated on right panel of Figure 4.1. Therefore, in this infinite-horizon equilibrium, banks are borrowing and lending to each other at the constant rate

$$A := a + q + (1 - \frac{1}{N})\overline{\eta}. \tag{4.2}$$

In Figure 4.2 we show the constant effective rates A (for infinite horizon) for the open-loop and closed-loop equilibria as N increases. Note that liquidity (quantified by the effective rate of lending/borrowing A) is higher under the open-loop equilibrium and increases with N.

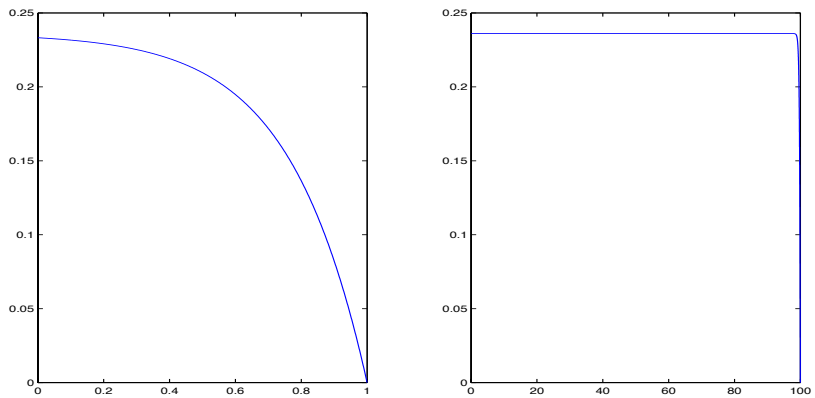

FIG. 4.1. *Plots of η_t with $c = 0$, $a = 1$, $q = 1$, $\epsilon = 2$ and $T = 1$ on the left, $T = 100$ on the right with $\overline{\eta} \sim 0.24$ (here we used $1/N \equiv 0$).*

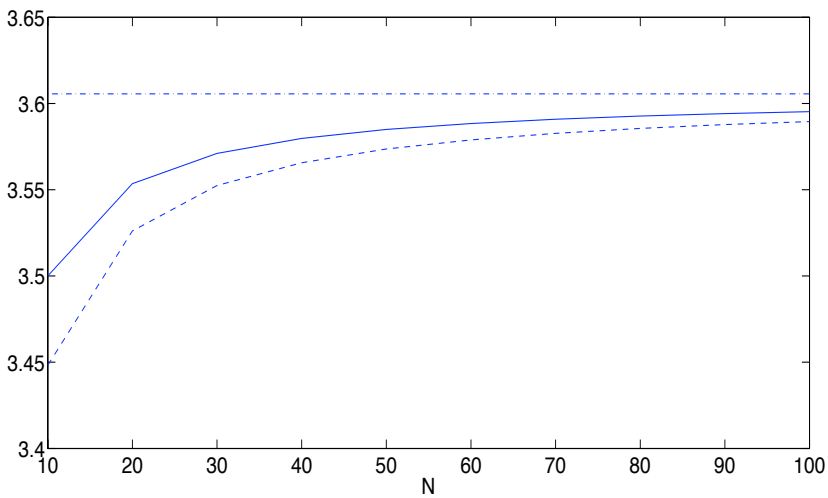

FIG. 4.2. *Plots of the effective rate A (4.2) for the open-loop equilibrium (solid line) and for the closed-loop equilibrium (dashed line) with $a = 1$, $q = 1$, $\epsilon = 10$, and as N increases. The dotted line shows the common limit as $N \to \infty$*

Remarks on possible extensions. We indicate here a few possible extensions of the model. From the beginning, we assumed for simplicity that all the banks had the same volatility σ. In fact, it is straightforward to generalize the results to the case where each bank has its own constant volatility σ_i. For the study in Section 3 with a finite number N of banks there is no restriction on these σ_i which will simply enter in the averaging of the Brownian motions in (3.11). In the analysis when $N \to \infty$,

one needs to impose a condition of the form

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \sigma_i^2 = \sigma^2,$$

with $0 < \sigma < \infty$, and that is the effective volatility which will appear in the limiting formulas. Treating the case of non constant volatilities such as $\sigma_i(t, X_t^i)$ is much more involved and no explicit solution can be expected in general.

Our analysis relies heavily on the symmetry of the model in the dynamics and in the cost functions. One way to depart from that symmetry would be to study models with groups of banks with symmetry within the groups. One would have to deal with coupled matrix Riccati equations. This is outside of the scope of this paper but a work in progress.

In the following section, we discuss asymptotics as $N \to \infty$.

5. Approximate Nash Equilibria

Why would we want an approximate Nash equilibrium when we can compute an exact one?

The game presented in the previous section was essentially designed to provide explicit Nash equilibria, and the fact that it does is already remarkable! However, slight modifications, even minor, dramatically change the equilibrium structures. For instance, the presence in the dynamics of the X_t^i or in the objective functions J^i of a nonlinear term in $\overline{X}_t$ renders the computation of exact Nash equilibria hopeless. The Mean Field Game strategy is based on the solution of effective equations in the limit $N \to \infty$, providing through the theory of the propagation of chaos, approximate Nash equilibria for the case N finite and large. We review this strategy in the context of the model presented in the previous section, and compare its output to the exact solutions derived earlier.

5.1. The Mean Field Game / FBSDE Approach

If it was not for the presence of a common noise, we could apply the results of [5] to obtain approximate Nash equilibria from the solution of the Mean Field Game (MFG). Notice that linear quadratic MFGs are studied in [2] and [6]. While the latter does not include the cross term $-q\alpha(\overline{x}-x^i)$ in the running cost of player i, the proofs of [6] apply *mutatis mutandis* to the model of the present paper when $\rho = 0$. We review the MFG strategy. It is based on the following three steps:

1. Fix $(m_t)_{t\geq 0}$, which should be thought of as a candidate for the limit of $\overline{X}_t$ as $N \to \infty$:

$$m_t = \lim_{N\to\infty} \overline{X}_t.$$

   Because of the presence of the common noise, $(m_t)_{t\geq 0}$ is a process adapted to the filtration generated by W^0 and one should think of m_t as a function of $(W_s^0)_{s\leq t}$;

2. Solve the one-player standard control problem

$$\inf_{\alpha=(\alpha_t)\in\mathcal{A}} \mathbb{E}\left\{\int_0^T \left[\frac{\alpha_t^2}{2} - q\alpha_t(m_t - X_t) + \frac{\epsilon}{2}(m_t - X_t)^2\right] dt + \frac{c}{2}(m_T - X_T)^2\right\},$$

   subject to the dynamics

$$dX_t = [a(m_t - X_t) + \alpha_t]\, dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2} dW_t\right),$$

where W_t^0 and W_t are independent Brownian motions, independent of the initial value X_0 which may be a square integrable random variable ξ.

3. Solve the fixed point problem: find m_t so that $m_t = \mathbb{E}[X_t \mid (W_s^0)_{s\le t}]$ for all t.

We treat the above stochastic control problem as a problem of control of non-Markovian dynamics with random coefficients. The Hamiltonian of the system is given by

$$H(t,x,y,\alpha) = [a(m_t - x) + \alpha]\, y + \frac{1}{2}\alpha^2 - q\alpha(m_t - x) + \frac{\epsilon}{2}(m_t - x)^2,$$

which is strictly convex in (x,α) under the condition $q^2 \le \epsilon$ and attains its minimum at

$$\frac{\partial H}{\partial \alpha} = 0 \longrightarrow \hat{\alpha} = q(m_t - x) - y.$$

The corresponding *adjoint forward-backward equations* are given by

$$\begin{aligned} dX_t &= \frac{\partial H}{\partial y}(\hat{\alpha})dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t\right) && (5.1)\\ &= [(a+q)(m_t - X_t) - Y_t]\, dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t\right), \quad X_0 = \xi \\ dY_t &= -\frac{\partial H}{\partial x}(\hat{\alpha})dt + Z_t^0 dW_t^0 + Z_t dW_t && (5.2)\\ &= \left[(a+q)Y_t + (\epsilon - q^2)(m_t - X_t)\right] dt + Z_t^0 dW_t^0 + Z_t dW_t, \\ &\qquad\qquad Y_T = c(X_T - m_T), \end{aligned}$$

for some adapted square integrable processes (Z_t^0, Z_t). Despite its simple looking structure, such linear systems do not always have solutions. The existence of a solution in the present situation is argued in [6] where a solution is shown to exist. To identify it in the present situation, we use the notation $m_t^X = \mathbb{E}[X_t \mid (W_s^0)_{s\le t}]$ and $m_t^Y = \mathbb{E}[Y_t \mid (W_s^0)_{s\le t}]$. Taking conditional expectation given $(W_s^0)_{s\le t}$ in the second equation (5.2), and using the fact that in equilibrium (i.e. after solving for the fixed point), we have $m_t = m_t^X$ for all $t \le T$ which in turn implies $m_T^Y = c(m_T^X - m_T) = 0$, we obtain:

$$m_t^Y = -\int_t^T e^{(a+q)(s-t)} Z_s^0 dW_s^0. \tag{5.3}$$

Next, taking conditional expectations of both sides of equation (5.1), we deduce that, in equilibrium, we shall have

$$dm_t^X = -m_t^Y dt + \rho\sigma dW_t^0. \tag{5.4}$$

Now, we make the (educated) ansatz

$$Y_t = -\eta_t(m_t - X_t), \tag{5.5}$$

for some deterministic function $t \hookrightarrow \eta_t$ to be determined. Differentiating this ansatz and using (5.1) and (5.4) leads to

$$\begin{aligned} dY_t &= -\dot{\eta}_t(m_t - X_t)dt - \eta_t d(m_t - X_t) && (5.6)\\ &= \left[(-\dot{\eta}_t + \eta_t(a+q+\eta_t))\,(m_t - X_t) + \eta_t m_t^Y\right] dt + \eta_t\sigma\sqrt{1-\rho^2}dW_t. \end{aligned}$$

Plugging the ansatz (5.5) in (5.2) gives

$$dY_t = \left[-(a+q)\eta_t + (\epsilon - q^2)\right](m_t - X_t)dt + Z_t^0 dW_t^0 + Z_t dW_t. \tag{5.7}$$

Identifying the two Itô decompositions (5.6) and (5.7), we deduce first from the martingale terms that $Z_t^0 \equiv 0$ and $Z_t = \eta_t \sigma\sqrt{1-\rho^2}$. Thus, from (5.3) we obtain $m_t^Y = 0$, and equating the drift terms we see that η_t must be solution to the Riccati equation

$$\dot{\eta}_t = 2(a+q)\eta_t + \eta_t^2 - (\epsilon - q^2), \tag{5.8}$$

with terminal condition $\eta_T = c$. As expected, the solution for η_t is given explicitly in (3.23) after taking the limit $N \to \infty$. Observe that from $m_t^Y = 0$ and (5.4) we deduce that $m_t^X = \mathbb{E}(\xi) + \rho\sigma W_t^0$ which will enter in the optimal control $(q+\eta_t)(m_t^X - X_t)$.

An important result of the theory of MFGs (see for example [5]) is the fact that, once a solution to the MFG is found, on can use it to construct approximate Nash equilibria for the finitely many players games. Here, if one assumes that each player is given the information $\overline{X}_t$, and if player i uses the strategy

$$\alpha_t^i = (q+\eta_t)(\overline{X}_t - X_t^i),$$

which is the limit as $N \to \infty$ of the strategy used in the finite players game, one sees how solving the limiting MFG problem can provide approximate Nash equilibria for which the financial implications are identical as the ones given in Section 4 for the exact Nash equilibria.

5.2. The Mean Field Game / HJB Approach

It is interesting to go through the derivation of the MFG solution using the HJB approach since it involves additional difficulties due to the presence of the common noise. In our toy model, it can be handled explicitly and we briefly outline this derivation.

For Markovian strategies of the form $\alpha(t,x)$, the dynamics are given as previously by

$$dX_t = \left[a(m_t - X_t) + \alpha(t, X_t)\right]dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t\right).$$

Here, we should think as (W_t^0) as given, so that the forward equation for the conditional density of X_t becomes a stochastic PDE (SPDE) which can be written as

$$dp_t = \left\{-\partial_x\left[(a(m_t - x) + \alpha(t,x))\,p_t\right] + \frac{1}{2}\sigma^2(1-\rho^2)\partial_{xx}p_t\right\}dt - \rho\sigma(\partial_x p_t)dW_t^0,$$

with the initial density p_0 being the density of ξ. Here $\alpha(t,x)$ is given and $m_t = \int x p_t(x)dx$. Consequently, m_t, the conditional mean of X_t given W^0 is a stochastic process which will turn out to be Markovian with its infinitesimal generator denoted by $\mathcal{L}^m dt + \rho\sigma(\partial_m)dW_t^0$. The HJB equation for the value function $V(t,x,m)$ can be written as

$$\begin{aligned} dV &+ \left[\frac{1}{2}\sigma^2(1-\rho^2)\partial_{xx}V + \mathcal{L}^m V + (\partial_{xm}V)\frac{d\langle m, X\rangle}{dt}\right]dt \\ &+ \inf_\alpha\left\{\left[a(m-x)+\alpha\right]\partial_x V + \frac{\alpha^2}{2} - q\alpha(m-x) + \frac{\epsilon}{2}(m-x)^2\right\}dt \\ &+ \rho\sigma(\partial_m V)dW_t^0 + \rho\sigma(\partial_x V)dW_t^0 = 0. \end{aligned}$$

Next, we minimize in α to get $\hat{\alpha} = q(m-x) - \partial_x V$, and we make the ansatz $V(t,x,m) = \frac{\eta_t}{2}(m-x)^2 + \mu_t$. Plugging in the forward equation for p_t, multiplying by x, and integrating with respect to x gives

$$dm_t = \rho\sigma dW_t^0.$$

Therefore, conditionally in W^0, $\mathcal{L}^m V = 0$ and $d\langle m, X\rangle = 0$. Then, verifying that the ansatz satisfies the HJB equation, by canceling terms in $(m-x)^2$ we obtain that η_t must satisfy the Riccati equation (5.8), and canceling state-independent terms leads to $\dot{\mu}_t = -\frac{1}{2}\sigma^2(1-\rho^2)\eta_t$ and therefore

$$\mu_t = \frac{1}{2}\sigma^2(1-\rho^2)\int_t^T \eta_s ds.$$

5.3. Controlling McKean-Vlasov Dyamics: Possibly a Different Solution

In this subsection we search for a different type of approximate equilibrium. As before, we require that all the banks use the same feedback function α to compute their lending and borrowing rate from their private information, but we now work with a different notion of equilibrium. We now base the notion of a critical point for the optimization on a simultaneous deviation of all the strategies at once. So instead of stress-testing by perturbing the lending and borrowing policy one at a time like in the search for Nash equilibria, we perturb them simultaneously. This form of equilibrium was tentatively called *franchised* equilibrium in [4]. See also [6] for a comparison with the results of the MFG models. Our interest in this model comes from the results of these two papers which show that, despite strong similarities, these two models can lead to different solutions, even in the case of linear - quadratic models.

The theory of the propagation of chaos implies that, when $N \to \infty$, the individual states X_t^i become independent and identically distributed, their common distribution being the law of the solution of an equation of the McKean-Vlasov type:

$$dX_t = [a(\mathbb{E}[X_t] - X_t) + \alpha(t, X_t)]dt + \sigma dW_t, \tag{5.9}$$

for some Wiener process (W_t). See for example [21] or [16]. Accordingly, the expected cost that the players try to minimize becomes

$$J(\phi) = \mathbb{E}\bigg[\int_0^T f(X_t, \mathbb{P}_{X_t}, \alpha(X_t))dt + g(X_T, \mathbb{P}_{X_T})\bigg],$$

where we use the notation $\mathbb{P}_Y$ to denote the law of the random variable Y. In the linear-quadratic case at hand, both functions f and g do not depend upon the entire distribution $\mathbb{P}_{X_t}$, but only upon the first two moments $\mathbb{E}[X_t]$ and $\mathbb{E}[X_t^2]$. The search for equilibrium considered in this subsection amounts to minimizing $J(\alpha)$ over a set of admissible feedback functions α under the constraint (5.9). The probabilistic approach to the solution of this problem is developed in [4] where a Pontryagin stochastic maximum principle is proved in full generality. Adjoint equations are identified and it is shown that there are different than in the MFG case. However, in the present situation, the fact that the cost functions $f(x,\mu,\alpha)$ and $g(x,\mu)$ depend only upon the distance from the argument x to the mean of the measure μ implies that the expectation $\mathbb{E}[Y_t]$ of the adjoint process vanishes, and the correction to the driver of the MFG adjoint equation vanishes as well. In the notation of [6], this corresponds

to the fact that $\overline{q} = -q$ and $\overline{m}_t = -m_t$, implying that the differences exhibited there between the solutions of linear quadratic MFG and McKean-Vlasov controlled dynamics do not exist in this instance. Again, this is due to the special form of interaction between the players through $\overline{X}_t - X_t^i$.

In any case, the two forms of approximate equilibria, though different in general, do coincide in our model of lending and borrowing!

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