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Systemic Risk and Stochastic Games with Delay

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Abstract We propose a model of inter-bank lending and borrowing which takes into account clearing debt obligations. The evolution of log-monetary reserves of banks is described by coupled diffusions driven by controls with delay in their drifts. Banks are minimizing their finite-horizon objective functions which take into account a quadratic cost for lending or borrowing and a linear incentive to borrow if the reserve is low or lend if the reserve is high relative to the average capitalization of the system. As such, our problem is a finite-player linear-quadratic stochastic differential game with delay. An open-loop Nash equilibrium is obtained using a system of fully coupled forward and advanced backward stochastic differential equations. We then describe how the delay affects liquidity and systemic risk characterized by a large number of defaults. We also derive a close-loop Nash equilibrium using a Hamilton-Jacobi-Bellman partial differential equation approach.

Keywords systemic risk · inter-bank borrowing and lending · stochastic game with delay · Nash equilibrium

Mathematics Subject Classification (2000) 91A15 · 91G80 · 60G99

1 Introduction

In [1], we proposed a stochastic game model of inter-bank lending and borrowing, where banks borrow from or lend to a central bank with no obligation to pay back their loans and no gain from lending. The main finding was that in equilibrium, the central bank is acting as a clearing house, liquidity is created, thus, leading to a more stable system. Systemic risk was analyzed as in [2] in the case of a linear model without control. Systemic risk being characterized as the rare event of a large number of defaults occurring, when the average capitalization reaches a prescribed level, the conclusion was that inter-bank lending and borrowing leads to stability through a flocking effect. For this type of interaction without control, we also refer to [3–5].

In order to make the toy model of [1] more realistic, we introduce delay in the controls. This forces banks to take responsibility for past lending and borrowing. In this paper, the evolution of the log-monetary reserves of the banks is described by a system of delayed stochastic differential equations, and banks try to minimize their costs or maximize their profits by controlling the rate of borrowing or lending. They interact via the average capitalization meaning that banks consider this average as a critical level to determine borrowing from or lending to the central bank.

We identify open-loop Nash equilibria by solving fully coupled forward and *advanced* backward stochastic differential equations (FABSDEs) introduced by [6]. Our conclusion is that the new effect, created by the need to *pay back* or *receive refunds* due to the presence of the delay in the controls, reduces the liquidity observed in the case without delay. However, despite these quantitative differences, the central bank is still acting as a clearing house. A closed-loop Nash equilibrium to this stochastic game with delay is derived from the Hamilton-Jacobi-Bellman (HJB) equation approach using the results in [7] and we provide a verification Theorem.

For a general introduction to BSDEs, stochastic control and stochastic differential games without delay, we refer to the recent monograph [8]. Stochastic control problems with delay have been studied from various points of view. When the delay only appears in the state variable, solutions to delayed optimal control problems were derived from variants of the Pontryagin-Bismut-Bensoussan stochastic maximum principle. See, for instance, [9, 10]. Alternatively, in order to use dynamic programming, [11, 12] reduce the system with delay to a finite-dimension problem, but still the delay does not appear in the control like in the case we want to study.

The general case of stochastic optimal control of stochastic differential equations with delay both in the state and the control is studied using an infinite-dimensional HJB equation in [7, 13]. The case with pointwise delayed control is studied in [14]. The general stochastic control problem in the case of delayed states and controls, both appearing in the forward equation. is studied in [15–17] by using the forward and advanced backward stochastic equations. Linear-Quadratic mean field Stackelberg games with delay and with a major player and many small players are studied in [18].

2 Setup of the Problem

The typical problem studied in this paper can be described as follows. The dynamics of the log-monetary reserves of N banks are given by the following diffusion processes X_t^i, $i = 1, \cdots, N$,

$$dX_t^i = \left(\alpha_t^i - \alpha_{t-\tau}^i\right) dt + \sigma dW_t^i, \quad 0 \le t \le T, \tag{1}$$

where W_t^i, $i = 1, \cdots, N$ are independent standard Brownian motions, and the rate of borrowing or lending α_t^i represents the control exerted by bank i on the system. In this example, we use the simplest possible form of delay, the delayed control $\alpha_{t-\tau}^i$ corresponding to repayments after a fixed time τ such that $0 \leq \tau \leq T$. We shall use deterministic initial conditions given by

$$X_0^i = \xi^i, \qquad \text{and} \qquad \alpha_t^i = 0, \quad t \in [-\tau, 0[. \tag{2}$$

For simplicity, we assume that the banks have the same volatility $\sigma > 0$. In what follows we use the notations $X = (X^1, \cdots, X^N)$, $x = (x^1, \cdots, x^N)$, $\alpha = (\alpha^1, \cdots, \alpha^N)$, and $\overline{x} = \frac{1}{N} \sum_{i=1}^N x^i$.

Before concentrating on the specific case (1), we prove a dedicated version of the sufficient condition of the Pontryagin stochastic maximum principle for a more general class of models for which the dynamics of the states are given by stochastic differential equations of the form:

$$dX_t^i = \left(\int_0^\tau \alpha_{t-s}^i \theta(ds) \right) dt + \sigma dW_t^i, \quad 0 \leq t \leq T, \tag{3}$$

where θ is a nonnegative measure on $[0, \tau]$. The special case (1) corresponds to $\theta = \delta_0 - \delta_\tau$.

Bank i chooses its own strategy α^i in order to minimize its objective function of the form:

$$J^i(\alpha^{-i}, \alpha^i) = \mathbb{E}\left\{ \int_0^T f_i(X_t, \alpha_t^i) dt + g_i(X_T) \right\}, \tag{4}$$

where the strategy α is denoted by (α^{-i}, α^i) to single out the control of bank i while J^i still depends on the strategies α^{-i} of the other banks through X_t.

In this paper, we concentrate on the running and terminal cost functions used in [1], namely:

$$f_i(x, \alpha^i) = \frac{(\alpha^i)^2}{2} - q\alpha^i(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2, \quad q \geq 0, \quad \epsilon > 0, \tag{5}$$

and

$$g_i(x) = \frac{c}{2}\left(\overline{x} - x^i\right)^2, \quad c \geq 0, \tag{6}$$

with $q^2 < \epsilon$ so that $f_i(x, \alpha)$ is convex in (x, α). Note that the case $\tau > T$ corresponds to no repayment and therefore no delay in the equations,. The case $\tau = 0$ corresponds to the case with no control and therefore no lending or borrowing. The term $q\alpha^i(\overline{x} - x^i)$ in the objective function (5) is an incentive to lend or borrow from a central bank which in this model does not make any decision and simply provides liquidity. However, we know that in the case with no delay [1], in equilibrium, the central bank acts as a clearing house. We will see in Section 7 that this is still the case with delay.

The paper is organized as follows. In Section 3, we briefly review the model without delay presented in [1]. The analysis of the stochastic differential games with delay is presented in Section 4 where we derive an exact open-loop Nash equilibrium using the FABSDE approach. In the process, we derive the *clearing house* role of the central bank in Remark 4.1. Section 5 is devoted to the derivation of a closed-loop equilibrium using an infinite-dimensional HJB equation approach with pointwise delayed control presented in [14]. In Section 6, we provide a verification Theorem. The effect of delay in term of financial implication is discussed in Section 7 where the main finding is that the introduction of delay in the model does not change the fact that in equilibrium, the central bank acts as a clearing house. However, liquidity is affected by the delay time.

3 Stochastic Games and Systemic Risk

The aim of this section is to briefly review the model of inter-bank lending or borrowing without delay studied in [1]. It is described by the model presented in the previous section, but with $\tau > T$, so that the delay term $\alpha^i_{t-\tau}$ in (1) is simply zero (note that in the model in [1], there is an additional drift term of the form $\frac{a}{N}\sum_{j=1}^N (X^j_t - X^i_t)$, which does not play a crucial role and that we ignore here by setting $a = 0$). The setup (4)–(6) of the stochastic game remains the same.

The open-loop problem consists in searching for an equilibrium among strategies $\{\alpha^i_t, i = 1, \cdots, N\}$ which are adapted processes satisfying an integrability condition such as $\mathbb{E}\left(\int_0^T |\alpha^i_t|^2 dt\right) < \infty$. The Hamiltonian for bank i is given by

$$H^i(x, y^i, \alpha) = \sum_{k=1}^N \alpha^k y^{i,k} + \frac{(\alpha^i)^2}{2} - q\alpha^i(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2, \qquad (7)$$

where $y^i = (y^{i,1}, \cdots, y^{i,N})$, $i = 1, \cdots, N$ are the adjoint variables.

For a given $\alpha = (\alpha^i)_{i=1,\cdots,n}$, the controlled forward dynamics of the states X^i_t are given by (1) without the delay term and with initial conditions $X^i_0 = \xi^i$. The adjoint processes $Y^i_t = (Y^{i,j}_t; j = 1, \cdots, N)$ and $Z^i_t = (Z^{i,j,k}_t;\ j = 1, \cdots, N,\ k = 1, \cdots, N)$ for $i = 1, \cdots, N$ are defined as the solutions of the backward stochastic differential equations (BSDEs):

$$dY_t^{i,j} = -\partial_{x^j} H^i(X_t, Y_t^i, \alpha_t)dt + \sum_{k=1}^{N} Z_t^{i,j,k} dW_t^k \tag{8}$$

with terminal conditions $Y_T^{i,j} = \partial_{x^j} g_i(X_T)$ for $i, j = 1, \cdots, N$ where g_i is given by (6). For each admissible strategy profile $\alpha = (\alpha^i)_{i=1,\cdots,n}$, standard existence and uniqueness results for BSDEs apply and the existence of the adjoint processes is guaranteed. Note that from (7), we have

$$\partial_{x^j} H^i = -q\alpha^i(\frac{1}{N} - \delta_{i,j}) + \epsilon(\overline{x} - x^i)(\frac{1}{N} - \delta_{i,j}).$$

The necessary condition of the Pontryagin stochastic maximum principle suggests that one minimizes the Hamiltonian H^i with respect to α^i which gives:

$$\hat{\alpha}^i = -y^{i,i} + q(\overline{x} - x^i). \tag{9}$$

With this choice for the controls α^i, the forward equation becomes coupled with the backward equation (8) to form a forward-backward coupled system. In the present linear-quadratic case, we make the ansatz

$$Y_t^{i,j} = \phi_t(\frac{1}{N} - \delta_{i,j})(\overline{X}_t - X_t^i), \tag{10}$$

for some deterministic scalar functions ϕ_t satisfying the terminal condition $\phi_T = c$. Using this ansatz, the backward equations (8) become

$$dY_t^{i,j} = (\frac{1}{N} - \delta_{i,j})(\overline{X}_t - X_t^i)\left[q(1 - \frac{1}{N})\phi_t - (\epsilon - q^2)\right] dt + \sum_{k=1}^{N} Z_t^{i,j,k} dW_t^k. \tag{11}$$

Using (9) and (10), the forward equation becomes

$$dX_t^i = \left[q + (1 - \frac{1}{N})\phi_t\right](\overline{X}_t - X_t^i)dt + \sigma dW_t^i. \tag{12}$$

Differentiating the ansatz (10) and identifying with the Ito's representation (11), one obtains from the martingale terms the deterministic adjoint variables

$$Z_t^{i,j,k} = \phi_t \sigma(\frac{1}{N} - \delta_{i,j})(\frac{1}{N} - \delta_{i,k}) \text{ for } \quad k = 1, \cdots, N,$$

and from the drift terms that the function ϕ_t must satisfy the scalar Riccati equation

$$\dot{\phi}_t = 2q(1 - \frac{1}{2N})\phi_t + (1 - \frac{1}{N})\phi_t^2 - (\epsilon - q^2), \tag{13}$$

with the terminal condition $\phi_T = c$. The explicit solution is given in [1]. Note that the form (9) of the control α_t^i, and the ansatz (10) combine to give:

$$\alpha_t^i = \left[q + (1 - \frac{1}{N})\phi_t \right] (\overline{X}_t - X_t^i), \tag{14}$$

so that, in this equilibrium, the forward equations become

$$dX_t^i = \left(q + (1 - \frac{1}{N})\phi_t \right) (\overline{X}_t - X_t^i)dt + \sigma dW_t^i. \tag{15}$$

Rewriting $(\overline{X}_t - X_t^i)$ as $\frac{1}{N}\sum_{j=1}^N (X_t^j - X_t^i)$, we see that the central bank is simply acting as a clearing house. From the form (15), we observe that the X^i's are mean-reverting to the average capitalization given by

$$d\overline{X}_t = \frac{\sigma}{N} \sum_{j=1}^N dW_t^j, \quad \overline{X}_0 = \frac{1}{N} \sum_{j=1}^N \xi^j.$$

In [2], we identified the systemic event as

$$\left\{ \min_{0 \le t \le T} (\overline{X}_t - \overline{X}_0) \le D \right\}$$

and we computed its probability

$$\mathbb{P}\left\{ \min_{0 \le t \le T} (\overline{X}_t - \overline{X}_0) \le D \right\} = 2\Phi\left(\frac{D\sqrt{N}}{\sigma\sqrt{T}} \right), \tag{16}$$

where Φ is the $\mathcal{N}(0,1)$-cdf. This systemic risk probability is exponentially small of order $\exp(-D^2 N/(2\sigma^2 T))$ as in the large deviation estimate.

4 Stochastic Games with Delay

Most often, a tailor made version of the stochastic maximum principle is used as a workhorse to construct open loop Nash equilibria for stochastic differential games. Here, we provide such a tool in a more general set up than used in the paper because we believe that this result is of independent interest on its own. We then specialize it to the model considered for systemic risk in Section 4.3.1.

4.1 The Model

We work with a finite horizon $T > 0$. Recall that we denote by $\tau > 0$ the delay length. As explained in the introduction, the delay is implemented with a (signed) measure θ on $[0, \tau]$, and in the case of interest, we shall use the particular case $\theta = \delta_0 - \delta_\tau$. All the stochastic processes are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a right continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$. The state and control processes are denoted by $\mathbf{X} = (X_t)_{0 \le t \le T}$ and $\alpha = (\alpha_t)_{0 \le t \le T}$. They are progressively measurable processes with values in $(\mathbb{R}^d)^N$ and a closed convex subset A of $(\mathbb{R}^d)^N$ respectively. They are linked by the dynamical equation:

$$dX_t =< \alpha_{[t]}, \theta > dt + \sigma dW_t \tag{17}$$

where $\mathbf{W} = (W_t)_{0 \le t \le T}$ is a $(d \times N)$-dimensional $\mathbb{F}$-Brownian motion, σ is a positive constant or a matrix. We use the notation $\alpha_{[t]} = \alpha_{[t-\tau,t]}$ for the restriction of the path of α to the interval $[t - \tau, t]$. By convention, and unless specified otherwise, we extend functions defined on the interval $[0, T]$ to functions on $[-\tau, T+\tau]$ by setting them equal to 0 outside the interval $[0, T]$. Also, we use the bracket notation $< f, \theta >$ to denote the integral $\int_0^\tau f(s)\theta(ds)$.

We assume that the dynamics of the state X_t of the system are given by a stochastic differential equation (17) which we can rewrite in coordinate form if we denote by X_t^i the N components of X_t, in which case we can interpret X_t^i as the private state of player i:

$$dX_t^i = \left(\int_0^\tau \alpha_{t-s}^i \theta(ds) \right) dt + \sigma dW_t^i, \quad 0 \le t \le T, \tag{18}$$

where the components W_t^i, $i = 1, \cdots, N$ of W_t are independent standard Wiener processes, and the component processes $(\alpha_t^i)_{t \ge 0}$ can be interpreted as the strategies used by the individual players. As explained in the introduction, θ is a nonnegative measure on $[0, \tau]$ implementing the impact of the delay on the dynamics. Recall that the special case of interest corresponds to $\theta = \delta_0 - \delta_\tau$. We assume the initial conditions:

$$X_0^i = \xi^i, \qquad \text{and} \qquad \alpha_t^i = 0, \quad t \in [-\tau, 0[. \tag{19}$$

The assumptions that the various states have the same volatility $\sigma > 0$ and the delay measure θ is the same for all the players are only made for convenience. These symmetry properties are important to derive mean field limits, but they are not really needed when we deal with finitely many players. The objective function of player i is given by (4) which we repeat here:

$$J^i(\alpha) = \mathbb{E}\left\{ \int_0^T f_i(X_t, \alpha_t^i) dt + g_i(X_T) \right\}.$$

For the sake of simplicity, we assume that the cost f_i to player i depends only upon the control α_t^i of player i, and not on the controls α_t^j for $j \neq i$ of the other players. In the case of games with mean field interactions, the cost functions are often of the form $f_i(x,\alpha) = f(x^i, \overline{x}, \alpha)$ and $g_i(x) = g(x^i, \overline{x})$, as in the particular case of the systemic risk model studied in this paper where:

$$f_i(x,\alpha^i) = f(x^i,\overline{x},\alpha^i) = \frac{(\alpha^i)^2}{2} - q\alpha^i(\overline{x}-x^i) + \frac{\epsilon}{2}(\overline{x}-x^i)^2,$$

for $q \geq 0$ and $\epsilon > 0$ as in (5), and:

$$g_i(x) = g(x^i,\overline{x}) = \frac{c}{2}\left(\overline{x}-x^i\right)^2, \quad c \geq 0,$$

as in (6) and with $q^2 < \epsilon$ to make sure that $f_i(x,\alpha)$ is convex in (x,α). Next, we introduce the system of adjoint equations.

4.2 The Adjoint Equations

For each player i and each given admissible control $\boldsymbol{\alpha}^i = (\alpha^i_t)_{0\leq t\leq T}$ for player i, we define the adjoint equation for player i as the Backward Stochastic Differential Equation (BSDE):

$$dY^i_t = -\partial_x f_i(X_t,\alpha^i_t)dt + Z^i_t dW_t, \qquad 0 \leq t \leq T \tag{20}$$

with terminal condition $Y^i_T = \partial_x g_i(X_T)$, and we call the processes $\mathbf{Y}^i = (Y^i_t)_{0\leq t\leq T}$ and $\mathbf{Z}^i = (Z^i_t)_{0\leq t\leq T}$ the adjoint processes corresponding to the strategy $\boldsymbol{\alpha}^i = (\alpha^i_t)_{0\leq t\leq T}$ of player i. Notice that each $\mathbf{Y}^i$ has the same dimension as $\mathbf{X}$, namely $N\times d$ if d is the dimension of each individual player private state X^i_t, while each $\mathbf{Z}^i$ has dimension $N^2\times d$. Accordingly, we shall use the notation $Y^i_t = (Y^{i,j}_t)_{j=1,\cdots,N}$ where each $Y^{i,j}_t$ has the same dimension

d as each of the private states X_t^j, and similarly, $Z_t^i = (Z_t^{i,j,k})_{j,k=1,\cdots,N}$. In the application of interest to us in this paper we have $d = 1$.

As before, the following notation will turn out to be helpful. If $\mathbf{Y} = (Y_t)_{0\le t\le T}$ is a progressively measurable process (scalar or multivariate) with continuous sample paths, we denote by $\tilde{\mathbf{Y}} = (\tilde{Y}_t)_{0\le t\le T}$ the process defined by:

$$\tilde{Y}_t = \mathbb{E}\left[\int_0^\tau Y_{t+s}\theta(ds)\ \big|\mathcal{F}_t\right] = \int_0^\tau \mathbb{E}[Y_{t+s}|\mathcal{F}_t]\ \theta(ds), \qquad 0 \le t \le T.$$

Moreover, for each $t \in [0, T]$, $x \in (\mathbb{R}^d)^N$ and $y \in \mathbb{R}^d$, we denote by $\hat{\alpha}^i(x, y)$ any $\alpha \in \mathbb{R}^d$ satisfying:

$$\partial_\alpha f_i(x, \alpha) = -y. \tag{21}$$

Under specific assumptions the implicit function theorem will provide existence of $\hat{\alpha}_i$, and regularity properties of this function with respect to the variables x and y.

4.3 Sufficient Condition for Optimality

Theorem 4.1 *Let us assume that the cost functions f_i are continuously differentiable in $(x, \alpha) \in (\mathbb{R}^d)^N \times \mathbb{R}^d$, and g_i are continuously differentiable on $(\mathbb{R}^d)^N$ with partial derivatives of (at most) linear growth, and that:*

(i) the functions g_i are convex;

(ii) the functions $(x, \alpha) \mapsto f_i(x, \alpha)$ are convex.

If $\boldsymbol{\alpha} = (\alpha_t^1, \cdots, \alpha_t^N)_{0\le t\le T}$ is an admissible adapted open-loop strategy profile (that is a function of the paths of the Brownian motions), and

$(\mathbf{X},\mathbf{Y},\mathbf{Z}) = \big((X_t^1,\cdots,X_t^N),(Y_t^1,\cdots,Y_t^N),(Z_t^1,\cdots,Z_t^N)\big)$ *are adapted process such that the dynamical equation* (17) *and the adjoint equations* (20) *are satisfied for the controls* $\alpha_t^i = \hat\alpha^i(X_t,\tilde Y_t^{i,i})$*, then the strategy profile* $\boldsymbol{\alpha} = (\alpha_t^1,\cdots,\alpha_t^N)_{0\le t\le T}$ *is an open loop Nash equilibrium.*

Proof We follow the proof given in [8] in the case without delay. We fix $i\in\{1,\cdots,N\}$, a generic admissible control strategy $(\beta_t)_{0\le t\le T}$ for player i, and for the sake of simplicity, we denote by X' the state $X_t^{(\hat\alpha^{-i},\beta)}$ controlled by the strategies $(\hat\alpha^{-i},\beta)$. The function g_i being convex, almost surely, we have:

$$
\begin{aligned}
&g_i(X_T)-g_i(X'_T)\\
&\quad\le (X_T-X'_T)\cdot\partial_x g_i(X_T)\\
&\quad= (X_T-X'_T)\cdot Y_T^i\\
&\quad= \int_0^T (X_t-X'_t)\,dY_t^i + \int_0^T Y_t^i\; d(X_t-X'_t)\\
&\quad= -\int_0^T (X_t-X'_t)\cdot\partial_x f_i(X_t,\alpha_t^i)\,dt + \int_0^T Y_t^i\cdot < \alpha_{[t]} - (\hat\alpha^{-i},\beta)_{[t]},\theta > \;dt\; + \text{ martingale}\\
&\quad= -\int_0^T (X_t-X'_t)\cdot\partial_x f_i(X_t,\alpha_t^i)\,dt + \int_0^T Y_t^{i,i}\cdot < \alpha_{[t]}^i - \beta_{[t]},\theta > \;dt\; + \text{ martingale.}
\end{aligned}
$$

Notice that we can use the classical form of integration by parts is due to the fact that the volatilities of all the states are the same constant σ. Taking expectations of both sides and plugging the result into

$$
J^i(\boldsymbol{\alpha})-J^i((\boldsymbol{\alpha}^{-i},\boldsymbol{\beta})) = \mathbb{E}\bigg\{\int_0^T [f_i(X_t,\alpha_t^i)-f_i(X'_t,\beta_t)]dt\bigg\}+\mathbb{E}\{g_i(X_T)-g_i(X'_T)\},
$$

we get:

$$
\begin{aligned}
&J^i(\boldsymbol{\alpha}) - J^i((\boldsymbol{\alpha}^{-i}, \boldsymbol{\beta})) \\
&\quad \leq \mathbb{E}\left\{\int_0^T [f_i(X_t, \alpha_t^i) - f_i(X'_t, \beta_t)]dt - \int_0^T (X_t - X'_t) \cdot \partial_x f_i(X_t, \alpha_t^i)\, dt\right\} \\
&\qquad\quad + \mathbb{E}\left\{\int_0^T Y_t^{i,i} \cdot < \alpha_{[t]}^i - \beta_{[t]}, \theta > \ dt\right\} \\
&\quad \leq \mathbb{E}\left\{\int_0^T [\alpha_t^i - \beta_t]\partial_\alpha f_i(X_t, \alpha_t^i) + Y_t^{i,i} \cdot < \alpha_{[t]}^i - \beta_{[t]}, \theta > \ dt\right\}. \qquad (22)
\end{aligned}
$$

Notice that:

$$
\begin{aligned}
&\mathbb{E}\bigg[\int_0^T Y_t^{i,i} \cdot < \alpha_{[t]}^i - \beta_{[t]}, \theta > dt\bigg] = \mathbb{E}\bigg[\int_0^\tau \bigg(\int_{-s}^{T-s} Y_{t+s}^{i,i}[\alpha_t^i - \alpha_t^i]dt\bigg)\,\theta(ds)\bigg] \\
&= \int_0^\tau \int_0^T \mathbb{E}[Y_{t+s}^{i,i}[\alpha_t^i - \beta_t]dt\,\theta(ds)] = \int_0^\tau \int_0^T \mathbb{E}[\mathbb{E}[Y_{t+s}^{i,i}|\mathcal{F}_t][\alpha_t^i - \beta_t]dt\,\theta(ds)] \\
&= \mathbb{E}\bigg[\int_0^\tau \int_0^T \bigg(\int_0^\tau \mathbb{E}[Y_{t+s}^{i,i}|\mathcal{F}_t]\theta(ds)\bigg)[\alpha_t^i - \beta_t]dt\bigg] = \mathbb{E}\bigg[\int_0^T \widetilde{Y}_t^{i,i} \cdot [\alpha_t^i - \beta_t]dt\bigg].
\end{aligned}
$$

Consequently:

$$
\begin{aligned}
J^i(\boldsymbol{\alpha}) - J^i((\boldsymbol{\alpha}^{-i}, \boldsymbol{\beta})) &\leq \mathbb{E}\left\{\int_0^T \bigg([\alpha_t^i - \beta_t]\partial_\alpha f_i(X_t, \alpha_t^i) + \widetilde{Y}_t^{i,i} \cdot [\alpha_t^i - \beta_t]\bigg)\, dt\right\} \\
&= 0
\end{aligned}
$$

by definition 21 of $\hat{\alpha}(t, \hat{X}_t, \widetilde{Y}_t^{i,i})$. □

4.3.1 Example

We shall use the above result when $d = 1$, $\theta = \delta_0 - \delta_{-\tau}$ so that

$< \alpha_{[t]}, \theta >= \int_0^\delta \alpha_{t-\tau}\, \theta(d\tau) = \alpha_t - \alpha_{t-\delta}$, and the cost functions are given by

(5) and (6), namely:

$$
f_i(x, \alpha) = \frac{1}{2}\alpha^2 - q\alpha(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2
$$

for some positive constants q and ϵ satisfying $q < \epsilon^2$ which guarantees that the functions f_i are convex. Notice that relation (21) gives $\hat{\alpha}^i(x,y) = -y - q(x^i - \overline{x})$. To derive the adjoint equations, we compute:

$$\partial_{x^i} f_i(x,\alpha) = \big(1 - \frac{1}{N}\big)[q\alpha + \epsilon(x^i - \overline{x})], \quad \text{and} \quad \partial_{x^j} f_i(x,\alpha) = -\frac{1}{N}[q\alpha + \epsilon(x^i - \overline{x})],$$

for $j \neq i$. Accordingly, the system of forward and advanced backward equations identified in the above theorem reads:

$$\begin{cases} dX_t^i = - < \widetilde{Y}_{[t]}^{i,i} + q(X_{[t]}^i - \overline{X}_{[t]}), \theta > dt + \sigma dW_t^i, \qquad i = 1, \cdots, N \\ dY_t^{i,j} = \big(\delta_{i,j} - \frac{1}{N}\big)[q\widetilde{Y}_t^{i,j} + (q^2 - \epsilon)(X_t^i - \overline{X}_t)]dt + \sum_{k=1}^N Z_t^{i,j,k} dW_t^k \quad i,j = 1, \cdots, N, \end{cases} \tag{23}$$

where we used the Kronecker symbol $\delta_{i,j}$ which is equal to 1 if $i = j$ and 0 if $i \neq j$. If we specialize this system to the case $\theta = \delta_0 - \delta_\tau$, we have $\widetilde{Y}_t^{i,j} = Y_t^{i,j} - \mathbb{E}[Y_{t+\tau}^{i,j} | \mathcal{F}_t]$, so that the forward advanced-backward system reads:

$$\begin{cases} dX_t^i = \big(-Y_t^{i,i} + Y_{t-\tau}^{i,i} + \mathbb{E}[Y_{t+\tau}^{i,i} | \mathcal{F}_t] - \mathbb{E}[Y_t^{i,i} | \mathcal{F}_{t-\tau}] \\ \qquad\qquad -q[X_t^i - X_{t-\tau}^i - \overline{X}_t + \overline{X}_{t-\tau}]\big)dt + \sigma dW_t^i, \qquad i = 1, \cdots, N \\ dY_t^{i,j} = \big(\delta_{i,j} - \frac{1}{N}\big)[qY_t^{i,j} - q\mathbb{E}[Y_{t+\tau}^{i,j} | \mathcal{F}_t] + (q^2 - \epsilon)(X_t^j - \overline{X}_t)]dt + \sum_{k=1}^N Z_t^{i,j,k} dW_t^k \\ \qquad\qquad i,j = 1, \cdots, N. \end{cases} \tag{24}$$

The version of the stochastic maximum principle proved in Theorem 4.1 reduces the problem of the existence of Nash equilibria for the system, to the solution of forward anticipated-backward stochastic differential equation. The following result can be used to resolve the existence issue but first we make the following remark which is key in term of financial interpretation.

Remark 4.1 ***(Clearing House Property)*** In the present situation, in contrast with the case without delay presented in Section 3, we will not be able to derive explicit formulas for the equilibrium optimal strategies such as (14). However, it is remarkable to see that the *clearing house property* $\sum \alpha^i = 0$ still holds. Indeed, setting $i = j$ in (23) and summing over N to derive an equation for $\overline{Y}_t = \frac{1}{N}\sum_{i=1}^N Y_t^{i,i}$ and $\overline{Z}_t^k = \frac{1}{N}\sum_{i=1}^N Z_t^{i,i,k}$, we find:

$$d\overline{Y}_t = -\left(\frac{1}{N} - 1\right) q\widetilde{\overline{Y}}_t dt + \sum_{k=1}^N \overline{Z}_t^k dW_t^k, \quad t \in [0,T],$$

with terminal condition $\overline{Y}_t = 0$ for $t \in [T, T+\tau]$. This equation admits the unique solution:

$$\overline{Y}_t = 0, \quad t \in [0, T+\tau], \quad \text{and} \quad \overline{Z}_t^k = 0,\, k = 1, \cdots, N,\, t \in [0,T].$$

and as a result,

$$\overline{\hat{\alpha}}_t = -\widetilde{\overline{Y}}_t = 0. \tag{25}$$

In what follows, on the top of $q^2 < \epsilon$, we further assume that

$$q^2(1 - \frac{1}{2N})^2 \leq \epsilon(1 - \frac{1}{N}), \tag{26}$$

which is satisfied for N large enough, or q small enough.

Theorem 4.2 *The FABSDE* (24) *has a unique solution.*

Remark 4.2 While this theorem gives existence of open-loop Nash equilibria for the model, it is unlikely that uniqueness holds. However, the cost functions f_i and g_i depending only upon x^i and $\overline{x}$, one could consider the mean field game problem corresponding to the limit $N \to \infty$, and in this limiting regime, it is likely that the strict convexity of the cost functions could be used to prove some form of uniqueness of the solution of the equilibrium problem.

Proof We first solve the system considering only the case $j = i$. Once this is done, we should be able to inject the process $X_t = (X_t^1, \cdots, X_t^N)$ so obtained into the equation for $dY_t^{i,j}$ for $j \neq i$, and solve this advanced equation with random coefficients.

Summing over $i = 1, \cdots, N$ the equations for X^i in (23), using the clearing house property of Remark 4.1, and denoting $\overline{\xi} = \frac{1}{N}\sum_{i=1}^N \xi^i$ give

$$\overline{X}_t = \overline{\xi} + \frac{\sigma}{N}\sum_{i=1}^N W_t^i, \quad t \in [0,T]. \tag{27}$$

Therefore, without loss of generality, we can work with the "centered" variables $X_t^{i,c} = X_t^i - \overline{X}_t$, $Y_t^{i,i,c} = Y_t^{i,i} - \overline{Y}_t = Y_t^{i,i}$, and $Z_t^{i,i,k,c} = Z_t^{i,i,k} - \overline{Z}_t^k = Z_t^{i,i,k}$ which must satisfy the system:

$$\begin{cases} dX_t^{i,c} = - < \widetilde{Y}_{[t]}^{i,i} + qX_{[t]}^{i,c}, \theta > dt + \sigma \sum_{k=1}^N \big(\delta_{i,k} - \frac{1}{N}\big) dW_t^k, \\ dY_t^{i,i} = \big(1 - \frac{1}{N}\big)[q\widetilde{Y}_t^{i,i} + (q^2 - \epsilon)X_t^{i,c}]dt + \sum_{k=1}^N Z_t^{i,i,k} dW_t^k \end{cases} \tag{28}$$

with $X_0^{i,c} = \xi^{i,c} := \xi^i - \overline{\xi}$, $Y_T^{i,i} = -c\left(\frac{1}{N} - 1\right) X_T^{i,c}$, and $Y_t^{i,i} = 0$ for $t \in]T, T+\tau]$ for $i = 1, \cdots, N$. We solve this system by extending the continuation method (see for example [19] and [6]) to the case of stochastic games. We consider a system which is written as a perturbation of the previous one without delay. Since we now work with $i \in \{1, \cdots, N\}$ fixed, we drop the exponent i from the notation for the sake of readability of the formulas.

$$\begin{cases} dX_t^\lambda = \big[-(1-\lambda)Y_t^\lambda - \lambda < \widetilde{Y}_{[t]}^\lambda + qX_{[t]}^\lambda, \theta > + \phi_t\big]dt \\ \qquad\qquad + \sum_{k=1}^N \big[-(1-\lambda)Z_t^{k,\lambda} + \lambda\sigma\big(\delta_{i,k} - \frac{1}{N}\big) + \psi_t^k\big]dW_t^k, \\ dY_t^\lambda = \big[-(1-\lambda)X_t^\lambda + \lambda\big(1-\frac{1}{N}\big)[q\widetilde{Y}_t^\lambda + (q^2-\epsilon)X_t^\lambda] + r_t\big]dt \\ \qquad\qquad + \sum_{k=1}^N Z_t^{k,\lambda} dW_t^k \end{cases} \tag{29}$$

with initial condition $X_0^\lambda = \xi^{i,c}$ and terminal condition $Y_T^\lambda = (1-\lambda)X_T^\lambda - \lambda c\left(\frac{1}{N} - 1\right)X_T^\lambda + \zeta^{i,i}$ and $Y_t^\lambda = 0$ for $t \in]T, T+\tau]$ in the case of $c > 0$, and $Y_T^\lambda = \zeta^{i,i}$ and $Y_t^\lambda = 0$ for $t \in]T, T+\tau]$ in the case of $c = 0$.

Here (recall that i is now fixed), ϕ_t, ψ_t^k, r_t are for $k = 1, \cdots, N$, square integrable processes which will be chosen at each single step of the induction procedure. Also ζ is a $L^2(\Omega, \mathcal{F}_T)$ random variable. Observe that if $\lambda = 0$, the system (29) is a particular case of the system in Lemma 2.5 in [19] for which existence and uniqueness is established, and it becomes the system (28) when setting $\lambda = 1$, $\zeta^{i,i} = 0$, $\phi_t^i = 0$, $\psi_t^{i,i,k} = 0$, $r_t^{i,i} = 0$, $i = 1, \cdots, N$ and $k = 1, \cdots, N$, for $0 \le t \le T$. We only give the proof of existence and uniqueness for the solution of the system (28) in the case of $c = 0$. The same arguments can be used to treat the case $c > 0$.

The proof relies on the following technical result which we prove in the appendix.

Lemma 4.1 *If there exists $\lambda_0 \in [0, 1[$ such that for any ζ and ϕ_t, r_t, ψ_t^k, $k = 1, \cdots, N$ for $0 \le t \le T$ the system* (29) *admits a unique solution for*

$\lambda = \lambda_0$, then there exists $\kappa_0 > 0$, such that for all $\kappa \in [0, \kappa_0[$, (29) admits a unique solution for any $\lambda \in [\lambda_0, \lambda_0 + \kappa[$.

Taking for granted the result of this lemma, we can prove existence and uniqueness for (29). Indeed, for $\lambda = 0$, the result is known. Using Lemma 4.1, there exists $\kappa_0 > 0$ such that (29) admits a unique solution for $\lambda = 0 + \kappa$ where $\kappa \in [0, \kappa_0[$. Repeating the inductive argument n times for $1 \leq n\kappa_0 < 1 + \kappa_0$ gives the result for $\lambda = 1$ and, therefore, the existence of the unique solution for (28). Since $X_t^{i,c} = X_t^i - \overline{X}_t$, $Y_t^{i,i,c} = Y_t^{i,i}$ and $Z_t^{i,i,k,c} = Z_t^{i,i,k}$, and $\overline{X}_t$ is given by (27), we obtain a unique solution $(X_t^i, Y_t^{i,i}, Z_t^{i,i,k})$ to the system (23). □

5 Hamilton-Jacobi-Bellman (HJB) Approach

In this section, we return to the particular case $\theta = \delta_0 - \delta_\tau$ of the drift given by the delayed control $\alpha_t - \alpha_{t-\tau}$. The HJB approach for delayed systems has been applied by [20] to a deterministic linear quadratic control problem. Later, [21] followed a similar approach for stochastic control problems. Here, we generalize the approach [21] based on an infinite dimensional representation and functional derivatives. We extend this approach to our stochastic game model with delay in order to identify a closed-loop Nash equilibrium.

Note that two specific features of our discussion require additional work for our argument to be fully rigorous at the mathematical level. First, the delayed control in the state equation appears as a mass at time $t - \tau$ and a smoothing argument as in [14] is needed. Second, we are using functional derivatives and proper function spaces should be introduced for our computations to be fully justified. However, since most of the functions we manipulate are linear or quadratic, we refrain from giving the details. In that sense, and for these two reasons, what follows is merely heuristic. A rigorous proof of the fact that the equilibrium identified in this section is actually a Nash equilibrium will be given in Section 6.

5.1 Infinite Dimensional Representation

Let $\mathbb{H}^N$ be the Hilbert space defined by

$$\mathbb{H}^N = \mathbb{R}^N \times L^2([-\tau, 0]; \mathbb{R}^N),$$

with the inner product

$$\langle z, \tilde{z} \rangle = z_0 \tilde{z}_0 + \int_{-\tau}^{0} z_1(\xi) \tilde{z}_1(\xi)\, d\xi,$$

where $z, \tilde{z} \in \mathbb{H}^N$, and z_0 and $z_1(.)$ correspond respectively to the $\mathbb{R}^N$-valued and $L^2([-\tau, 0]; \mathbb{R}^N)$-valued components.

By reformulating the system of coupled diffusions (1) in the Hilbert space $\mathbb{H}^N$, the system of coupled Abstract Stochastic Differential Equations (ASDE) for $Z = (Z^1, \cdots, Z^N) \in \mathbb{H}^N$ appears as

$$\begin{aligned} dZ_t &= (AZ_t + B\alpha_t)\, dt + GdW_t, \quad 0 \le t \le T, \\ Z_0 &= (\xi, 0) \in \mathbb{H}^N. \end{aligned} \tag{30}$$

where $W_t = (W_t^1, \cdots, W_t^N)$ is a standard N-dimensional Brownian motion and $\xi = (\xi^1, \cdots, \xi^N)$.

Here $Z_t = (Z_{0,t}, Z_{1,t,r})$, $r \in [-\tau, 0]$ corresponds to $(X_t, \alpha_{t-\tau-r})$ in the system of diffusions (1). In other words, for each time t, in order to find the dynamics of the states X_t, it is necessary to have X_t itself, and the past of the control $\alpha_{t-\tau-r}$, $r \in [-\tau, 0]$.

The operator $A : D(A) \subset \mathbb{H}^N \to \mathbb{H}^N$ is defined as

$$A : (z_0, z_1(r)) \to (z_1(0), -\frac{dz_1(r)}{dr}) \quad a.e., \quad r \in [-\tau, 0],$$

and its domain is

$$D(A) = \{(z_0, z_1(.)) \in \mathbb{H}^N : z_1(.) \in W^{1,2}([-\tau, 0]; \mathbb{R}^N), z_1(-\tau) = 0\}.$$

The adjoint operator of A is $A^* : D(A^*) \subset \mathbb{H}^N \to \mathbb{H}^N$ and is defined by

$$A^* : (z_0, z_1(r)) \to (0, \frac{dz_1(r)}{dr}) \quad a.e., \quad r \in [-\tau, 0],$$

with domain

$$D(A^*) = \{(w_0, w_1(.)) \in \mathbb{H}^N : w_1(.) \in W^{1,2}([-\tau, 0]); \mathbb{R}^N), w_0 = w_1(0)\}.$$

The operator $B : \mathbb{R}^N \to \mathbb{R}^N \times C^*([-\tau, 0], \mathbb{R}^N)$ is defined by

$$B : u \to (u, -\delta_{-\tau}(r)u), \quad r \in [-\tau, 0],$$

where $\delta_{-\tau}(.)$ is the Dirac measure at $-\tau$.

Remark 5.1 Note that in [21], the case of pointwise delay is not considered as the above operator B becomes unbounded because of the dirac measure. Here, we still use the unbounded operator B (in a heuristic sense!) and for a rigorous treatment, we refer to [14] where they use partial smoothing.

Finally, the operator $G : \mathbb{R}^N \to \mathbb{H}^N$ is defined by

$$G : z_0 \to (\sigma z_0, 0).$$

Remark 5.2 Let Z_t be a weak solution of the system of coupled ASDEs (30) and X_t be a continuous solution of the system of diffusions (1), then, with a similar line of reasoning as in Proposition 2 in [21], it can be proved that $X_t = Z_{0,t}$, a.s. for all $t \in [0,T]$ (for the infinite dimensional representation of problems with pointwise delay in the control, see also [22]).

5.2 System of Coupled HJB Equations

In order to use the dynamic programming principle for stochastic games (we refer to [8]) in search of closed-loop Nash equilibrium, the initial time is varied (closed-loop means that the control at time t is a function of the state at time t and of the past of the control). At time $t \in [0,T]$, given initial state $Z_t = z$ (whose second component is the past of the control), bank i chooses the control α^i to minimize its objective function $J^i(t,z,\alpha)$.

$$J^i(t,z,\alpha) = \mathbb{E}\left\{ \int_t^T f_i(Z_{0,s}, \alpha_s^i)dt + g_i(Z_{0,T}) \mid Z_t = z \right\}. \tag{31}$$

A Nash equilibrium α^* is such that for any i and any admissible α^i in feedback form, one has $J^i(\alpha^*) \leq J^i(\alpha^{*-i}, \alpha^i)$. In equilibrium, that is all other banks $j \neq i$ have optimized their objective function, bank i's value function $V^i(t,z)$ is

$$V^i(t,z) = \inf_{\alpha^i} J^i(t,z,\alpha). \tag{32}$$

The set of value functions $V^i(t,z)$, $i = 1, \cdots, N$ is a classical solution in the sense of Definition (5.1) of the following system of coupled HJB equations (we refer to [23] Chapter 2, for further discussion in this regard):

$$\partial_t V^i + \frac{1}{2} Tr(Q\partial_{zz} V^i) + \langle Az, \partial_z V^i \rangle + H_0^i(\partial_z V^i) = 0, \tag{33}$$
$$V^i(T) = g_i,$$

where $Q = G^*G$, and the Hamiltonian function $H_0^i(p^i) : \mathbb{H}^N \to \mathbb{R}$ is defined by

$$H_0^i(p^i) = \inf_{\alpha^i} [\langle B\alpha, p^i \rangle + f_i(z_0, \alpha^i)]. \tag{34}$$

Definition 5.1 (Classical Solution) A set of functions $V^i : [0,T] \times \mathbb{H}^N \to \mathbb{R}$, $i = 1, \dots N$ is a classical solution of coupled equations (33) if for $i = 1, \dots, N$, $V^i \in \mathbb{C}^{\{1,2\}}([0,T), \mathbb{H}^N) \cap \mathbb{C}([0,T), \mathbb{H}^N)$; $\partial_z V^i : [0,T] \times \mathbb{H}^N \to D(A^*)$; $A^*(\partial_z V^i) \in \mathbb{C}([0,T), \mathbb{H}^N)$; and V^i satisfies (33) pointwise.

Here, $p^i \in \mathbb{H}^N$ and can be written as $p^i = (p^{i,1}, \cdots, p^{i,N})$ where $p^{i,k} \in \mathbb{H}^1$, $k = 1, \cdots, N$. Given that $f_i(z_0, \alpha^i)$ is convex in (z_0, α^i),

$$\hat{\alpha}^i = -\langle B, p^{i,i} \rangle - q(z_0^i - \bar{z}_0). \tag{35}$$

Therefore,

$$\begin{aligned} H_0^i(p) &= \langle B\hat{\alpha}, p^i \rangle + f_i(z_0, \hat{\alpha}^i), \\ &= \sum_{k=1}^{N} \langle B, p^{i,k} \rangle \left(-\langle B, p^{k,k} \rangle - q(z_0^k - \bar{z}_0) \right) \\ &+ \frac{1}{2} \langle B, p^{i,i} \rangle^2 + \frac{1}{2} (\epsilon - q^2)(\bar{z}_0 - z_0^i)^2. \end{aligned} \tag{36}$$

We then make the ansatz

$$V^i(t,z) = E_0(t)(\bar{z}_0 - z_0^i)^2 - 2(\bar{z}_0 - z_0^i)\int_{-\tau}^{0} E_1(t,-\tau-s)(\bar{z}_{1,s} - z_{1,s}^i)ds$$
$$+ \int_{-\tau}^{0}\int_{-\tau}^{0} E_2(t,-\tau-s,-\tau-r)(\bar{z}_{1,s} - z_{1,s}^i)(\bar{z}_{1,r} - z_{1,r}^i)dsdr + E_3(t), \tag{37}$$

where $E_0(t)$, $E_1(t,s)$, $E_2(t,s,r)$ and $E_3(t)$ are some deterministic functions to be determined. It is assumed that $E_2(t,s,r) = E_2(t,r,s)$.

Remark 5.3 Note that the ansatz (37) depends on $z \in \mathbb{H}^N$ whose second component is the past of all banks' controls α. In other words, the value function $V^i(t,z)$ is an explicit function of the past of all banks' controls $\alpha_{t-\tau-r}$, $r \in [-\tau, 0]$.

The derivatives of the ansatz (37) are as follows

$$\partial_t V^i = \frac{dE_0(t)}{dt}(\bar{z}_0 - z_0^i)^2 - 2(\bar{z}_0 - z_0^i)\int_{-\tau}^{0} \frac{\partial E_1(t,-\tau-s)}{\partial t}(\bar{z}_{1,s} - z_{1,s}^i)ds$$
$$+ \int_{-\tau}^{0}\int_{-\tau}^{0} \frac{\partial E_2(t,-\tau-s,-\tau-r)}{\partial t}(\bar{z}_{1,s} - z_{1,s}^i)(\bar{z}_{1,r} - z_{1,r}^i)dsdr + \frac{dE_3(t)}{dt}, \tag{38}$$

$$\partial_{z^j} V^i =$$
$$\begin{bmatrix} 2E_0(t)(\bar{z}_0 - z_0^i) - 2\int_{-\tau}^{0} E_1(t,-\tau-s)(\bar{z}_{1,s} - z_{1,s}^i)ds \\ -2(\bar{z}_0 - z_0^i)E_1(t,-\tau-s) + \\ 2\int_{-\tau}^{0} E_2(t,-\tau-s,-\tau-r)(\bar{z}_{1,r} - z_{1,r}^i)dr \end{bmatrix}\left(\frac{1}{N} - \delta_{i,j}\right), \tag{39}$$

$$\partial_{z^j z^k} V^i =$$
$$\begin{bmatrix} 2E_0(t) & -2E_1(t,-\tau-s) \\ -2E_1(t,-\tau-s) & 2E_2(t,-\tau-s,-\tau-r) \end{bmatrix} \left(\frac{1}{N}-\delta_{i,j}\right)\left(\frac{1}{N}-\delta_{i,k}\right). \quad (40)$$

By plugging the ansatz (37) into the HJB equation (33), and collecting all the corresponding terms, the following set of equations are derived for $t \in [0,T]$ and $s, r \in [-\tau, 0]$.

The equation corresponding to the *constant* terms is

$$\frac{dE_3(t)}{dt} + (1-\frac{1}{N})\sigma^2 E_0(t) = 0, \quad (41)$$

The equation corresponding to the $(\bar{z}_0 - z_0^i)^2$ terms is

$$\frac{dE_0(t)}{dt} + \frac{\epsilon}{2} = 2(1-\frac{1}{N^2})(E_1(t,0)+E_0(t))^2 + 2q(E_1(t,0)+E_0(t)) + \frac{q^2}{2}. \quad (42)$$

The equation corresponding to the $(\bar{z}_0 - z_0^i)(\bar{z}_1 - z_1^i)$ terms is

$$\frac{\partial E_1(t,s)}{\partial t} - \frac{\partial E_1(t,s)}{\partial s} =$$
$$2(1-\frac{1}{N^2})\left(E_1(t,0)+E_0(t)+\frac{q}{2(1-\frac{1}{N^2})}\right)(E_2(t,s,0)+E_1(t,s)). \quad (43)$$

The equation corresponding to the $(\bar{z}_1 - z_1^i)(\bar{z}_1 - z_1^i)$ terms is

$$\frac{\partial E_2(t,s,r)}{\partial t} - \frac{\partial E_2(t,s,r)}{\partial s} - \frac{\partial E_2(t,s,r)}{\partial r} =$$
$$2(1-\frac{1}{N^2})(E_2(t,s,0)+E_1(t,s))(E_2(t,r,0)+E_1(t,r)). \quad (44)$$

The boundary conditions are

$$E_0(T) = \frac{c}{2}, \quad E_1(T,s) = 0, \quad E_2(T,s,r) = 0, \quad E_2(t,s,r) = E_2(t,r,s),$$
$$E_1(t,-\tau) = -E_0(t), \quad \forall t \in [0,T[, \quad E_2(t,s,-\tau) = -E_1(t,s), \quad \forall t \in [0,T[,$$
$$E_3(T) = 0. \quad (45)$$

Note that with these boundary conditions (at $t = T$), we have

$V^i(T, z) = g_i(z_0) = \frac{c}{2}(\bar{z}_0 - z_0^i)^2$, as desired.

Define $D_\theta = \{(t, s, r) : \theta \le t \le T, -\tau \le s \le 0, -\tau \le r \le 0\}$, and

$D = \cup_{0 \le \theta \le T} D_\theta$.

Remark 5.4 The set of equations (41–44) with boundary conditions (45) has a unique solution in the domain D.

Proof Here we just provide a sketch of the proof, which involves several steps. We refer to [24] for full details of each step.

Step 1: The system of equations (41–44) is rewritten in integral form.

$$
\begin{aligned}
E_3(t) &= E_3(0) - \int_0^t (1 - \frac{1}{N})\sigma^2 E_0(\theta) d\theta, \\
E_0(t) &= E_0(0) + \int_0^t \Big[-\frac{\epsilon}{2} + 2(1 - \frac{1}{N^2})(E_1(\theta, 0) + E_0(\theta))^2 \\
&\qquad\qquad + 2q(E_1(\theta, 0) + E_0(\theta)) + \frac{q^2}{2} \Big] d\theta, \\
E_1(t, s) &= -E_0(\min(T, t + s + \tau)) + E_0(T)\mathbf{1}_{\{t=T\}} + \\
&\int_{\min(T, t+s+\tau)}^t \Bigg[(1 - \frac{1}{N^2}) \left(E_1(\theta, 0) + E_0(\theta) + \frac{q}{2(1 - \frac{1}{N^2})} \right) \\
&\qquad\qquad \times (E_2(\theta, -\theta + t + s, 0) + E_1(\theta, -\theta + t + s)) \Bigg] d\theta. \\
E_2(t, s, r) &= -E_1(\min(T, t + s + \tau), r) + \\
&\int_{\min(T, t+s+\tau)}^t \Bigg[2(1 - \frac{1}{N^2}) (E_2(\theta, -\theta + t + s, 0) + E_1(\theta, -\theta + t + s)) \\
&\qquad\qquad \times (E_2(\theta, r, 0) + E_1(\theta, r)) \Bigg] d\theta, \quad s \le r, \\
E_2(t, s, r) &= E_2(t, r, s).
\end{aligned} \tag{46}
$$

Step 2: There exists a $\gamma > 0$ such that the system (46) has a unique solution for $\gamma \le t \le T$ and $-\tau \le s, r \le 0$. The idea of the proof is to define $\mathcal{B}$ as the Banach space of the quadruples of continuous functions
$\beta = (E_0(.), E_1(.,.), E_2(.,.,.), E_3(.))$ on D_γ with the norm
$||\beta|| = \max_{t,s,r}[|E_0(t)| + |E_1(t,s)| + |E_2(t,s,r)| + |E_3(t)|]$. Then, we find a $\gamma > 0$ such that the operator $\mathcal{J} = (\mathcal{J}_0, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3)$, defined as follows, becomes a contraction of the unit ball of $\mathcal{B}$ into itself, so by fixed point theorem, there exists a unique solution.

$$
\begin{aligned}
(\mathcal{J}_3\beta)(t) &= E_3(0) - \int_0^t (1 - \frac{1}{N})\sigma^2 E_0(\theta) d\theta, \\
(\mathcal{J}_0\beta)(t) &= E_0(0) + \int_0^t \Big[-\frac{\epsilon}{2} + 2(1 - \frac{1}{N^2})(E_1(\theta,0) + E_0(\theta))^2 \\
&\qquad + 2q(E_1(\theta,0) + E_0(\theta)) + \frac{q^2}{2} \Big] d\theta, \\
(\mathcal{J}_1\beta)(t,s) &= -(\mathcal{J}_0\beta)(\min(T, t+s+\tau)) + E_0(T)\mathbf{1}_{\{t=T\}} + \\
&\int_{\min(T,t+s+\tau)}^t \left(1 - \frac{1}{N^2}\right) \left(E_1(\theta,0) + E_0(\theta) + \frac{q}{2(1-\frac{1}{N^2})} \right) \\
&\qquad \times (E_2(\theta, -\theta+t+s, 0) + E_1(\theta, -\theta+t+s)) \Big] d\theta. \\
(\mathcal{J}_2\beta)(t,s,r) &= -(\mathcal{J}_1\beta)(\min(T, t+s+\tau), r) + \\
&\int_{\min(T,t+s+\tau)}^t \Big[2(1 - \frac{1}{N^2}) (E_2(\theta, -\theta+t+s, 0) + E_1(\theta, -\theta+t+s)) \\
&\qquad \times (E_2(\theta, r, 0) + E_1(\theta, r)) \Big] d\theta, \quad s \le r, \\
(\mathcal{J}_2\beta)(t,s,r) &= (\mathcal{J}_2\beta)(t,r,s). \qquad (47)
\end{aligned}
$$

Step 3: The solution is extended beyond γ. □

Theorem 5.1 *The ansatz* $V^i(t,z)$ *in* (37) *is a* classical solution *of the system of coupled HJB equations* (33).

Proof Given the functions $E_0 - E_3$ defined in (41)-(44) with boundary conditions (45), it is straightforward to check that the ansatz $V^i(t,z)$ satisfy all the conditions set forth in Definition (5.1).

If all the other banks choose their candidate optimal controls, then the bank i's candidate optimal strategy $\hat{\alpha}^i$, $i = 1, \cdots, N$ follows

$$\begin{aligned}\hat{\alpha}^i_t &= -\langle B, \partial_{z^i} V^i\rangle - q(z^i_0 - \bar{z}_0),\\ &= 2\left(1-\frac{1}{N}\right)\left[\left(E_1(t,0) + E_0(t) + \frac{q}{2\left(1-\frac{1}{N}\right)}\right)(\bar{z}_0 - z^i_0)\right.\\ &\left. - \int_{-\tau}^{0} \left(E_2(t,-\tau-s,0) + E_1(t,-\tau-s)\right)(\bar{z}_{1,s} - z^i_{1,s})ds\right]. \qquad (48)\end{aligned}$$

In terms of the original system of coupled diffusions (1), the candidate closed-loop Nash equilibrium corresponds to

$$\begin{aligned}\hat{\alpha}^i_t &= 2\left(1-\frac{1}{N}\right)\left[\left(E_1(t,0) + E_0(t) + \frac{q}{2\left(1-\frac{1}{N}\right)}\right)(\bar{X}_t - X^i_t)\right.\\ &\left. + \int_{t-\tau}^{t} \left[E_2(t,s-t,0) + E_1(t,s-t)\right](\bar{\hat{\alpha}}_s - \hat{\alpha}^i_s)ds\right], \quad i = 1, \cdots, N.\end{aligned} \qquad (49)$$

In the next Section 6, we provide a verification theorem which proves that the candidate optimal controls in (48) and (49) are indeed the optimal controls corresponding to the closed-loop Nash equilibrium.

Remark 5.5 As pointed out in Remark 4.1 of Section 4, in the present situation we still have $\sum_{i=1}^N \hat{\alpha}^i_t = 0$ as can be seen by summing (49) and using

$\sum_{i=1}^{N}(\bar{X}_t - X_t^i) = 0$ and $\sum_{i=1}^{N}(\bar{\hat{\alpha}}_s - \hat{\alpha}_s^i) = 0$. Therefore, in this equilibrium, the central bank serves as a clearing house (see also Section 7).

6 A Verification Theorem

In this section, we provide a verification theorem establishing that the strategies given by (49) correspond to a Nash equilibrium. Our solution is only *almost explicit* because the equilibrium strategies are given by the solution of a system of integral equations. This approach has been used by [24] to find the optimal control in a deterministic delayed linear quadratic control problem. Recently, [15] and [25] have applied this approach to delayed linear quadratic stochastic control problems. In this section, we generalize it to delayed linear -quadratic stochastic differential games.

We recall that at time $t \in [0, T]$, given $x = (x^1, \cdots, x^N)$, which should be viewed as the state of the N banks at time t, and an A-valued function α on $[0, \tau[$, which should be viewed as their collective controls over the time interval $[t-\tau, t[$, bank i chooses the strategy α^i to minimize its objective function

$$J^i(t, x, \alpha, (\alpha^{i,t}, \alpha^{-i,t})) = \mathbb{E}\left\{\int_t^T f_i(X_s, \alpha_s^i)ds + g_i(X_T) \mid X_t = x, \alpha_{[t)} = \alpha\right\}. \quad (50)$$

Here $\alpha_{[t)}$ is defined as the restriction of the path $s \mapsto \alpha_s$ to the interval $[t-\tau, t[$ and α^t is an admissible control strategy for the N banks over the time interval $[t, T]$. We denote by $\mathbb{A}^t$ this set of admissible strategies.

In the search for Nash equilibria, for each bank i, we assume that the banks $j \neq i$ chose their strategies $\alpha^{-i,t}$ for the *future* $[t,T]$, in which case, bank i's should choose a strategy $\alpha^{i,t} \in \mathbb{A}^{i,t}$ in order to try to minimize its objective function $J^i(t,x,\alpha,(\alpha^{i,t},\alpha^{-i,t}))$. As a result, we define the value function $V^i(t,x,\alpha,\alpha^{-i,t})$ of bank i by:

$$V^i(t,x,\alpha,\alpha^{-i,t}) = \inf_{\alpha^{i,t}\in\mathbb{A}^{i,t}} J^i(t,x,\alpha,(\alpha^{i,t},\alpha^{-i,t})). \tag{51}$$

Because of the linear nature of the dynamics of the states, together with the quadratic nature of the costs, we expect that in equilibrium, the functions J^i and V^i to be quadratic functions of the state x and the past α of the control. This is consistent with the choices we made in the previous section. Accordingly, we write the functions V^i as

$$\begin{aligned} V^i(t,x,\alpha) = E_0(t)(\bar{x}-x^i)^2 + 2(\bar{x}-x^i)\int_{t-\tau}^{t} E_1(t,s-t)(\bar{\alpha}_s-\alpha^i_s)ds \\ + \int_{t-\tau}^{t}\int_{t-\tau}^{t} E_2(t,s-t,r-t)(\bar{\alpha}_s-\alpha^i_s)(\bar{\alpha}_r-\alpha^i_r)dsdr + E_3(t), \end{aligned} \tag{52}$$

where we dropped the dependence of V^i upon its fourth parameter $\alpha^{-i,t}$ because the right hand side of (52) does not depend upon $\alpha^{-i,t}$. The deterministic functions $E_i\,(i=0,\cdots,3)$, are the solutions of the system (41–44) with the boundary conditions (45).

The main result of this section is Proposition 6.1 below which says that any solution of the system (49) of integral equations provides a Nash equilibrium. For that reason, we first prove existence and uniqueness of solutions of these integral equations when they are recast as a fixed point problem in classical spaces of adapted processes. This is done in Lemma 6.1 below. We simplify the notation and we rewrite equation (49) for the purpose of the proof of the lemma. We set:

$$\varphi(t) = 2\left(1 - \frac{1}{N}\right)\left(E_1(t,0) + E_0(t) + \frac{q}{2\left(1-\frac{1}{N}\right)}\right)$$

and

$$\bar{\psi}(t,s) = [E_2(t,s-t,0) + E_1(t,s-t)]\mathbf{1}_{[t-\tau,t]}(s)$$

so that equation (49) can be rewritten as:

$$\begin{aligned}
\hat{\alpha}^i_t &= \varphi(t)(\bar{X}_t - X^i_t) + \int_0^t \bar{\psi}(t,s)(\bar{\hat{\alpha}}_s - \hat{\alpha}^i_s)ds \\
&= \varphi(t)\left((\bar{\xi} - \xi^i) - \int_0^t [(\bar{\hat{\alpha}}_s - \hat{\alpha}^i_s) - (\bar{\hat{\alpha}}_{s-\tau} - \hat{\alpha}^i_{s-\tau})]ds + \sigma[\bar{W}_t - W^i_t]\right) \\
&+ \int_0^t \bar{\psi}(t,s)(\bar{\hat{\alpha}}_s - \hat{\alpha}^i_s)ds.
\end{aligned} \tag{53}$$

Summing these equations for $i = 1, \cdots, N$, we see that any solution should necessarily satisfy $\sum_{1\le i\le N} \hat{\alpha}^i = 0$, so that if we look for a solution of the system (49), we might as well restrict our search to processes satisfying $\bar{\hat{\alpha}}_t = 0$ for all $t \in [0,T]$.

So we denote by $\mathbb{R}^N_0$ the set of elements $x = (x^1, \cdots, x^N)$ of $\mathbb{R}^N$ satisfying $\sum_{1\le i\le N} x^i = 0$, and by $\mathcal{H}^{2,N}_0$ the space of $\mathbb{R}^N_0$-valued adapted processes $a = (a_t)_{0\le t\le T}$ satisfying

$$\|a\|_0^2 := \mathbb{E}\left[\int_0^T |a_t|^2 dt\right] < \infty.$$

Clearly, $\mathcal{H}^{2,N}_0$ is a real separable Hilbert space for the scalar product derived from the norm $\|\cdot\|_0$ by polarization. For $a \in \mathcal{H}^{2,N}_0$ we define the $\mathbb{R}^N_0$-valued process $\Psi(a)$ by:

$$\Psi(a)^i_t = \varphi(t)(\bar{\xi} - \xi^i) + \sigma\varphi(t)[\bar{W}_t - W^i_t] + \int_0^t \psi(t,s)a^i_s ds,\ 0 \le t \le T,\ i = 1, \cdots, N. \tag{54}$$

where the function ψ is defined by $\psi(t,s) = 1 - \mathbf{1}_{[0,0\vee(t-\tau)]}(s) - \bar{\psi}(t,s)$. We shall use the fact that the functions φ and ψ are bounded.

Given the above set-up, existence and uniqueness of a solution to (49) is given by the following lemma whose proof mimics the standard proofs of existence and uniqueness of solutions of stochastic differential equations.

Lemma 6.1 *The map Ψ defined by* (54) *has a unique fixed point in $\mathcal{H}^{2,N}_0$.*

Sketch of Proof We first check that Ψ maps $\mathcal{H}^{2,N}_0$ into itself. Indeed, if $a \in \mathcal{H}^{2,N}_0$,

$$\begin{aligned}
\|\Psi(a)\|_0^2 &= \mathbb{E}\int_0^T |\Psi(a)_t|^2 dt \\
&\le C\sum_{i=1}^N \Bigg[\mathbb{E}[|\bar{\xi} - \xi^i|^2]\int_0^T \varphi(t)^2 dt + \sigma^2\int_0^T \varphi(t)^2\mathbb{E}[|\bar{W}_t - W^i_t|^2]dt \\
&\quad + \int_0^T \mathbb{E}\left[\Big(\int_0^t \psi(t,s)a^i_s ds\Big)^2\right] dt\Bigg] \\
&\le C' + C''\int_0^T \mathbb{E}[|a_s|^2]ds < \infty,
\end{aligned} \tag{55}$$

where we have used that the functions φ and ψ are bounded. That proves that $\Psi(a) \in \mathcal{H}^{2,N}_0$. Existence and uniqueness of a fixed point is obtained by proving that Ψ is a strict contraction for a norm equivalent to the original norm $\|\cdot\|_0$ of $\mathcal{H}^{2,N}_0$. One can use the equivalent norm $\|\cdot\|_\epsilon$ defined by:

$$\|a\|_\epsilon^2 = \mathbb{E}\left[\int_0^T e^{-\epsilon t}|a_t|^2 dt\right]$$

for a positive number $\epsilon > 0$ to be chosen appropriately (we omit the remaining details). □

We now prove existence of Nash equilibria for the system.

Proposition 6.1 *The strategies* $(\hat{\alpha}^i_t)_{0\le t\le T,\, i=1,\cdots,N}$ *given by the solution of the system of integral equations* (49) *form a Nash equilibrium, and the corresponding value functions are given by* (52)*.*

In other words, we prove that

$$V^i(0,\xi^i,\alpha_{[0)}) \le J^i(0,\xi^i,\alpha_{[0)},(\alpha^i,\hat{\alpha}^{-i})),$$

for any α^i, and choosing $\alpha^i = \hat{\alpha}^i$ gives:

$$V^i(0,\xi^i,\alpha_{[0)}) = J^i(0,\xi^i,\alpha_{[0)},(\hat{\alpha}^i,\hat{\alpha}^{-i})).$$

Notice that the equilibrium strategies, which we identified, are in feedback form in the sense that each $\hat{\alpha}^i_t$ is a deterministic function of the trajectory $X_{[0,t]}$ of the past of the state. Notice also that there is absolutely nothing special with the time $t = 0$ and the initial condition $X_0 = \xi, \alpha_{[0)} = 0$. Indeed, for any $t \in [0,T]$ and $\mathbb{R}^N$-valued square integrable random variable ζ, the same proof can be used to construct a Nash equilibrium for the game over the interval $[t,T]$ and any initial condition $(X_t = \zeta, \alpha_{[t)})$.

Proof We fix an arbitrary $i \in \{1,\cdots,N\}$, an admissible control $\alpha^i \in \mathbb{A}^{-i}$ for player i, and we assume that the state process $(X_t)_{0\le t\le T}$ for the N banks is controlled by $(\alpha^i_t,\hat{\alpha}^i_t)_{0\le t\le T}$ where $(\hat{\alpha}^k_t)_{0\le t\le T,\, k=1,\cdots,N}$ solves the system of integral equations (49). Next, we apply Itô's formula to $V^i(t,X_t,\alpha_{[t)})$ where the function V^i is defined by (52) (see [26] Section 4.4 for infinite dimensional Itô's formula, and note that here V^i is differentiable in t and quadratic in $(x, \alpha_{[t)})$). We obtain

$$
\begin{aligned}
& dV^i(t, X_t, \alpha_{[t)}) = \\
& \Bigg\{ \frac{dE_0(t)}{dt}(\bar{X}_t - X_t^i)^2 + 2E_0(t)(\bar{X}_t - X_t^i)\left(\bar{\alpha}_t - \alpha_t^i - (\bar{\alpha}_{t-\tau} - \alpha_{t-\tau}^i)\right) \\
& + \sum_{j=1}^{N} \sigma^2 E_0(t)(\frac{1}{N} - \delta_{i,j})^2 + 2\left(\bar{\alpha}_t - \alpha_t^i - (\bar{\alpha}_{t-\tau} - \alpha_{t-\tau}^i)\right) \\
& \qquad\qquad\qquad \times \int_{t-\tau}^{t} E_1(t, s-t)(\bar{\alpha}_s - \alpha_s^i) ds \\
& + 2(\bar{X}_t - X_t^i) \int_{t-\tau}^{t} \left[\frac{\partial E_1(t, s-t)}{\partial t} - \frac{\partial E_1(t, s-t)}{\partial s} \right] (\bar{\alpha}_s - \alpha_s^i) ds \\
& + 2(\bar{X}_t - X_t^i) E_1(t, 0)(\bar{\alpha}_t - \alpha_t^i) - 2(\bar{X}_t - X_t^i) E_1(t, -\tau)(\bar{\alpha}_{t-\tau} - \alpha_{t-\tau}^i) \\
& + \int_{t-\tau}^{t} \int_{t-\tau}^{t} \Bigg[\frac{\partial E_2(t, s-t, r-t)}{\partial t} - \frac{\partial E_2(t, s-t, r-t)}{\partial s} \\
& - \frac{\partial E_2(t, s-t, r-t)}{\partial r} \Bigg] (\bar{\alpha}_s - \alpha_s^i)(\bar{\alpha}_r - \alpha_r^i) ds dr \\
& + (\bar{\alpha}_t - \alpha_t^i) \Bigg(\int_{t-\tau}^{t} E_2(t, s-t, 0)(\bar{\alpha}_s - \alpha_s^i) ds \\
& \qquad\qquad\qquad + \int_{t-\tau}^{t} E_2(t, 0, r-t)(\bar{\alpha}_r - \alpha_r^i) dr \Bigg) \\
& - (\bar{\alpha}_{t-\tau} - \alpha_{t-\tau}^i) \Bigg(\int_{t-\tau}^{t} E_2(t, s-t, -\tau)(\bar{\alpha}_s - \alpha_s^i) ds \\
& \qquad\qquad + \int_{t-\tau}^{t} E_2(t, -\tau, r-t)(\bar{\alpha}_r - \alpha_r^i) dr \Bigg) + \frac{dE_3(t)}{dt} \Bigg\} dt \\
& + 2 \sum_{j=1}^{N} (\frac{1}{N} - \delta_{i,j}) \Bigg\{ E_0(t)(\bar{X}_t - X_t^i) + \int_{t-\tau}^{t} E_1(t, s-t)(\bar{\alpha}_s - \alpha_s^i) ds \Bigg\} \sigma dW_t^j.
\end{aligned}
\tag{56}
$$

Then, integrating between 0 and T, using $V^i(T, X_T) = g_i(X_T)$ (ensured by the boundary conditions at $t = T$ for E_k, $k = 0, 1, 2, 3$), taking expectation, using the differential equations (41-44), using the short notation $A_1 = 1 - \frac{1}{N}$, $A_2 = 1 - \frac{1}{N^2}$, and adding $\mathbb{E}\int_0^T f_i(X_s, \alpha_s^i)dt$ on both sides, one obtains:

$$
\begin{aligned}
&-V^i(0,\xi^i,\alpha_{[0)}) + \mathbb{E}(g_i(X_T)) + \mathbb{E}\int_0^T f_i(X_s,\alpha_s^i)dt = \\
&-V^i(0,\xi^i,\alpha_{[0)}) + J^i(0,\xi^i,\alpha_{[0)},\alpha) = \\
&\mathbb{E}\int_0^T \Bigg\{ \left[-\frac{\epsilon}{2} + 2A_2(E_1(t,0)+E_0(t))^2 + 2q(E_1(t,0)+E_0(t)) + \frac{q^2}{2}\right](\bar{X}_t - X_t^i)^2 \\
&+2E_0(t)(\bar{X}_t - X_t^i)\big((\bar{\alpha}_t - \alpha_t^i) - (\bar{\alpha}_{t-\tau} - \alpha_{t-\tau}^i)\big) + \sigma^2 E_0(t)\sum_{j=1}^N (\frac{1}{N} - \delta_{i,j})^2 \\
&+2\left(\bar{\alpha}_t - \alpha_t^i - (\bar{\alpha}_{t-\tau} - \alpha_{t-\tau}^i)\right)\int_{t-\tau}^t E_1(t,s-t)(\bar{\alpha}_s - \alpha_s^i)ds \\
&+2(\bar{X}_t - X_t^i)\int_{t-\tau}^t \Bigg[2A_2\left(E_1(t,0) + E_0(t) + \frac{q}{2A_2}\right) \\
&\qquad\qquad \times (E_2(t,s-t,0) + E_1(t,s-t))\Bigg](\bar{\alpha}_s - \alpha_s^i)ds \\
&+2(\bar{X}_t - X_t^i)E_1(t,0)(\bar{\alpha}_t - \alpha_t^i) - 2(\bar{X}_t - X_t^i)E_1(t,-\tau)(\bar{\alpha}_{t-\tau} - \alpha_{t-\tau}^i) \\
&+\int_{t-\tau}^t\int_{t-\tau}^t \Big[2A_2\left(E_2(t,s-t,0) + E_1(t,s-t)\right)\left(E_2(t,r-t,0) + E_1(t,r-t)\right)\Big] \\
&\qquad\qquad \times(\bar{\alpha}_s - \alpha_s^i)(\bar{\alpha}_r - \alpha_r^i)dsdr \\
&+(\bar{\alpha}_t - \alpha_t^i)\left(\int_{t-\tau}^t E_2(t,s-t,0)(\bar{\alpha}_s - \alpha_s^i)ds + \int_{t-\tau}^t E_2(t,0,r-t)(\bar{\alpha}_r - \alpha_r^i)dr\right) \\
&-(\bar{\alpha}_{t-\tau} - \alpha_{t-\tau}^i)\Bigg(\int_{t-\tau}^t E_2(t,s-t,-\tau)(\bar{\alpha}_s - \alpha_s^i)ds \\
&\qquad\qquad + \int_{t-\tau}^t E_2(t,-\tau,r-t)(\bar{\alpha}_r - \alpha_r^i)dr\Bigg)
\end{aligned}
$$

$$-A_1\sigma^2 E_0(t) + \frac{1}{2}(\alpha_t^i)^2 - q\alpha_t^i(\bar{X}_t - X_t^i) + \frac{\epsilon}{2}(\bar{X}_t - X_t^i)^2 \Bigg\} dt. \qquad (57)$$

Observe that the terms in ϵ cancel, the terms in σ^2 cancel, and the terms involving delayed controls cancel using symmetries and boundary conditions (45) for the functions E_k's.

Next, motivated by (49), we rearrange the terms left in (57) so that the square of

$$\alpha_t^i - 2A_1\Bigg[\left(E_1(t,0) + E_0(t) + \frac{q}{2A_1}\right)(\bar{X}_t - X_t^i) + \int_{t-\tau}^{t}\left[E_2(t,s-t,0) + E_1(t,s-t)\right](\bar{\hat{\alpha}}_s - \hat{\alpha}_s^i)ds\Bigg]$$

appears first. We obtain

$$\begin{aligned}
&-V^i(0,\xi^i,\alpha_{[0)}) + J^i(0,\xi^i,\alpha_{[0)},\alpha) = \\
&\mathbb{E}\int_0^T \Bigg\{ \frac{1}{2}\Bigg(\alpha_t^i - 2A_1\Bigg[\left(E_1(t,0) + E_0(t) + \frac{q}{2A_1}\right)(\bar{X}_t - X_t^i) \\
&\qquad + \int_{t-\tau}^{t}\left[E_2(t,s-t,0) + E_1(t,s-t)\right](\bar{\hat{\alpha}}_s - \hat{\alpha}_s^i)ds\Bigg]\Bigg)^2 \\
&+(\bar{X}_t - X_t^i)^2\Bigg[-2[A_1(E_1(t,0) + E_0(t) + \frac{q}{2}]^2 \\
&\qquad +2A_2(E_1(t,0) + E_0(t))^2 + 2q(E_1(t,0) + E_0(t)) + \frac{q^2}{2}\Bigg] \\
&+(\bar{X}_t - X_t^i)\left[2\alpha_t^i[A_1(E_1(t,0) + E_0(t)] + 2(E_1(t,0) + E_0(t))(\bar{\alpha}_t - \alpha_t^i)\right] \\
&+(\bar{X}_t - X_t^i)\left(\int_{t-\tau}^{t}(E_2(t,s-t,0) + E_1(t,s-t)(\bar{\alpha}_s - \alpha_s^i)ds\right) \\
&\qquad \times\left[-4A_1\left(A_1(E_1(t,0) + E_0(t) + \frac{q}{2}\right) + 4A_2\left(E_1(t,0) + E_0(t) + \frac{q}{2A_2}\right)\right] \\
&+\left(\int_{t-\tau}^{t}(E_2(t,s-t,0) + E_1(t,s-t)(\bar{\alpha}_s - \alpha_s^i)ds\right)\left[2A_1\alpha_t^i + 2(\bar{\alpha}_t - \alpha_t^i)\right] \\
&+\left(\int_{t-\tau}^{t}(E_2(t,s-t,0) + E_1(t,s-t)(\bar{\alpha}_s - \alpha_s^i)ds\right)^2\left[-2A_1^2 + 2A_2\right]\Bigg\} dt. \qquad (58)
\end{aligned}$$

Using $A_2 = A_1^2 + \frac{2}{N}A_1$ and the relation $\bar{\alpha}_t - \alpha_t^i = \frac{1}{N}\sum_{j\neq i}\alpha_t^j - A_1\alpha_t^i$, we simplify (58) to obtain:

$$
\begin{aligned}
&-V^i(0,\xi^i,\alpha_{[0)}) + J^i(0,\xi^i,\alpha_{[0)},\alpha) = \\
&\mathbb{E}\int_0^T \Bigg\{ \frac{1}{2}\Bigg(\alpha_t^i - 2A_1\Bigg[\Bigg(E_1(t,0) + E_0(t) + \frac{q}{2A_1}\Bigg)(\bar{X}_t - X_t^i) \\
&\qquad\qquad + \int_{t-\tau}^t \left[E_2(t,s-t,0) + E_1(t,s-t)\right](\bar{\hat{\alpha}}_s - \hat{\alpha}_s^i)ds\Bigg]\Bigg)^2 \\
&+(\bar{X}_t - X_t^i)^2\left[\frac{4}{N}A_1(E_1(t,0)+E_0(t))^2 + \frac{2q}{N}(E_1(t,0)+E_0(t))\right] \\
&+(\bar{X}_t - X_t^i)\left[\frac{2}{N}\sum_{j\neq i}\alpha_t^j(E_1(t,0)+E_0(t))\right] \\
&+(\bar{X}_t - X_t^i)\left(\int_{t-\tau}^t (E_2(t,s-t,0) + E_1(t,s-t)(\bar{\alpha}_s - \alpha_s^i)ds\right) \\
&\qquad\qquad \times\left[\frac{8}{N}A_1(E_1(t,0)+E_0(t)) + \frac{2q}{N}\right] \\
&+\left(\int_{t-\tau}^t (E_2(t,s-t,0) + E_1(t,s-t)(\bar{\alpha}_s - \alpha_s^i)ds\right)\left[\frac{2}{N}\sum_{j\neq i}\alpha_t^j\right] \\
&+\left(\int_{t-\tau}^t (E_2(t,s-t,0) + E_1(t,s-t)(\bar{\alpha}_s - \alpha_s^i)ds\right)^2\left[\frac{4}{N}A_1\right]\Bigg\}dt. \qquad (59)
\end{aligned}
$$

Now, assuming that the players $j \neq i$ are using the strategies $\hat{\alpha}_t^j$ given by (49), the quantity $\sum_{j\neq i}\alpha_t^j$ becomes

$$
\begin{aligned}
\sum_{j\neq i}\hat{\alpha}_t^j = -2A_1\Bigg[&\Bigg(E_1(t,0) + E_0(t) + \frac{q}{2A_1}\Bigg)(\bar{X}_t - X_t^i) \\
&+ \int_{t-\tau}^t \left[E_2(t,s-t,0) + E_1(t,s-t)\right](\bar{\hat{\alpha}}_s - \hat{\alpha}_s^i)ds\Bigg].
\end{aligned}
$$

Plugging this last expression in (59), one sees that the terms after the square cancel and we get

$$
\begin{aligned}
&-V^i(0,\xi^i,\alpha_{[0)}) + J^i(0,\xi^i,\alpha_{[0)},(\alpha^i,\hat{\alpha}^{-i})) = \\
&\mathbb{E}\int_0^T \left\{ \frac{1}{2}\left(\alpha_t^i - 2A_1\left[\left(E_1(t,0)+E_0(t)+\frac{q}{2A_1}\right)(\bar{X}_t - X_t^i)\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. + \int_{t-\tau}^{t}\left[E_2(t,s-t,0)+E_1(t,s-t)\right](\bar{\hat{\alpha}}_s - \hat{\alpha}_s^i)ds\right]\right)^2\right\}dt.
\end{aligned}
\tag{60}
$$

Consequently $V^i(0,\xi^i,\alpha_{[0)}) \le J^i(0,\xi^i,\alpha_{[0)},(\alpha^i,\hat{\alpha}^{-i}))$, and choosing $\alpha^i = \hat{\alpha}^i$ leads to $V^i(0,\xi^i,\alpha_{[0)}) = J^i(0,\xi^i,\alpha_{[0)},(\hat{\alpha}^i,\hat{\alpha}^{-i}))$. □

Remark 6.1 While we obtained the existence of a closed-loop Nash equilibrium for the model, it is unlikely that uniqueness holds. However, like in Remark 4.2 for open-loop Nash equilibria, one could consider the mean-field game problem corresponding to the limit $N \to \infty$, and in this limiting regime, it is likely that the strict convexity of the cost functions could be used to prove some form of uniqueness of the solution of the equilibrium problem.

7 Financial Implications and Numerical Illustration

The main finding is that taking into account repayment with delay does not change the fact that the central bank providing liquidity is acting as a *clearing house* in all the Nash equilibria we identified (open-loop in Section 4 or closed-loop in Sections 5 and 6).

The delay time, that is the single repayment maturity τ that we considered in this paper, controls the liquidity provided by borrowing and lending. The two extreme cases are:

1. No borrowing/lending: $\tau = 0$:

   In that case, no liquidity is provided and the log-reserves X_t^i follow independent Brownian motions.

2. No repayment: $\tau \geq T$:

   This is the case studied previously in [1] and summarized in Section 3. The rate of liquidity (the speed at which money is flowing through the system) is given by $\left[q + (1 - \frac{1}{N})\phi_t\right]$ as shown in equation (12).

3. Intermediate regime $0 < \tau < T$:

   We conjecture that in the regime T large and zero terminal condition ($c = 0$), for fixed t the rate of liquidity is monotone in τ in a fixed range $[0, \tau_{max}]$. For instance, in the case of the close-loop equilibrium obtained in Section 6 given by (49), the rate of liquidity is $[2E_1(t, 0) + 2E_0(t) + q]$, where the function E_1 and E_0 are solutions to the system (41–43). These solutions are not given by closed-form formulas. We computed them numerically. We show in Figure 1 that as expected, liquidity increases as τ increases. This is clear for values of τ which are small relative to the time horizon T. For values of τ, which are large and comparable with T, the boundary effect becomes more important as oscillations propagate backward.

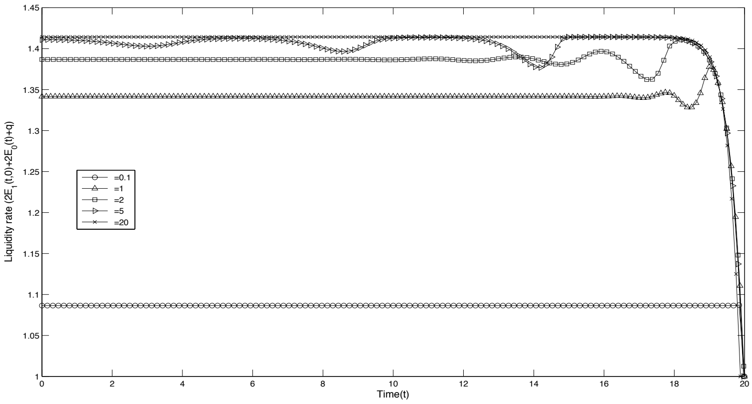

Fig. 1 Liquidity as a function of the delay time τ. The parameters are $T = 20$, $q = 1$, $\varepsilon = 2$, and $c = 0$.

8 Conclusion

We proposed a continuous-time model for interbank borrowing and lending which takes into account clearing debt obligations. By controlling their rate of borrowing/lending, banks minimize an objective function comprising a quadratic cost and an incentive to stay close to the average capitalization. Our model is a finite-player linear-quadratic stochastic differential game with delay. The novelty is in the presence of the delay, and especially, delay in the controls. We characterized an open-loop Nash equilibrium using a system of forward advanced backward stochastic differential equations (FABSDEs), and a closed-loop Nash equilibrium using a system of infinite-dimensional Hamilton-Jacobi-Bellman equations and a verification argument. We do not expect uniqueness of these equilibria. Still, we show that the equilibria we identified satisfy the desirable "clearing house condition" which ensures that the overall sum of lending and borrowing is zero, so that the central bank acts only as a clearing house. The question of the existence of other equilibria satisfying this condition remains open. Accordingly, the case of more general (non linear-quadratic) stochastic differential games with delay is open for further study.

Our model is solved by a construction of the "mean field" type. Part of our ongoing research is to derive the master equation for the corresponding mean field game with delay. Such an equation involves naturally the law of the past of the control and therefore, falls in the category of the so-called "extended mean field games". In our model, the derivation of the master equation and its solution will offer a practical tool to approximate the solution of the finite-player games, and hopefully, to derive large deviation estimates related to systemic risk.

A Proof of Lemma 4.1

Proof Assuming that $(\check{X}, \check{Y}, (\check{Z}^k)_{k=1,\cdots,N})$ is given as an input, we solve the system (29) for $\lambda = \lambda_0$ and the processes ϕ_t, ψ_t^k, r_t and the random variable ζ replaced according to the prescriptions:

$$
\begin{aligned}
\phi_t &\leftarrow \phi_t + \kappa\big[\check{Y}_t - < \tilde{\check{Y}}_{[t]} + q\check{X}_{[t]}, \theta >\big] \\
\psi_t^k &\leftarrow \psi_t^k + \kappa\big[\check{Z}_t^k + \sigma(\frac{1}{N} - \delta_{i,k})\big], \quad k = 1, \cdots, N \\
r_t &\leftarrow r_t + \kappa\big[\check{X}_t + \big(1 - \frac{1}{N}\big)\big[q\tilde{\check{Y}}_t + \big(q^2 - \epsilon\big)\,\check{X}_t\big]\big] \\
\zeta &\leftarrow \zeta + \kappa\big[-\check{X}_T + c(1 - \frac{1}{N})\check{X}_T\big],
\end{aligned}
$$

and denote the solution by $(X, Y, (Z^k)_{k=1,\cdots,N})$. In this way, we defined a mapping

$$
\Phi : (\check{X}, \check{Y}, (\check{Z}^k)_{k=1,\cdots,N}) \to \Phi(\check{X}, \check{Y}, (\check{Z}^k)_{k=1,\cdots,N}) = (X, Y, (Z^k)_{k=1,\cdots,N}),
$$

and the proof consists in proving that the latter is a contraction for small enough $\kappa > 0$.

Consider $(\widehat{X}, \widehat{Y}, (\widehat{Z}^k)_{k=1,\cdots,N}) = (X - X', Y - Y', (Z^k - Z^{k\prime})_{k=1,\cdots,N})$ where $(X, Y, (Z^k)_{k=1,\cdots,N})$ and $(X', Y', (Z^{k\prime})_{k=1,\cdots,N})$ are the corresponding image using inputs $(\check{X}, \check{Y}, (\check{Z}^k)_{k=1,\cdots,N})$ and $(\check{X}', \check{Y}', (\check{Z}^{k\prime})_{k=1,\cdots,N})$. We obtain

$$
\begin{aligned}
d\widehat{X}_t &= \big[-(1-\lambda_0)\widehat{Y}_t - \lambda_0 < \widetilde{\widehat{Y}}_{[t]} + q\widehat{X}_{[t]}, \theta > + \kappa\big[\widehat{\check{Y}}_t - < \widetilde{\widehat{\check{Y}}}_{[t]} + q\widehat{\check{X}}_{[t]}, \theta >\big]\big]dt \\
&\quad + \sum_{k=1}^{N} [-(1-\lambda_0)\widehat{Z}_t^k + \kappa\widehat{\check{Z}}_t^k] dW_t^k \\
d\widehat{Y}_t &= \Big[-(1-\lambda_0)\widehat{X}_t + \lambda_0\big(1 - \frac{1}{N}\big)\big[q\widetilde{\widehat{Y}}_t + (q^2 - \epsilon)\widehat{X}_t\big] \\
&\qquad + \kappa\big[\widehat{\check{X}}_t + \big(1 - \frac{1}{N}\big)\big[q\widetilde{\widehat{\check{Y}}}_t + (q^2 - \epsilon)\widehat{\check{X}}_t\big]\big]\Big]dt + \sum_{k=1}^{N} \widehat{Z}_t^k dW_t^k, \qquad (61)
\end{aligned}
$$

with initial condition $\widehat{X}_0 = 0$ and terminal conditions $\widehat{Y}_T = (1-\lambda_0)\widehat{X}_T + \lambda_0 c\big(1 - \frac{1}{N}\big)\widehat{X}_T - \kappa\widehat{\check{X}}_T + \kappa c(1 - \frac{1}{N})\widehat{\check{X}}_T$ and $\widehat{Y}_t = 0$ for $t \in (T, T+\tau]$ in the case of $c > 0$, and $\widehat{Y}_T = 0$ and $\widehat{Y}_t = 0$ for $t \in (T, T+\tau]$ in the case of $c = 0$. As we stated in the text, we only give the proof in the case $c = 0$ to simplify the notation. The proof of the case $c > 0$ is a easy modification. Using the form of the terminal condition and Itô's formula, we get

$$
\begin{aligned}
0 &= \mathbb{E}[\widehat{Y}_T\widehat{X}_T] \\
&= \mathbb{E}\int_0^T \Bigg\{\widehat{Y}_t\Big[-(1-\lambda_0)\widehat{Y}_t - \lambda_0 < \widetilde{\widehat{Y}}_{[t]} + q\widehat{X}_{[t]}, \theta > + \kappa\big[\widehat{\check{Y}}_t - < \widetilde{\widehat{\check{Y}}}_{[t]} + q\widehat{\check{X}}_{[t]}, \theta >\big]\Big] \\
&\quad + \widehat{X}_t\Big[-(1-\lambda_0)\widehat{X}_t + \lambda_0\big(1 - \frac{1}{N}\big)\big[q\widetilde{\widehat{Y}}_t + (q^2 - \epsilon)\widehat{X}_t\big] \\
&\quad + \kappa\big[\widehat{\check{X}}_t + \big(1 - \frac{1}{N}\big)\big[q\widetilde{\widehat{\check{Y}}}_t + (q^2 - \epsilon)\widehat{\check{X}}_t\big]\big]\Big] - (1-\lambda_0)\sum_{k=1}^{N}|\widehat{Z}_t^k|^2 + \kappa\sum_{k=1}^{N}\widehat{Z}_t^k\widehat{\check{Z}}_t^k\Bigg\}dt \qquad (62) \\
&= -(1-\lambda_0)\mathbb{E}\int_0^T |\widehat{Y}_t|^2 dt - \lambda_0\mathbb{E}\int_0^T \widehat{Y}_t < \widetilde{\widehat{Y}}_{[t]} + q\widehat{X}_{[t]}, \theta > dt \\
&\qquad + \kappa\mathbb{E}\int_0^T \widehat{Y}_t\big[\widehat{\check{Y}}_t - < \widetilde{\widehat{\check{Y}}}_{[t]} + q\widehat{\check{X}}_{[t]}, \theta >\big]dt \\
&\qquad - (1-\lambda_0)\mathbb{E}\int_0^T |\widehat{X}_t|^2 dt + \lambda_0\big(1 - \frac{1}{N}\big)\mathbb{E}\int_0^T \widehat{X}_t\big[q\widetilde{\widehat{Y}}_t + (q^2 - \epsilon)\widehat{X}_t\big]dt \\
&\qquad + \kappa\mathbb{E}\int_0^T \widehat{X}_t\big[\widehat{\check{X}}_t + \big(1 - \frac{1}{N}\big)\big[q\widetilde{\widehat{\check{Y}}}_t + (q^2 - \epsilon)\widehat{\check{X}}_t\big]\big]dt \\
&\qquad - (1-\lambda_0)\mathbb{E}\int_0^T \sum_{k=1}^{N}|\widehat{Z}_t^k|^2 dt + \kappa\sum_{k=1}^{N}\widehat{Z}_t^k\widehat{\check{Z}}_t^k dt \qquad (63)
\end{aligned}
$$

and rearranging the terms we find:

$$
\begin{aligned}
&(1-\lambda_0)\big[\mathbb{E}\int_0^T |\widehat{X}_t|^2dt + \mathbb{E}\int_0^T |\widehat{Y}_t|^2dt + \mathbb{E}\int_0^T \sum_{k=1}^N |\widehat{Z}_t^k|^2\ dt\big] \\
&= \kappa\mathbb{E}\int_0^T \widehat{X}_t\widehat{\tilde{X}}_t dt - \lambda_0\mathbb{E}\int_0^T \widehat{Y}_t < \widehat{\tilde{Y}}_{[t]} + q\widehat{X}_{[t]}, \theta > dt \\
&\quad +\kappa\mathbb{E}\int_0^T \widehat{Y}_t\big[\widehat{\tilde{Y}}_t - < \widehat{\tilde{\tilde{Y}}}_{[t]} + q\widehat{\tilde{X}}_{[t]}, \theta >\big]dt + \lambda_0\big(1-\frac{1}{N}\big)\mathbb{E}\int_0^T \widehat{X}_t\big[q\widehat{\tilde{\tilde{Y}}}_t + (q^2-\epsilon)\widehat{X}_t\big]dt \\
&\quad +\kappa\big(1-\frac{1}{N}\big)\mathbb{E}\int_0^T \widehat{X}_t\big[q\widehat{\tilde{\tilde{Y}}}_t + (q^2-\epsilon)\widehat{\tilde{X}}_t\big]\big]dt + \kappa\mathbb{E}\int_0^T \sum_{k=1}^N \widehat{Z}_t^k\widehat{\tilde{Z}}_t^k\ dt
\end{aligned}
$$

Letting $\mu = \epsilon(1-\frac{1}{N}) - q^2(1-\frac{1}{2N})^2 > 0$, we obtain:

$$
\begin{aligned}
&(1-\lambda_0+\lambda_0\mu)\mathbb{E}\int_0^T |\widehat{X}_t|^2dt + (1-\lambda_0)\mathbb{E}\int_0^T |\widehat{Y}_t|^2dt + (1-\lambda_0)\mathbb{E}\int_0^T \sum_{k=1}^N |\widehat{Z}_t^k|^2dt \\
&\leq \kappa\mathbb{E}\int_0^T \widehat{Y}_t\big[\widehat{\tilde{Y}}_t - < \widehat{\tilde{\tilde{Y}}}_{[t]} + q\widehat{\tilde{X}}_{[t]}, \theta >\big]dt \\
&\quad +\kappa\big(1-\frac{1}{N}\big)\mathbb{E}\int_0^T \Big(\big(q^2-\epsilon\big)\,\widehat{\tilde{X}}_t + q\widehat{\tilde{\tilde{Y}}}_t\big)\widehat{X}_t dt + \kappa\mathbb{E}\int_0^T \sum_{k=1}^N \widehat{Z}_t^k\widehat{\tilde{Z}}_t^k dt,
\end{aligned}
$$

and a straightforward computation using repeatedly Cauchy–Schwarz and Jensen's inequalities leads to the existence of a positive constant K_1 such that

$$
\begin{aligned}
&(1-\lambda_0+\lambda_0\mu)\mathbb{E}\int_0^T |\widehat{X}_t|^2dt + (1-\lambda_0)\mathbb{E}\int_0^T |\widehat{Y}_t|^2dt + (1-\lambda_0)\mathbb{E}\int_0^T \sum_{k=1}^N |\widehat{Z}_t^k|^2dt \\
&\quad \leq \kappa K_1\Big\{\mathbb{E}\int_0^T |\widehat{X}_t|^2dt + \mathbb{E}\int_0^T |\widehat{Y}_t|^2dt + \mathbb{E}\int_0^T \sum_{k=1}^N |\widehat{Z}_t^k|^2dt \\
&\qquad +\mathbb{E}\int_0^T |\widehat{\tilde{X}}_t|^2dt + \mathbb{E}\int_0^T |\widehat{\tilde{Y}}_t|^2dt + \mathbb{E}\int_0^T \sum_{k=1}^N |\widehat{\tilde{Z}}_t^k|^2dt\Big\}.
\end{aligned}
$$

Referring to [27], applying Itô's formula to $|\widehat{X}_t|^2$ and $|\widehat{Y}_t|^2$, Gronwall's inequality, and again Cauchy-Schwarz and Jensen's inequalities, owing to $0 \leq \lambda_0 \leq 1$, we obtain a constant $K_2 > 0$ independent of λ_0 so that

$$
\sup_{0\leq t\leq T} \mathbb{E}|\widehat{X}_t|^2 \leq \kappa K_2\left\{\mathbb{E}\int_0^T |\widehat{\tilde{X}}_t|^2 + |\widehat{\tilde{Y}}_t|^2 + \sum_{k=1}^N |\widehat{\tilde{Z}}_t^k|^2dt\right\} + K_2\left\{\mathbb{E}\int_0^T |\widehat{Y}_t|^2 + \sum_{k=1}^N |\widehat{Z}_t^k|^2dt\right\},
$$

$$
\begin{aligned}
\mathbb{E}\int_0^T |\widehat{X}_t|^2dt \leq \kappa K_2 T&\left\{\mathbb{E}\int_0^T |\widehat{\tilde{X}}_t|^2 + |\widehat{\tilde{Y}}_t|^2 + \sum_{k=1}^N |\widehat{\tilde{Z}}_t^k|^2dt\right\} \\
&+K_2T\left\{\mathbb{E}\int_0^T |\widehat{Y}_t|^2 + \sum_{k=1}^N |\widehat{Z}_t^k|^2dt\right\},
\end{aligned}
$$

$$\mathbb{E}\int_0^T |\widehat{Y}_t|^2 + \sum_{k=1}^N |\widehat{Z}_t^k|^2 dt \le \kappa K_2 \left\{ \mathbb{E}\int_0^T |\hat{\widehat{X}}_t|^2 + |\hat{\widehat{Y}}_t|^2 + \sum_{k=1}^N |\hat{\widehat{Z}}_t^k|^2 dt \right\}$$
$$+ K_2 \mathbb{E}\int_0^T |\widehat{X}_t|^2 dt. \tag{64}$$

By using (64), there exists $0 < \mu' < \mu/K_2$ such that

$$\begin{aligned}
&\lambda_0 \mu' K_2 \mathbb{E}\int_0^T |\widehat{X}_t|^2 dt \\
&\ge \lambda_0 \mu' \left(\mathbb{E}\int_0^T |\widehat{Y}_t|^2 + \sum_{k=1}^N |\widehat{Z}_t^k|^2 dt \right) - \lambda_0 \mu' \kappa K_2 \left\{ \mathbb{E}\int_0^T |\hat{\widehat{X}}_t|^2 + |\hat{\widehat{Y}}_t|^2 + \sum_{k=1}^N |\hat{\widehat{Z}}_t^k|^2 dt \right\} \\
&\ge \lambda_0 \mu' \left(\mathbb{E}\int_0^T |\widehat{Y}_t|^2 + \sum_{k=1}^N |\widehat{Z}_t^k|^2 dt \right) - \mu' \kappa K_2 \left\{ \mathbb{E}\int_0^T |\hat{\widehat{X}}_t|^2 + |\hat{\widehat{Y}}_t|^2 + \sum_{k=1}^N |\hat{\widehat{Z}}_t^k|^2 dt \right\}
\end{aligned} \tag{65}$$

Therefore, we have

$$\begin{aligned}
&\Big(1 - \lambda_0 + \lambda_0(\mu - K_2\mu')\Big) \mathbb{E}\int_0^T |\widehat{X}_t|^2 dt \\
&+ (1 - \lambda_0 + \lambda_0\mu') \mathbb{E}\int_0^T |\widehat{Y}_t|^2 dt + (1 - \lambda_0 + \lambda_0\mu') \mathbb{E}\int_0^T \sum_{k=1}^N |\widehat{Z}_t^k|^2 dt \\
&\le \kappa K_1 \Big\{ \mathbb{E}\int_0^T |\widehat{X}_t|^2 dt + \mathbb{E}\int_0^T |\widehat{Y}_t|^2 dt + \mathbb{E}\int_0^T \sum_{k=1}^N |\widehat{Z}_t^k|^2 dt \\
&\quad + \mathbb{E}\int_0^T |\hat{\widehat{X}}_t|^2 dt + \mathbb{E}\int_0^T |\hat{\widehat{Y}}_t|^2 dt + \mathbb{E}\int_0^T \sum_{k=1}^N |\hat{\widehat{Z}}_t^k|^2 \Big\} \\
&+ \kappa K_2 \mu' \left\{ \mathbb{E}\int_0^T |\hat{\widehat{X}}_t|^2 dt + \mathbb{E}\int_0^T |\hat{\widehat{Y}}_t|^2 dt + \mathbb{E}\int_0^T \sum_{k=1}^N |\hat{\widehat{Z}}_t^k|^2 dt \right\}.
\end{aligned} \tag{66}$$

Note that since $\mu - K_2\mu'$ and μ' stay in positive, we have $(1 - \lambda_0 + \lambda_0(\mu - K_2\mu')) \ge \mu''$ and $(1 - \lambda_0 + \lambda_0\mu') \ge \mu''$ where for some $\mu'' > 0$. Combining the inequalities (64-66), we obtain

$$\begin{aligned}
&\mathbb{E}\int_0^T |\widehat{X}_t|^2 dt + \mathbb{E}\int_0^T |\widehat{Y}_t|^2 dt + \mathbb{E}\int_0^T \sum_{k=1}^N |\widehat{Z}_t^k|^2 dt \\
&\qquad \le \kappa K \left(\mathbb{E}\int_0^T |\hat{\widehat{X}}_t|^2 dt + \mathbb{E}\int_0^T |\hat{\widehat{Y}}_t|^2 dt + \mathbb{E}\int_0^T \sum_{k=1}^N |\hat{\widehat{Z}}_t^k|^2 dt \right),
\end{aligned} \tag{67}$$

where the constant K depends upon μ', μ'', K_1, K_2, and T. Hence, Φ is a strict contraction for sufficiently small κ. ◻

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