

# Calibration of Stock Betas from Skews of Implied Volatilities

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## Abstract

We develop call option price approximations for both the market index and an individual asset using a singular perturbation of a continuous time Capital Asset Pricing Model (CAPM) in a stochastic volatility environment. These approximations show the role played by the asset's beta parameter as a component of the parameters of the call option price of the asset. They also show how these parameters, in combination with the parameters of the call option price for the market, can be used to extract the beta parameter. Finally, a calibration technique for the beta parameter is derived using the estimated option price parameters of both the asset and market index. The resulting estimator of the beta parameter is not only simple to implement but has the advantage of being *forward-looking* as it is calibrated from skews of implied volatilities.

## 1 Introduction

The concept of stock betas was developed in the context of the Capital Asset Pricing Model of Sharpe [13] and was based on previous portfolio theory in

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Markowitz [11]. The beta of a stock represents the scale of the risk of the asset relative to the systematic risk of the market and is critical in the development and performance of stock portfolios. This paper examines the role of a stocks beta parameter in option prices on the stock in the presence of stochastic volatility and develops a calibration technique for the beta parameter using the option prices, or equivalently, implied volatilities.

The estimation of the beta parameter is an important issue in financial practices that deal with CAPM models and is used, for amongst other things, portfolio construction and performance measurement. The original discrete time CAPM model defined the log price return on individual asset  $R_a$  as a linear function of the risk free interest rate  $R_f$ , the log return of the market  $R_M$ , and a Gaussian error term  $\epsilon_a$ :

$$R_a = R_f + \beta_a(R_M - R_f) + \epsilon_a. \quad (1)$$

The beta coefficient was originally estimated using historical returns on the asset and market index. The classic approach used a simple linear regression of asset returns on market returns as implied by (1). This regression approach leads to a simple estimation of beta as the ratio of the covariance of historical market and asset returns to the variance of historical market returns. Other approaches have accounted for the fact that the beta parameter may not be constant in time. To this end, Scholes and Williams [12] provided an approach to estimating the beta using historical non-synchronous data. However a fundamental flaw in estimating the beta parameter using historical data is that it is inherently backward looking, which can be a major drawback for the use of betas in forward looking portfolio construction. As such, many studies on beta estimation such as French, Goth, and Kolari [3] and Siegel [14], and more recently Christoffersen, Jacobs, and Vainberg [2] have attempted to extract the parameter from option prices on the underlying market and asset processes. In fact, in [2] the authors derive the following formula

$$\beta_a = \left( \frac{SKEW_a}{SKEW_M} \right)^{\frac{1}{3}} \left( \frac{VAR_a}{VAR_M} \right)^{\frac{1}{2}}, \quad (2)$$

where  $VAR_a$  (resp.  $VAR_M$ ), and  $SKEW_a$  (resp.  $SKEW_M$ ) are the variance, and the risk-neutral skewness of returns of the asset (resp. of the market). Then, they use results from Carr and Madan [1] which relate these moments to options prices (*Quad* and *Cubic*) on the asset (resp. on the market).

The advantage of this approach is that option prices are inherently forward looking on the underlying price process.

Our main result, formula (33), can be viewed as a simplified version of (2) allowing for a direct calibration to the skews of implied volatilities.

In the first part of this paper we explore the effects of the introduction of stochastic volatility on a continuous time CAPM model. The model we propose is similar in spirit to the original CAPM model in (1) but in continuous time and with a more realistic error process driven by stochastic volatility. Stochastic volatility is a well established empirical characteristic of option prices, see for instance [9, 4]. Stochastic volatility models are able to explain the “smile/skew” of option prices. In addition, it has been shown directly that a fast mean reverting stochastic volatility process is also directly observable in stock prices themselves [10, 4]. It follows then that any return models, including CAPM type models, should account for a stochastic volatility factor. We will show that in the presence of fast mean reverting stochastic volatility in stock price movements, the beta parameter of an asset flows through to the parameters of option price approximations similar to the one presented in [5]. These price approximations are based on a singular perturbation expansion with respect to the rate of mean reversion of the stochastic volatility process.

In the second part of the paper we show how the presence of the beta parameter in the option price approximations allows us to calibrate this beta parameter using those option prices. The estimator we develop is simple and easy to implement using a spread of call option prices on the asset of interest and the market.

The organization of the paper is as follows. The continuous-time CAPM model with an underlying stochastic volatility is presented in Section 2. In Section 3 we develop an asymptotic price approximation for call option prices on both an individual asset and the market, and we show the role the beta parameter plays in these prices. A further parameter reduction is given in the Appendix. A technique for the calibration of the beta parameter using the options prices is developed in Section 4. Empirical results are provided in Section 5.

## 2 Introduction of Stochastic Volatility in a Continuous Time CAPM Model

In order to develop our beta estimation technique it is convenient to consider a continuous time CAPM model. Moving to continuous time is more accurate for modeling, but with the possible drawback of increasing the complexity of model characteristics. To start, a simple continuous time CAPM model for market price  $M_t$  and an asset price  $X_t$  would evolve as follows:

$$\frac{dM_t}{M_t} = \mu_m dt + \sigma_m dW_t^{(1)}, \quad (3)$$

$$\frac{dX_t}{X_t} = r dt + \beta \left( \frac{dM_t}{M_t} - r dt \right) + \sigma dW_t^{(2)}, \quad (4)$$

where  $\mu_m$  is the rate of return of the market,  $\sigma_m$  and  $\sigma$  are positive constant volatilities, and  $W^{(1)}$ ,  $W^{(2)}$  are independent Brownian motions

$$d\langle W^{(1)}, W^{(2)} \rangle_t = 0.$$

The model (3,4) is consistent with (1) in that the excess return of the asset  $\frac{dX_t}{X_t} - r dt$  is an affine function of the excess return of the market  $\frac{dM_t}{M_t} - r dt$  through the  $\beta$  coefficient and a Brownian driven noise process. Most importantly, the process preserves the definition of the  $\beta$  coefficient as the covariance of the asset and market returns divided by the market variance, that is formally:

$$\begin{aligned} \frac{\langle \frac{dX_t}{X_t}, \frac{dM_t}{M_t} \rangle}{\langle \frac{dM_t}{M_t}, \frac{dM_t}{M_t} \rangle} &= \frac{\langle r dt + \beta \left( \frac{dM_t}{M_t} - r dt \right) + \sigma dW_t^{(2)}, \frac{dM_t}{M_t} \rangle}{\langle \frac{dM_t}{M_t}, \frac{dM_t}{M_t} \rangle} \\ &= \frac{\langle \beta \frac{dM_t}{M_t}, \frac{dM_t}{M_t} \rangle}{\langle \frac{dM_t}{M_t}, \frac{dM_t}{M_t} \rangle} = \beta, \end{aligned} \quad (5)$$

where the second equality holds due to the independence of  $M_t$  and  $W_t^{(2)}$ . Observe that the evolution of  $X_t$  is given by

$$\frac{dX_t}{X_t} = (r + \beta(\mu_m - r)) dt + \beta\sigma_m dW_t^{(1)} + \sigma dW_t^{(2)},$$

that is a geometric Brownian motion with volatility  $\sqrt{\beta^2\sigma_m^2 + \sigma^2}$ . Even if this quantity is known, along with the volatility  $\sigma_m$  of the market process, one cannot disentangle  $\beta$  and  $\sigma$ .

Moreover, the assumption of constant volatility in asset markets has been shown to be inconsistent with both price and option data. As such, following [4], we introduce a stochastic volatility component to the market price process, that is we replace in (3)  $\sigma_m$  by a stochastic process  $\sigma_t = f(Y_t)$ :

$$\frac{dM_t}{M_t} = \mu_m dt + f(Y_t) dW_t^{(1)}, \quad (6)$$

$$\frac{dX_t}{X_t} = r dt + \beta \left( \frac{dM_t}{M_t} - r dt \right) + \sigma dW_t^{(2)}, \quad (7)$$

$$dY_t = \frac{1}{\epsilon} (m - Y_t) dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} dZ_t. \quad (8)$$

In this model, the volatility process is driven by a mean-reverting Ornstein-Uhlenbeck process  $Y_t$  with a large mean-reversion rate  $1/\epsilon$  and the invariant (long-run) distribution  $\mathcal{N}(m, \nu^2)$ . The model is completely flexible in the function form of  $f$  which defines the way in which the volatility level acts on the market price process. This model also implies stochastic volatility in the asset price through its dependence on the market return, and assumes correlation between the Brownian motions driving the market returns and volatility process:

$$d\langle W^{(1)}, Z \rangle_t = \rho dt.$$

In fact, in what follows we will assume that  $\rho \neq 0$ . However, we continue to assume independence between  $W_t^{(2)}$  and the other two Brownian motions  $W_t^{(1)}$  and  $Z_t$  in order to preserve the interpretation of  $\beta$  in (5). We could suppose that the volatility  $\sigma$  is also fluctuating with  $Y_t$  so that  $\sigma = \sigma(Y_t)$ . The results presented below would remain essentially the same. The introduction of another independent fast mean-reverting volatility factor is more involved but can be treated as well. Note that, in this paper, we do not try to work with the most elaborate model, but rather to present the main idea, that is calibration of  $\beta$ , in the context of the simplest model which allows it. In particular, the asymptotic analysis that we will use does not depend crucially on the particular choice of an OU process for  $Y$ , but more on its ergodic properties. Another choice could be a CIR (Cox-Ingersoll-Ross) process often used in financial modeling combined with a function  $f$  of the form  $f(y) = \sqrt{y}$ . In that case,  $(M, Y)$  would simply be a Heston model. More importantly, we are assuming  $\epsilon$  to be small, that is the volatility is fast mean-reverting. Evidence of the presence of such a fast time-scale has

been given, for instance in [4] and [10]. In fact, there are slower volatility time-scales which we do not account for here in order to keep the presentation simple. A Generalization of our result to the case of fast and slow time-scales can easily be derived following [7].

### 3 Option Price Approximation in a Continuous-Time CAPM Model

In this section we generalize the approach presented in [4] to the case of the CAPM model with stochastic volatility introduced in the previous section. We show how to calculate an approximation of option prices up to an error of order  $\epsilon$  which is the inverse of the rate of mean reversion of the stochastic volatility driving the market returns. The approximation of the option prices results from a singular perturbation with respect to the mean reversion time scale  $\epsilon$  of the fast mean-reverting volatility process. As we will show later it has the advantage of using a parsimonious set of parameters which will allow us to estimate the beta parameter using only call options on the market  $M$  and on the asset  $X$ .

The first step is to rewrite the dynamics of the market and of the asset under a pricing risk-neutral measure.

#### 3.1 Pricing Risk-Neutral Measure

The market (or index) and the asset being both tradable, their discounted prices need to be martingales under a pricing risk-neutral measure. In order to achieve that, we first write

$$Z_t = \rho dW_t^{(1)} + \sqrt{1 - \rho^2} dW_t^{(3)},$$

with now  $(W_t^{(1)}, W_t^{(2)}, W_t^{(3)})$  being three independent standard Brownian motions, and then we rewrite the system (6, 7, 8) as:

$$\begin{aligned} \frac{dM_t}{M_t} &= rdt + f(Y_t) \left( dW_t^{(1)} + \frac{\mu_m - r}{f(Y_t)} dt \right), \\ \frac{dX_t}{X_t} &= rdt + \beta f(Y_t) \left( dW_t^{(1)} + \frac{\mu_m - r}{f(Y_t)} dt \right) + \sigma dW_t^{(2)}, \end{aligned}$$

$$\begin{aligned}
dY_t &= \frac{1}{\epsilon}(m - Y_t)dt - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} \Lambda(Y_t)dt \\
&\quad + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} \left[ \rho \left( dW_t^{(1)} + \frac{\mu_m - r}{f(Y_t)} dt \right) + \sqrt{1 - \rho^2} \left( dW_t^{(3)} + \gamma(Y_t)dt \right) \right],
\end{aligned}$$

where  $\gamma(Y_t)$  is a market price of volatility risk, which we suppose to depend on  $Y_t$  only, and we defined

$$\Lambda(Y_t) = \rho \frac{\mu_m - r}{f(Y_t)} + \sqrt{1 - \rho^2} \gamma(Y_t).$$

Setting

$$\begin{aligned}
dW_t^{(1)*} &= dW_t^{(1)} + \frac{\mu_m - r}{f(Y_t)} dt, \\
dW_t^{(2)*} &= dW_t^{(2)}, \\
dW_t^{(3)*} &= dW_t^{(3)} + \gamma(Y_t)dt,
\end{aligned}$$

by Girsanov's theorem, there is an equivalent probability  $\mathbb{P}^{*(\gamma)}$  such that  $(W_t^{(1)*}, W_t^{(2)*}, W_t^{(3)*})$  are independent standard Brownian motions under  $\mathbb{P}^{*(\gamma)}$ , called the pricing equivalent martingale measure and determined by the market price of volatility risk  $\gamma$ . We assume here that the Sharpe ratio  $\frac{\mu_m - r}{f(Y_t)}$  and the volatility premium  $\gamma(Y_t)$  are bounded, which, depending on the choice of function  $f$ , may require that  $\mu_m$  depends on  $Y_t$ . Finally, under  $\mathbb{P}^{*(\gamma)}$ , the dynamics (6, 7, 8) become:

$$\frac{dM_t}{M_t} = rdt + f(Y_t)dW_t^{(1)*}, \quad (9)$$

$$\frac{dX_t}{X_t} = rdt + \beta f(Y_t)dW_t^{(1)*} + \sigma dW_t^{(2)*}, \quad (10)$$

$$dY_t = \frac{1}{\epsilon}(m - Y_t)dt - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} \Lambda(Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} dZ_t^*, \quad (11)$$

$$Z_t^* = \rho W_t^{(1)*} + \sqrt{1 - \rho^2} W_t^{(3)*}.$$

In what follows, we take the point of view that by pricing options on the index  $M$  and on the particular asset  $X$ , the market is “completing” itself and indirectly choosing the market price of volatility risk  $\gamma$ .

### 3.2 Market Option Prices

In looking first at option prices on the market index we will only focus on the autonomous evolution of  $(M_t, Y_t)$  described by equations (9,11) under the risk-neutral pricing measure. A singular perturbation approach to option pricing on the model described in (9,11) was developed in [4]. Here we use this approximation technique but with an additional parameter reduction to allow for the estimation of our beta parameter using option data only (see also [8]). The details of this derivation can be found in Appendix 7, and lead to the following price approximation for call option prices on the market. Let  $P^{M,\epsilon} = P(t, \xi; T, K)$  denote the price of a European call option written on the market index  $M$ , with maturity  $T$  and strike  $K$ , evaluated at time  $t < T$  with current value  $M_t = \xi$ , where we explicitly show the dependence on the small volatility mean-reversion time  $\epsilon$ . Then, we have the following approximation which depends on a parameter  $V_3^{M,\epsilon}$  described below

$$P^{M,\epsilon} \sim P^{M*} + (T - t)V_3^{M,\epsilon}\xi \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial^2 P^{M*}}{\partial \xi^2} \right), \quad (12)$$

where  $P^{M*}$  is the corresponding Black Scholes call price with constant volatility equal to the *adjusted effective volatility*  $\sigma^{M*}$ :

$$P^{M*} = P_{BS}(\sigma^{M*}). \quad (13)$$

Here

$$\sigma^{M*} = \sqrt{\bar{\sigma}^2 + 2V_2^{M,\epsilon}}, \quad (14)$$

where  $\bar{\sigma}$  is the *effective volatility* defined by

$$\bar{\sigma}^2 = \langle f^2 \rangle \equiv \int f(y)^2 \frac{1}{\sqrt{2\pi\nu}} e^{-\frac{(y-m)^2}{2\nu^2}} dy, \quad (15)$$

with the average being taken with respect to the invariant distribution of the OU process  $Y$ . The small parameter  $V_2^{M,\epsilon}$ , proportional to  $\sqrt{\epsilon}$  and defined by (35), accounts for a volatility adjustment due to the market price of volatility risk. The small parameter  $V_3^{M,\epsilon}$  appearing in (12), is defined by (36). It is proportional to  $\sqrt{\epsilon}$  and to the correlation coefficient  $\rho$ , and accounts for the skew of implied volatility. It is shown in [6] that the accuracy of the approximation (12) is  $\mathcal{O}(\epsilon \log |\epsilon|)$ .



### 3.3 Asset Option Prices

We now proceed analogously in showing an approximation for an option price written on the asset, the evolution of which under the risk neutral measure is described by (10,11). Note that from (10), the effective volatility of the asset denoted by  $\bar{\sigma}_a$  is given by

$$\bar{\sigma}_a^2 = \langle \beta^2 f^2 + \sigma^2 \rangle = \beta^2 \bar{\sigma}^2 + \sigma^2. \quad (16)$$

The details of the derivation of the following approximation of a call option price on the asset can be found in Appendix 7. Let  $P^{a,\epsilon} = P(t, x; T, K)$  denote the price of a European call option written on the asset  $X$ , with maturity  $T$  and strike  $K$ , evaluated at time  $t < T$  with current value  $X_t = x$ , where we explicitly show the dependence on the small volatility mean-reversion time  $\epsilon$ . Then, we have the following approximation

$$P^{a,\epsilon} \sim P^{a*} + (T - t)V_3^{a,\epsilon} x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P^{a*}}{\partial x^2} \right), \quad (17)$$

where  $P^{a*}$  is the corresponding Black Scholes call price with constant volatility equal to the *adjusted effective volatility*  $\sigma^{a*}$ :

$$P^{a*} = P_{BS}(\sigma^{a*}). \quad (18)$$

Here

$$\sigma^{a*} = \sqrt{\bar{\sigma}_a^2 + 2V_2^{a,\epsilon}}, \quad (19)$$

where  $\bar{\sigma}_a$  is the *effective volatility* given by (16), and the small parameter  $V_2^{a,\epsilon}$ , proportional to  $\sqrt{\epsilon}$  and defined by (44), accounts for a volatility adjustment due to the market price of volatility risk. The small parameter  $V_3^{a,\epsilon}$  appearing in (17), is defined by (45). It is proportional to  $\sqrt{\epsilon}$  and to the correlation coefficient  $\rho$ , and accounts for the skew of implied volatility. As in the case of market option prices, the accuracy of the approximation (17) is  $\mathcal{O}(\epsilon \log |\epsilon|)$ .

### 3.4 Beta Estimation

From the expressions for  $V_3^{M,\epsilon}$  and  $V_3^{a,\epsilon}$  given respectively in (36) and (45), one deduces that for  $\rho \neq 0$ ,  $V_3^{M,\epsilon} \neq 0$ ,  $V_3^{a,\epsilon} \neq 0$ , and

$$V_3^{a,\epsilon} = \beta^3 V_3^{M,\epsilon}.$$

It is then natural to propose the following estimator for  $\beta$ :

$$\beta = \left( \frac{V_3^{a,\epsilon}}{V_3^{M,\epsilon}} \right)^{\frac{1}{3}}. \quad (20)$$

Therefore in order to estimate the market beta parameter in a forward looking fashion using the implied skew parameters from option prices we must calibrate our two parameters  $V_3^{a,\epsilon}$  and  $V_3^{M,\epsilon}$ . In the next section we will show how to calibrate these two parameters and therefore the beta parameter using the implied volatility surfaces from options data.

## 4 Calibration of Option Price Parameters

In this section we will show how to calibrate our two option price approximation parameters  $V_3^\epsilon$  and  $\sigma^*$ . While the true value of these parameters differ between market price options and asset price options, the calibration approach does not, and hence we proceed in general terms with the calibration approach. In the end we will re-express everything with respect to their specific price series. We follow [4] or [5], and we show that, in fact, there is no need to estimate  $\bar{\sigma}$ , making our procedure fully *forward looking*.

### 4.1 General Calibration Approach

We have shown in the previous section that a first order approximation of an option price with time to maturity  $\tau = T - t$ , and in the presence of fast mean-reverting stochastic volatility, takes the following form:

$$P^\epsilon \sim P_{BS}^* + \tau V_3^\epsilon x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_{BS}^*}{\partial x^2} \right), \quad (21)$$

where  $P_{BS}^*$  is the Black-Scholes option price with volatility  $\sigma^*$  which was defined in (14) for options on the market index price and defined in (19) for options on the individual asset, and where  $V_3^\epsilon$  was defined in (36) for options on market and in (45) for options on the asset.

The European call option price  $P_{BS}^*$  with current price  $x$ , time to maturity  $\tau$ , and strike price  $K$  is given by the Black-Scholes formula

$$P_{BS}^* = xN(d_1^*) - Ke^{-r\tau}N(d_2^*), \quad (22)$$

where  $N$  is the cumulative standard normal distribution and

$$d_{1,2}^* = \frac{\log(x/K) + (r \pm \frac{1}{2}\sigma^{*2})\tau}{\sigma^* \sqrt{\tau}}. \quad (23)$$

Before proceeding, we recall the following relationship between European call option *Vega* and *Gamma*:

$$\frac{\partial P_{BS}^*}{\partial \sigma} = \tau \sigma^* x^2 \frac{\partial^2 P_{BS}^*}{\partial x^2}, \quad (24)$$

and we rewrite our price approximation in (21) as

$$P^\epsilon \sim P_{BS}^* + \frac{V_3^\epsilon}{\sigma^*} x \frac{\partial}{\partial x} \left( \frac{\partial P_{BS}^*}{\partial \sigma} \right). \quad (25)$$

Using the definition of the implied volatility  $P_{BS}(I) = P^\epsilon$ , and expanding the implied volatility as

$$I = \sigma^* + \sqrt{\epsilon} I_1 + \epsilon I_2 + \dots, \quad (26)$$

we obtain:

$$P_{BS}(\sigma^*) + \sqrt{\epsilon} I_1 \frac{\partial P_{BS}(\sigma^*)}{\partial \sigma} + \dots = P_{BS}^* + \frac{V_3^\epsilon}{\sigma^*} x \frac{\partial}{\partial x} \left( \frac{\partial P_{BS}^*}{\partial \sigma} \right) + \dots$$

By definition  $P_{BS}(\sigma^*) = P_{BS}^*$ , so that

$$\sqrt{\epsilon} I_1 = \frac{V_3^\epsilon}{\sigma^*} \left( \frac{\partial P_{BS}^*}{\partial \sigma} \right)^{-1} x \frac{\partial}{\partial x} \left( \frac{\partial P_{BS}^*}{\partial \sigma} \right). \quad (27)$$

Using the explicit computation of the *Vega*

$$\frac{\partial P_{BS}^*}{\partial \sigma} = \frac{x \sqrt{\tau} e^{-d_1^{*2}/2}}{\sqrt{2\pi}},$$

and consequently

$$x \frac{\partial}{\partial x} \left( \frac{\partial P_{BS}^*}{\partial \sigma} \right) = \left( 1 - \frac{d_1^*}{\sigma^* \sqrt{\tau}} \right) \frac{\partial P_{BS}^*}{\partial \sigma},$$

we deduce by using the definition (23) of  $d_1^*$ :

$$\sqrt{\epsilon} I_1 = \frac{V_3^\epsilon}{\sigma^*} \left( 1 - \frac{d_1^*}{\sigma^* \sqrt{\tau}} \right) = \frac{V_3^\epsilon}{2\sigma^*} \left( 1 - \frac{2r}{\sigma^{*2}} \right) + \frac{V_3^\epsilon}{\sigma^{*3}} \frac{\log(K/x)}{\tau}.$$

Finally, introducing the *Log-Moneyness to Maturity Ratio* (*LMMR*)

$$LMMR = \frac{\log(K/x)}{\tau},$$

we obtain from (26) the *affine LMMR formula*

$$I \sim b^* + a^\epsilon LMMR, \quad (28)$$

with the intercept  $b^*$  and the slope  $a^\epsilon$  to be fitted to the skew of options data, and related to our model parameters  $\sigma^*$  and  $V_3^\epsilon$  by:

$$b^* = \sigma^* + \frac{V_3^\epsilon}{2\sigma^{*2}} \left(1 - \frac{2r}{\sigma^{*2}}\right), \quad (29)$$

$$a^\epsilon = \frac{V_3^\epsilon}{\sigma^{*3}}. \quad (30)$$

From (20) we know that in order to estimate  $\beta$  we need  $V_3^\epsilon$ . In other words, we need to invert (29, 30) for  $V_3^\epsilon$ . From (29), we know that  $b^*$  and  $\sigma^*$  differ from a quantity of order  $\sqrt{\epsilon}$ . Therefore by replacing  $\sigma^*$  by  $b^*$  in (30), the order of accuracy for  $V_3^\epsilon$  is still  $\epsilon$  since  $a^\epsilon$  is also of order  $\sqrt{\epsilon}$ . Consequently we deduce

$$V_3^\epsilon = a^\epsilon \sigma^{*3} \sim a^\epsilon b^{*3} \equiv \widehat{V}_3^\epsilon. \quad (31)$$

It is indeed also possible to extract  $\sigma^*$  as follows. First, using (30), the relation (29) becomes:

$$b^* = \sigma^* + \frac{a^\epsilon \sigma^{*2}}{2} \left(1 - \frac{2r}{\sigma^{*2}}\right) = \sigma^* - a^\epsilon \left(r - \frac{\sigma^{*2}}{2}\right).$$

Using again the argument that  $b^*$  and  $\sigma^*$  differ by a quantity of order  $\sqrt{\epsilon}$  and  $a^\epsilon$  is also of order  $\sqrt{\epsilon}$ , by replacing  $\sigma^*$  by  $b^*$  in the last term in the relation above, the order of accuracy is still  $\epsilon$ . We then conclude that

$$\sigma^* \sim b^* + a^\epsilon \left(r - \frac{b^{*2}}{2}\right) \equiv \widehat{\sigma}^*. \quad (32)$$

## 4.2 Beta Calibration

The final step to the estimation of our assets beta parameter is to use our general calibration formula (31) on call option prices on both the market

index and asset prices. In order to do this, we index the parameter estimates fitted to market call option prices with  $M$  and parameters fitted to asset call option prices with  $a$ .

Defining the market fitted parameters as  $a^{M,\epsilon}$  and  $b^{M*}$  and the asset parameters as  $a^{a,\epsilon}$  and  $b^{a*}$ , we use the relationship between these parameters and our  $V_3^\epsilon$  parameter and the relation between the beta parameter and the two market and asset  $V_3^\epsilon$  parameters in (20) to establish our final beta parameter estimate:

$$\hat{\beta} = \left( \frac{\widehat{V_3^{a,\epsilon}}}{\widehat{V_3^{M,\epsilon}}} \right)^{1/3} = \left( \frac{a^{a,\epsilon}}{a^{M,\epsilon}} \right)^{1/3} \left( \frac{b^{a*}}{b^{M*}} \right), \quad (33)$$

where  $b^{a*} + a^{a,\epsilon} LMMR$  (resp.  $b^{M*} + a^{M,\epsilon} LMMR$ ) is the linear fit to the skew of implied volatilities for call options on the individual asset (resp. on the market index).

Observe the similarity between formula (2) and our formula (33) where  $a^{a,\epsilon}$ ,  $a^{M,\epsilon}$  are skews, and  $b^{a*}$ ,  $b^{M*}$  are *at-the-money* volatilities.

## 5 Empirical Results

In this section we examine the stability and accuracy of our beta calibration approach developed in (33) on a sample of S&P 500 stocks and the S&P 500 market index. The sample of stock betas calibrated includes Alcoa (AA), Amgen (AMGN), Amazon (AMZN), Allegheny Technologies (ATI), Constellation Energy Group (CEG), General Electric (GE), Google (GOOG), Goldman Sachs (GS), International Business Machines (IBM), Pepsi (PEP), and Exxon Mobil (XOM). Table 1 shows the calibrated beta values for each of the 11 stocks over the course of 10 market days from February 9, 2009 to February 23, 2009 (February 16 is a national holiday). As an example, we present in Figure 1, the implied volatility skews and their affine *LMMR* fits, for the index and AMGN on the particular day of February 18, 2009 (around the money options with *LMMR* values between  $-1$  and  $1$  are used in the fits). It is interesting to note the quality of the fits, and also the fact that the coefficients  $V_3$ 's are small which justify a posteriori the validity of our model with fast mean-reverting stochastic volatility. The result for this particular firm and particular day is bold-faced in Table 1.

For comparison purposes, the beta values in the table are also plotted in Figure 2 along with the stocks beta calibrated using historical log returns.

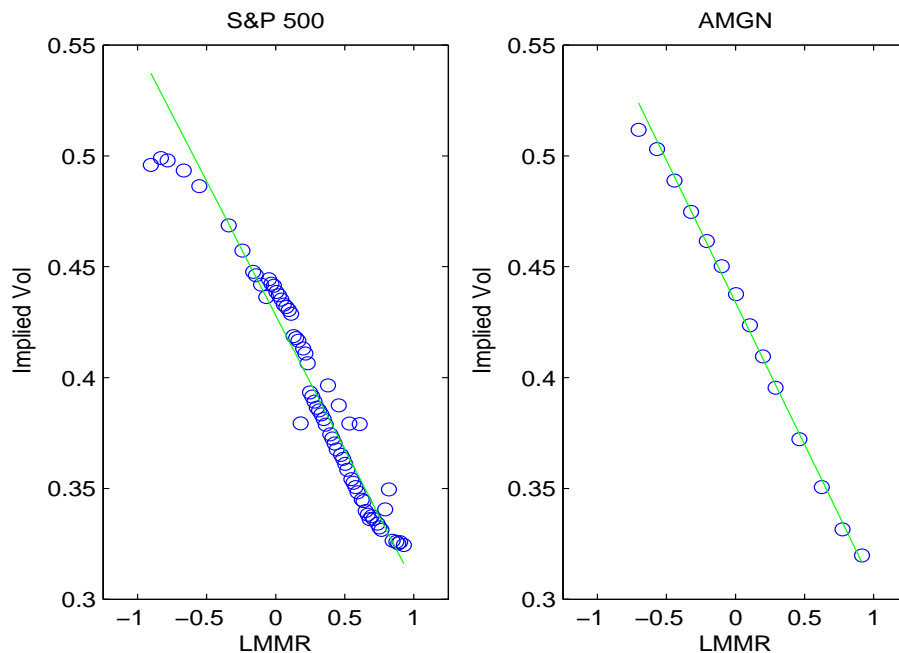


Figure 1: Implied volatility (y-axis) of June/July 2009 maturity options for the S&P 500 and Amgen, plotted against the option's *Log-Moneyness to Maturity Ratio* (LMMR). These are for February 18, 2009 option prices. The line is the affine fit of implied volatilities on *LMMR* by which the  $V_3$  parameter is fit. The parameters fit for each series are  
 S&P 500 Fit:  $a^{M,\epsilon} = -0.121$  and  $b^{M*} = 0.428 \Rightarrow V_3^{M,\epsilon} = -0.0095$   
 Amgen Fit:  $a^{a,\epsilon} = -0.0128$  and  $b^{a*} = 0.434 \Rightarrow V_3^{a,\epsilon} = -0.0105$   
 From (33), the beta estimate for Amgen is 1.03, and this example is bold-faced in Table 1.

The historical betas are derived from the slope coefficient of a simple linear regression of 40 daily log returns of each stock regressed on the log returns of the market, consistent with (1). The beta estimates are also compared to a *blended* beta. The *blended* beta is estimated using parameters from both the forward looking option data and historical prices. Specifically the market

and asset  $V_3$ 's are estimate as

$$V_3^\epsilon = a^\epsilon \bar{\sigma}^3,$$

where  $a^\epsilon$  is estimated from the *affine LMMR formula* of option data and  $\bar{\sigma}$  is estimated from the same historical prices used to estimate the historical beta. This approach is consistent with the approach proposed in [4]. In all cases the beta calibrated on historical data is the most stable. However this estimator is by default very stable over the course of several days, as the estimator for each day uses the same historical log returns as the previous day but with the oldest return in the series replaced by the most recent return. This can in fact be a drawback of the stability of the estimator as rapid changes in a stock's beta will take several days to detect. The beta calibrated in a forward looking fashion on the other hand can adapt to changing market conditions in a single day and is not reliant on previous days options prices. Finally, the *blended* beta moves very closely with the forward looking beta in a majority of cases. This is indicative of the effect of the affine slope on the magnitude of both the *forward looking* and *blended* beta estimates

The forward looking beta for the majority of the stocks fluctuate around their historical beta. In some cases however, such as AMZN and CEG, the forward looking beta is consistently higher than the historical beta indicating a potential shift in the markets expectation for the beta of those stocks going forward.

	2/9/09	2/10/09	2/11/09	2/12/09	2/13/09	2/17/09	<b>2/18/09</b>	2/19/09	2/20/09	2/23/09	2 Week Avg (St Dev)
AA	2.04	1.33	2.50	2.18	1.87	1.44	2.23	2.01	2.40	1.41	1.94 (0.42)
<b>AMGN</b>	1.23	0.72	1.26	1.20	0.91	0.71	<b>1.03</b>	0.98	1.17	0.72	0.99 (0.22)
AMZN	1.87	1.04	1.96	1.80	1.45	1.10	1.65	1.51	1.83	1.13	1.53 (0.34)
ATI	2.22	1.32	2.41	2.31	1.90	1.41	2.08	1.92	2.34	1.40	1.93 (0.42)
CEG	1.10	0.71	1.06	0.93	1.04	0.78	1.00	1.11	1.10	1.12	1.00 (0.14)
GE	1.77	0.89	1.68	1.75	1.45	1.13	1.72	2.13	2.62	1.82	1.70 (0.48)
GOOG	1.43	0.83	1.44	1.33	1.11	0.84	1.22	1.15	1.36	0.87	1.16 (0.24)
GS	2.62	1.51	2.79	2.83	2.06	1.65	2.65	2.28	3.03	1.83	2.32 (0.54)
IBM	1.08	0.72	1.32	1.17	0.99	0.75	1.16	1.10	1.34	0.81	1.04 (0.22)
PEP	0.78	0.44	0.76	0.72	0.56	0.44	0.70	0.65	0.82	0.49	0.64 (0.14)
XOM	0.94	0.57	1.12	1.21	0.77	0.63	1.00	0.88	1.07	0.64	0.88 (0.22)

Table 1: This table contains the betas for a sample of 11 S&P 500 stocks. The betas are calibrated on June/July 2009 expiration call options over the course of 10 market days from February 9, 2009 to February 23, 2009, using the forward looking calibration approach presented in section 4.2



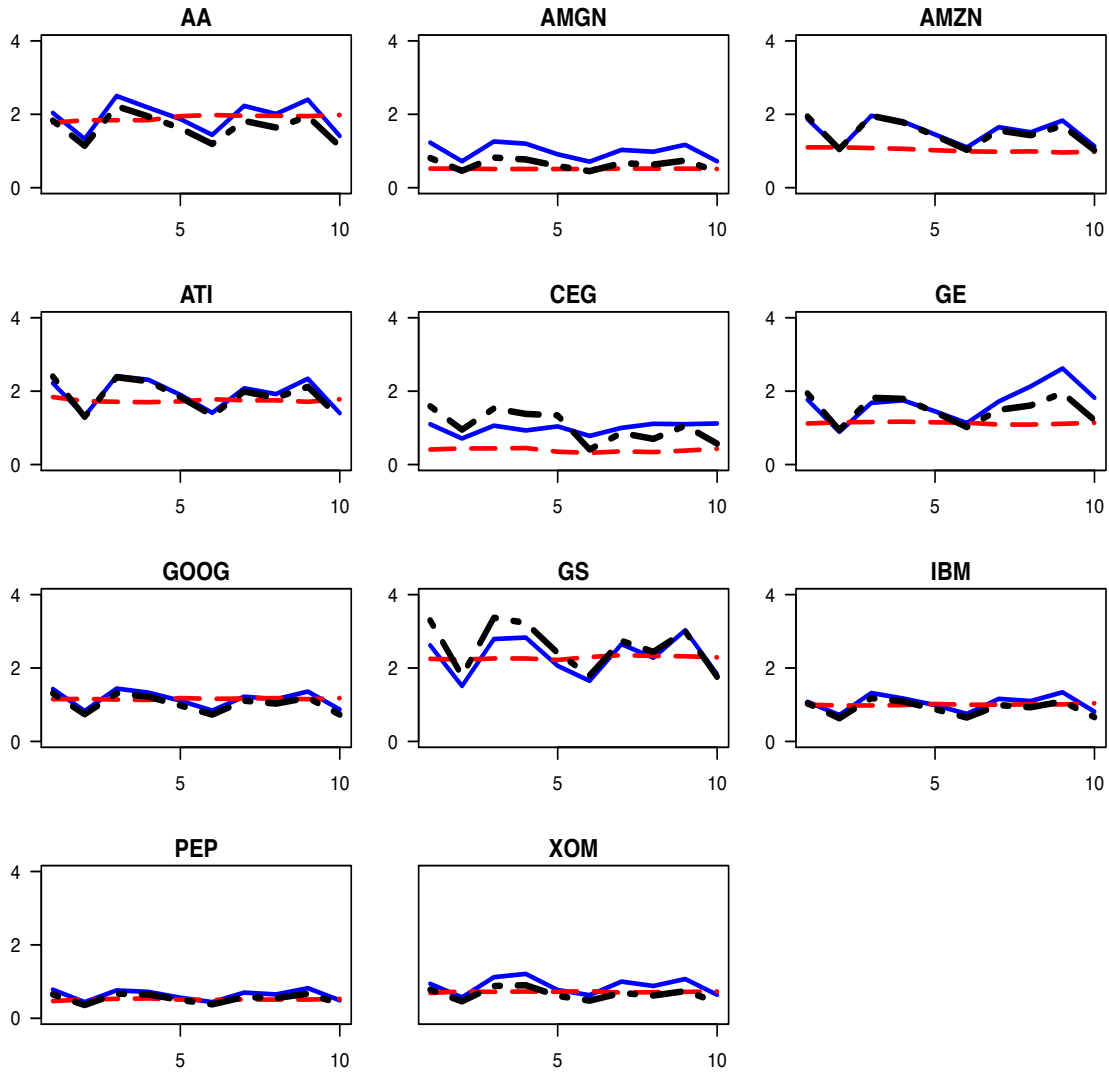


Figure 2: The solid line is the forward looking beta (y-axis) calibrated on June/July 2009 expiration call options over the course of 10 market days (x-axis) from February 9, 2009 to February 23, 2009. The dashed line is the corresponding historical beta calibrated on a series of historical prices from the 40 days prior to each option price date. The dot-dash line is the blended beta estimate using the affine slope parameter on option prices and  $\bar{\sigma}$  from historical prices

## 6 Conclusion

In this paper, we have explored the possibility of estimating the CAPM  $\beta$ -parameter of an asset in a forward-looking way as proposed in [2]. We have done so in the context of continuous time fast mean-reverting stochastic volatility models. Using approximation formulas for call options on the market index and on the asset, we have shown that the  $\beta$  parameter can be related by the simple formula (33) to the *skews* of implied volatilities estimated from a linear regression of implied volatilities with respect to the *log-moneyness-to-maturity-ratio* of the options. A further parameter reduction derived in the Appendix allows us to estimate  $\beta$  by using only options data. Empirical results are presented showing an excellent fit of implied volatilities, and for several firms, a comparison with the historical  $\beta$ . We suggest that this forward-looking  $\beta$ -estimate can be used to anticipate a change in the  $\beta$ -level of a firm, even during highly volatile period of times such as the one presented.

## 7 Appendix

### Market Option Approximation

We recall the approximation to option prices derived in Chapter 5 of [4]. It is based on a singular perturbation expansion method as  $\varepsilon \rightarrow 0$  for option prices generated by the model (9, 11). Let  $P^{M,\varepsilon}$  denote the price of a European option written on the market index  $M$ , with maturity  $T$  and payoff  $h$ , evaluated at time  $t < T$  with current value  $M_t = \xi$ , where we explicitly show the dependence on the small volatility mean-reversion time  $\varepsilon$ . Then, we have

$$P^{M,\varepsilon} = \mathbb{E}^{*(\gamma)} \left\{ e^{-r(T-t)} h(M_T) \mid \mathcal{F}_t \right\} = P^{M,\varepsilon}(t, M_t, Y_t),$$

since  $(M_t, Y_t)$  is markovian, and where, by the Feynman-Kac formula, the function  $P^{M,\varepsilon}(t, \xi, y)$  satisfies the partial differential equation:

$$\begin{aligned} \mathcal{L}^\varepsilon P^{M,\varepsilon} &= 0, \\ P^{M,\varepsilon}(T, \xi, y) &= h(\xi), \end{aligned}$$

where

$$\mathcal{L}^\varepsilon = \frac{1}{\varepsilon} \mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}} \mathcal{L}_1 + \mathcal{L}_2,$$

with

$$\begin{aligned}\mathcal{L}_0 &= \nu^2 \frac{\partial^2}{\partial y^2} + (m - y) \frac{\partial}{\partial y}, \\ \mathcal{L}_1 &= \rho\nu\sqrt{2}f(y)\xi \frac{\partial^2}{\partial \xi \partial y} - \nu\sqrt{2}\Lambda(y) \frac{\partial}{\partial y}, \\ \mathcal{L}_2 &= \frac{\partial}{\partial t} + \frac{1}{2}f(y)^2\xi^2 \frac{\partial^2}{\partial \xi^2} + r\left(\xi \frac{\partial}{\partial \xi} - \cdot\right) \equiv \mathcal{L}_{BS}(f(y)).\end{aligned}$$

Here,  $\mathcal{L}_{BS}(\sigma)$  denotes the Black-Scholes operator with volatility parameter  $\sigma$ . By expanding  $P^{M,\epsilon}$  in powers of  $\sqrt{\epsilon}$

$$P^{M,\epsilon} = P_0^M + \sqrt{\epsilon}P_1^M + \epsilon P_2^M + \epsilon^{3/2}P_3^M + \dots,$$

it is shown in [4] that

$$P^{M,\epsilon} = P_0^M + (T - t) \left( V_2^{M,\epsilon} \xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} + V_3^{M,\epsilon} \xi \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} \right) \right) + \mathcal{O}(\epsilon), \quad (34)$$

for smooth payoffs  $h$ . Here,  $P_0^M = P_{BS}(\bar{\sigma})$  is the Black-Scholes price of the option computed at the volatility level  $\bar{\sigma} = \sqrt{\langle f^2 \rangle}$  as introduced in (15), and the two parameters  $V_2^{M,\epsilon}, V_3^{M,\epsilon}$  are given by:

$$V_2^{M,\epsilon} = \frac{\sqrt{\epsilon}\nu}{\sqrt{2}} \langle \phi' \Lambda \rangle, \quad (35)$$

$$V_3^{M,\epsilon} = -\frac{\sqrt{\epsilon}\rho\nu}{\sqrt{2}} \langle \phi' f \rangle. \quad (36)$$

For a call option, that is  $h(\xi) = (\xi - K)^+$ , the accuracy is in fact  $\mathcal{O}(\epsilon \log |\epsilon|)$  as was shown in [6].

## Parameter Reduction

One of the inherent advantages of price estimation under our approximation approach is parameter reduction. While the stochastic volatility model (9,11) requires the four parameters  $(\epsilon, \nu, \rho, m)$  and the two functions  $f$  and  $\gamma$ , our approximated option price requires only the three group parameters  $(\bar{\sigma}, V_2^{M,\epsilon}, V_3^{M,\epsilon})$ . We can further reduce to only two parameters by noting that  $V_2^{M,\epsilon}$  is associated with a second order derivative with respect to the current market

price  $\xi$ . As such, it can be considered as a *volatility level correction* and absorbed into the volatility of the Black-Scholes price of the leading order term. As in (14) we introduce

$$\sigma^{M*} = \sqrt{\bar{\sigma}^2 + 2V_2^{M,\varepsilon}},$$

which is an effective volatility adjusted by the market price of volatility risk through the parameter  $V_2^{M,\varepsilon}$  proportional to  $\Lambda$  as can be seen in (35). Then, as in (13), we introduce  $P^{M*}$ , the Black-Scholes price of the option computed with the constant adjusted effective volatility  $\sigma^{M*}$  as defined in (14) and recalled above. Therefore

$$\begin{aligned} \mathcal{L}_{BS}(\sigma^{M*})P^{M*} &= 0, \\ P^{M*}(T, \xi) &= h(\xi). \end{aligned} \tag{37}$$

Next, we define the first order correction  $\sqrt{\varepsilon}P_1^{M*}$  as the solution to the problem

$$\begin{aligned} \mathcal{L}_{BS}(\sigma^{M*})(\sqrt{\varepsilon}P_1^{M*}) + V_3^{M,\varepsilon}\xi \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial^2 P_0^{M*}}{\partial \xi^2} \right) &= 0, \\ (\sqrt{\varepsilon}P_1^{M*})(T, \xi) &= 0, \end{aligned} \tag{38}$$

which is indeed given explicitly by

$$\sqrt{\varepsilon}P_1^{M*} = (T-t)V_3^{M,\varepsilon}\xi \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial^2 P_0^{M*}}{\partial \xi^2} \right). \tag{39}$$

With these definitions we obtain the same accuracy as in (34):

$$P^{M,\varepsilon} = P_0^{M*} + (T-t)V_3^{M,\varepsilon}\xi \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial^2 P_0^{M*}}{\partial \xi^2} \right) + \mathcal{O}(\varepsilon), \tag{40}$$

for smooth payoffs, and  $\mathcal{O}(\varepsilon \log |\varepsilon|)$  for call options. The derivation of this result goes as follows. We first observe from the definitions of  $\mathcal{L}_{BS}$  and  $\sigma^{M*}$  that

$$\mathcal{L}_{BS}(\sigma^{M*}) = \mathcal{L}_{BS}(\bar{\sigma}) + \frac{1}{2}(2V_2^{M,\varepsilon})\xi^2 \frac{\partial^2}{\partial \xi^2},$$

and therefore, it follows that

$$\mathcal{L}_{BS}(\bar{\sigma})(P_0^M - P_0^{M*}) = V_2^{M,\varepsilon}\xi^2 \frac{\partial^2 P_0^{M*}}{\partial \xi^2}, \tag{41}$$

$$(P_0^M - P_0^{M*})(T, \xi) = 0. \tag{42}$$

Since the source term is  $\mathcal{O}(\sqrt{\varepsilon})$  because of the  $V_2^{M,\varepsilon}$  factor, the difference  $P_0^M - P_0^{M*}$  is also  $\mathcal{O}(\sqrt{\varepsilon})$ . Note that by taking derivatives with respect to  $\xi$  in (41,42) we obtain similarly that the derivatives of the difference  $P_0^M - P_0^{M*}$  are also  $\mathcal{O}(\sqrt{\varepsilon})$ . Next we write

$$|P^{M,\varepsilon} - (P_0^{M*} + \sqrt{\varepsilon} P_1^{M*})| \leq |P^{M,\varepsilon} - (P_0^M + \sqrt{\varepsilon} P_1^M)| \\ + |(P_0^M + \sqrt{\varepsilon} P_1^M) - (P_0^{M*} + \sqrt{\varepsilon} P_1^{M*})|,$$

which, combined with (34), shows that the only quantity left to be controlled is the *residual*

$$R \equiv (P_0^M + \sqrt{\varepsilon} P_1^M) - (P_0^{M*} + \sqrt{\varepsilon} P_1^{M*}). \quad (43)$$

From the equations satisfied by  $P_0^M, \sqrt{\varepsilon} P_1^M, P_0^{M*}, \sqrt{\varepsilon} P_1^{M*}$ , it follows that

$$\mathcal{L}_{BS}(\bar{\sigma})(P_0^M + \sqrt{\varepsilon} P_1^M) + V_2^{M,\varepsilon} \xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} + V_3^{M,\varepsilon} \xi \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial^2 P_0^M}{\partial \xi^2} \right) = 0 \\ \mathcal{L}_{BS}(\sigma^{M*})(P_0^{M*} + \sqrt{\varepsilon} P_1^{M*}) + V_3^{M,\varepsilon} \xi \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial^2 P_0^{M*}}{\partial \xi^2} \right) = 0.$$

Denoting by

$$\mathcal{H}^\varepsilon = V_2^{M,\varepsilon} \xi^2 \frac{\partial^2}{\partial \xi^2} + V_3^{M,\varepsilon} \xi \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial^2}{\partial \xi^2} \right), \\ \mathcal{H}^{\varepsilon*} = V_3^{M,\varepsilon} \xi \frac{\partial}{\partial \xi} \left( \xi^2 \frac{\partial^2}{\partial \xi^2} \right),$$

the residual  $R$  satisfies the equation

$$\mathcal{L}_{BS}(\bar{\sigma})(R) = -\mathcal{H}^\varepsilon P_0^M - \left( \mathcal{L}_{BS}(\sigma^{M*}) - V_2^{M,\varepsilon} \xi^2 \frac{\partial^2}{\partial \xi^2} \right) (P_0^{M*} + \sqrt{\varepsilon} P_1^{M*}) \\ = -\mathcal{H}^\varepsilon P_0^M + \mathcal{H}^{\varepsilon*} P_0^{M*} + V_2^{M,\varepsilon} \xi^2 \frac{\partial^2}{\partial \xi^2} (P_0^{M*} + \sqrt{\varepsilon} P_1^{M*}) \\ = \mathcal{H}^{\varepsilon*} (P_0^{M*} - P_0^M) + V_2^{M,\varepsilon} \xi^2 \frac{\partial^2}{\partial \xi^2} (P_0^{M*} - P_0^M + \sqrt{\varepsilon} P_1^{M*}) \\ = \mathcal{O}(\varepsilon),$$

where we used in the last equality that  $\mathcal{H}^{\varepsilon*} = \mathcal{O}(\sqrt{\varepsilon}), V_2^{M,\varepsilon} = \mathcal{O}(\sqrt{\varepsilon}), P_0^{M*} - P_0^M = \mathcal{O}(\sqrt{\varepsilon}), \frac{\partial^k}{\partial \xi^k} (P_0^{M*} - P_0^M) = \mathcal{O}(\sqrt{\varepsilon}), \sqrt{\varepsilon} \frac{\partial^k}{\partial \xi^k} P_1^{M*} = \mathcal{O}(\sqrt{\varepsilon})$ . Since  $R$

vanishes at the terminal time  $T$ , as can be seen directly from (43), we obtain that  $R = \mathcal{O}(\varepsilon)$  which concludes the derivation of the accuracy in (40).

The new approximation (40) has now only two parameters to be calibrated  $\sigma^{M*}$  and  $V_3^{M,\varepsilon}$ , and has the same error of order  $\varepsilon$  as the initial approximation (34). This parameter reduction is essential in the forward-looking calibration procedure presented in Section 4.

## Asset Option Approximation

The approximation presented above for options on the market index  $M$  can be easily generalized to options on the asset  $X$ . The only changes are that  $\rho$  is multiplied by  $\beta$  in  $\mathcal{L}_1$ , and  $f(y)^2$  is replaced by  $(\beta^2 f(y)^2 + \sigma^2)$  in  $\mathcal{L}_2$ . Therefore

1.  $\bar{\sigma}^2$  in (15) is replaced by  $\bar{\sigma}_a^2 = \beta^2 \bar{\sigma}^2 + \sigma^2$  introduced in (16).

2.  $V_2^{M,\varepsilon}$  in (35) is replaced by

$$V_2^{a,\varepsilon} = \beta^2 V_2^{M,\varepsilon}. \quad (44)$$

3.  $V_3^{M,\varepsilon}$  in (36) is replaced by

$$V_3^{a,\varepsilon} = \beta^3 V_3^{M,\varepsilon}. \quad (45)$$

4.  $\sigma^{M*}$  in (14) is replaced by  $\sigma^{a*}$  given in (19) so that

$$\sigma_a^* = \sqrt{\beta^2 \bar{\sigma}^2 + \sigma^2 + 2V_2^{a,\varepsilon}} \quad (46)$$

5. The option price approximation becomes

$$P^{a,\varepsilon} = P_0^{a*} + (T - t)V_3^{a,\varepsilon} x \frac{\partial}{\partial x} \left( x^2 \frac{\partial^2 P_0^{a*}}{\partial x^2} \right) + \mathcal{O}(\varepsilon), \quad (47)$$

where  $P_0^{a*}$  is the Black-Scholes price with volatility  $\sigma^{a*}$  given by (46).

6. The parameters  $V_3^{a,\varepsilon}$  and  $\sigma^{a*}$  are calibrated to the implied volatility data  $(a^\varepsilon, b^*)$  according to the formulas (31) and (32).

Combining the fits from the skew of implied volatility in the market index and the skew of implied volatility in the individual asset, we deduce our main result (33).

## References

- [1] Carr, P. , and D. Madan (2001): “Optimal Positioning in Derivative Securities.” *Quantitative Finance*, 1, 19-37.
- [2] Christoffersen, P. , K. Jacobs, and G. Vainberg (2008): “Forward Looking Betas.” Manuscript, McGill University.
- [3] French, D. , J. Goth, and J. Kolari (1983): “Current Investor Expectations and Better Betas.” *Journal of Portfolio Management*, 12-17.
- [4] Fouque, J.-P. , G. Papanicolaou, and R. Sircar (2000): *Derivatives in Financial Markets with Stochastic Volatility*. Cambridge University Press.
- [5] Fouque, J.-P. , G. Papanicolaou, and R. Sircar (2000): “Stochastic Volatility: Calibrating Random Volatility.” *RISK Magazine*, 89-92.
- [6] Fouque, J.-P. , G. Papanicolaou, R. Sircar, and K. Sølna (2003): “Singular Perturbations in Option Pricing.” *SIAM Journal on Applied Mathematics*, 63, 1648-1665.
- [7] Fouque, J.-P. , G. Papanicolaou, R. Sircar, and K. Sølna (2004): “Multi-scale Stochastic Volatility Asymptotics.” *SIAM Journal Multiscale Modeling and Simulation*, 2(1): 22-42.
- [8] Fouque, J.-P. , G. Papanicolaou, R. Sircar, and K. Sølna (2004): “Timing the Smile.” *Wilmott Magazine*.
- [9] Heston, S. (1993): “A closed-form solution for options with stochastic volatility with applications to bonds and currency options.” *Review of Financial Studies*, 6, 327-343.
- [10] Merville, L. , and Dan R. Pieptea (1988): “Stock-price volatility, mean-reverting diffusion, and noise.” *Journal of Financial Economics*, 24, 193-214.
- [11] Markowitz, H, (1952): “Portfolio Selection.” *Journal of Finance*, 7 , 77-99.
- [12] Scholes, M. , and J. Williams (1977): “Estimating Betas from Nonsynchronous Data.” *Journal of Financial Economics*, 5, 309-327.

- [13] Sharpe, W. (1966): “Mutual Fund Performance.” *Journal of Business*, 39, 119-138.
- [14] Siegel, A. (1995): “Measuring Systematic Risk Using Implicit Beta.” *Management Science*, 41, 124-128.