

# PORTFOLIO OPTIMIZATION WITH AMBIGUOUS CORRELATION AND STOCHASTIC VOLATILITIES

JEAN-PIERRE FOUQUE\*, CHI SENG PUN†, AND HOI YING WONG‡

**Abstract.** In a continuous-time economy, we investigate the asset allocation problem among a risk-free asset and two risky assets with an ambiguous correlation between the two risky assets. The portfolio selection that is robust to the uncertain correlation is formulated as the utility maximization problem over the worst-case scenario with respect to the possible choice of correlation. Thus, it becomes a maximin problem. We solve the problem under the Black–Scholes model for risky assets with an ambiguous correlation using the theory of  $G$ -Brownian motions. We then extend the problem to stochastic volatility models for risky assets with an ambiguous correlation between risky asset returns. An asymptotic closed-form solution is derived for a general class of utility functions, including constant relative risk aversion and constant absolute risk aversion utilities, when stochastic volatilities are fast mean reverting. We propose a practical trading strategy that combines information from the option implied volatility surfaces of risky assets through the ambiguous correlation.

**Key words.** Ambiguous correlation,  $G$ -Brownian motion, Hamilton–Jacobi–Bellman–Isaacs equation, Stochastic volatility

**1. Introduction.** Parameter ambiguity is closely related to the concept of robust statistics introduced by [23] in the context of statistical estimation. For instance, if one wants to estimate the mean of a symmetrical distribution but the observations are contaminated by outliers, then Huber [23] shows that an optimal estimate is the maximum likelihood (ML) estimator for the least favorable distribution. In other words, the classical ML estimation is replaced by a maximin problem.

A similar notion also appears in mathematical finance. When model parameters are uncertain or ambiguous, financial economists are used to consider the financial decision under the worst-case scenario that corresponds to the least favorable distribution implied by the ambiguous parameter space. The optimal decision is then formulated as a stochastic optimal control problem. Related works include [2, 4, 9] for general discussions of uncertain volatility models in the context of option pricing, and [18] for an asymptotic analysis in the regime where the volatility uncertainty vanishes; [6] for modeling multiple-priors utility in continuous-time settings for maximin problems, [12] for the role of uncertainty in financial markets; [10, 11] for a model of utility maximization that incorporates the ambiguity about both volatility and drift, [30] for robust utility maximization on a single risky asset whose price is described by a diffusion process with misspecified trend and volatility coefficients, and a nontradable asset with known parameters, [29] for the investment and reinsurance problems on a single risky asset with model ambiguity, whose formulation starts with a reference model and penalizes the other models based on their distances (entropies) to the reference model; see [22] for the similar formulation for derivatives pricing, and references therein.

This paper focuses on the optimal allocation among a risk-free asset and two risky assets when the correlation between the two risky assets is uncertain. Unlike the existing literature, including the aforementioned works, which deals with the ambiguous

---

\*Department of Statistics and Applied Probability, University of California, Santa Barbara, CA 93106-3110 (fouque@pstat.ucsb.edu).

†Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore (cspun@ntu.edu.sg).

‡Corresponding author. Department of Statistics, The Chinese University of Hong Kong, Shatin, New Territories, Hong Kong (hywong@cuhk.edu.hk).

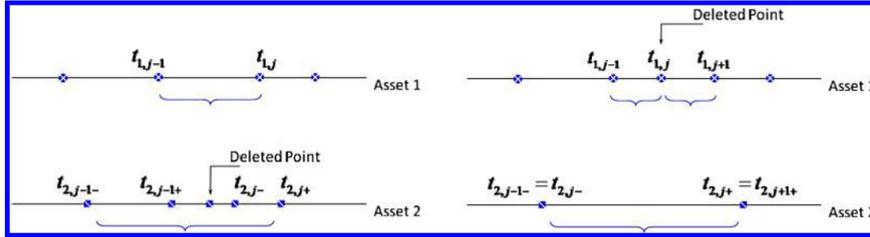


FIG. 1.1. *Synchronization in high-frequency trading; the figure is taken from [1].*

drift and volatility of a risky asset process, we are interested in a multivariate setting in which the drifts and volatilities of asset returns, but not the correlation, can be inferred from market information. This consideration is motivated by the fact that the correlation is in general difficult to calibrate even when using high-frequency data. For instance, in the context of high-frequency trading of multiple assets, Ait-Sahalia, Fan, and Xiu [1] find that it is necessary to synchronize asset returns because different stocks are traded at different time points. The synchronization involves removing some data points from the original data set, as shown in Fig 1.1. Therefore, the correlation estimate converges to its true value less rapidly than the estimates of volatilities that are based on the full sets of marginal observations. Despite the asynchronous issue, the spurious correlation reported in [13] further ensures the difficulty in estimating the correlation, where the spurious correlation refers to the feature whereby the sample correlation of independent data can be “accidentally” very high, whereas that of dependent data can be close to zero.

It is common practice to infer parameters by the calibrating model to derivative prices. In this approach, parameters are “forward looking”; see [8] for calibration of risk premia and [19] for forward-looking calibration of volatility parameters for optimal investment in a single risky asset. Fig 1.2 shows the option implied volatility surfaces of two different stocks. However, the optimal portfolio strategy on two risky assets requires the correlation as an input parameter, but the market generally lacks basket instruments to calibrate it.

In this paper, we treat the correlation as an ambiguous parameter but volatilities as stochastic processes with known parameters. The existing literature concentrates on ambiguous drift and volatility and assumes that they fall into a region such as  $[\underline{\mu}, \bar{\mu}] \times [\underline{\sigma}, \bar{\sigma}]$ . Estimating bounds for the drift and volatility is difficult. Recently, the correlation risk has become a great concern in finance. Buraschi, Porchia, and Trojani [5] and Chiu and Wong [7] attempt to explain the effect of correlation risk using a stochastic correlation model. However, the estimation of the stochastic correlation model is hardly made in practice. Fortunately, the correlation coefficient  $\rho$  has the natural bounds of  $[-1, 1]$ , which enables us to consider  $\rho \in [\underline{\rho}, \bar{\rho}] \subseteq [-1, 1]$  or the confidence interval.

The contribution of this paper is threefold: the formulation of portfolio optimization problems with an ambiguous correlation; the solution of the problem under the Black–Scholes (BS) economy with constant volatilities; and the characterization of the problem when volatilities are stochastic, so that an asymptotic solution to the robust portfolio problem under the fast mean-reverting stochastic volatility (FMRSV) model (see [17, 19] for details of this model) is derived. Specifically, we derive the closed-form explicit solutions to the optimal trading strategies robust to the uncertain

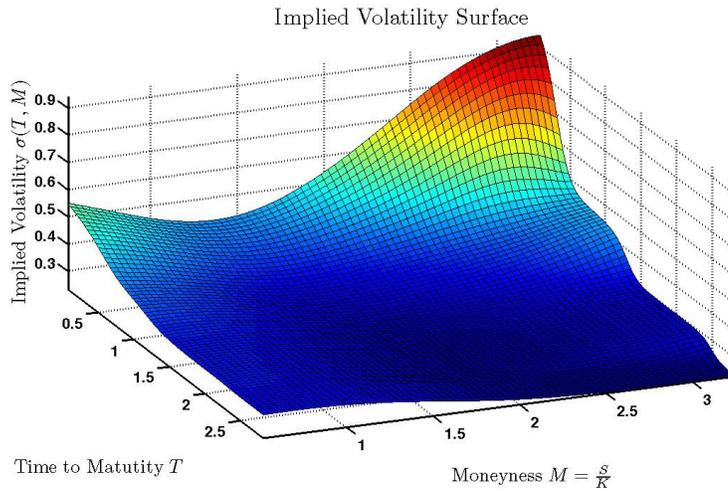
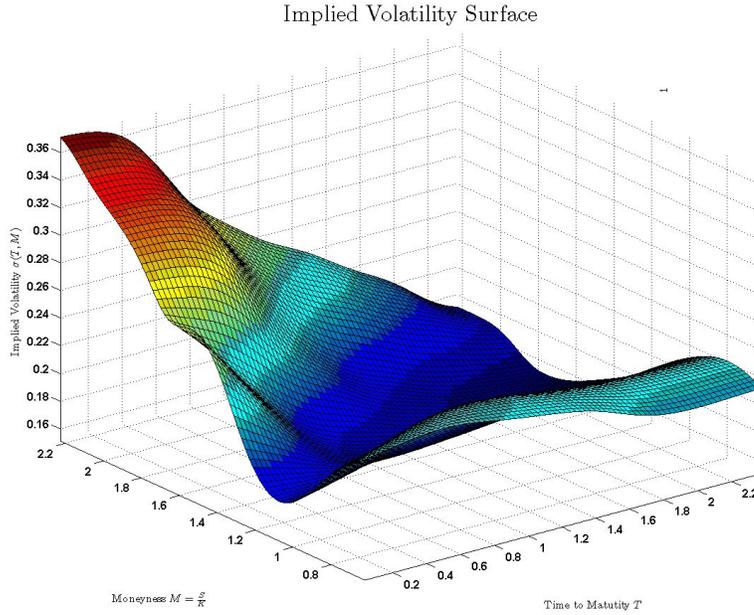


FIG. 1.2. Implied volatility surfaces of two stocks.

correlation under the BS model for specific classes of utility functions. The solution corresponding to the BS model turns out to be the zeroth-order approximation of the solution for the FMRSV model. The asymptotic solution has a practical use that enables portfolio managers to combine forward-looking information from the volatility surfaces of options to determine an optimal investment among risky and risk-free assets such that the investment strategy is robust to the imprecise estimate of the correlation. To the best of our knowledge, all of the results obtained in this paper, including those for the BS model, are new.

The rest of the paper is organized as follows. Section 2.2 presents the formula-

tion of the portfolio optimization among a risk-free asset and two risky assets where the correlation between the two risky assets is uncertain. The closed-form explicit solution to the problem under the BS model is derived in Section 2. We also provide a financial interpretation of the solution in this section. Section 3 investigates the optimal portfolio strategy with stochastic volatilities and an ambiguous correlation. We characterize the solution in a general stochastic volatility (SV) setting and derive asymptotic approximations to the FMRSV model. We offer both a theoretical full feedback strategy and a practical partial feedback strategy to the portfolio optimization problem. We also provide an algorithm that integrates the information from the implied volatility surfaces of risky assets into the partial feedback strategy. Section 4 discusses the results and future work. Section 5 concludes.

## 2. Merton problem under the BS model with uncertain correlation.

Consider a continuous-time economy with a risk-free asset and two risky assets, and time  $t$  varies over a finite (investment) horizon  $[0, T]$ . Let  $r$  be the constant instantaneous interest rate. Then the risk-free asset has the dynamic

$$dS_0(t) = rS_0(t) dt. \quad (2.1)$$

As the concept of an ambiguous correlation is a new feature, we start with the BS models with constant parameters for each of the risky assets. The generalization to stochastic volatility will be detailed in Section 3. The paths  $\omega = \{\omega(t) \in \mathbb{R}^2\}_{t \in [0, T]}$  of processes, driving the two risky assets, are assumed to be continuous and begin at 0. We define the canonical state space as  $\Omega = \{\omega = \{\omega(t)\}_{t \in [0, T]} : \omega(0) = 0\}$ . Let  $\mathcal{F}$  and  $\mathbb{P}^0$  be a  $\sigma$ -field and a (reference) measure, respectively on  $\Omega$ ,  $\mathcal{F}_t$  be the Borel  $\sigma$ -field on  ${}^t\Omega = \{\omega = \{\omega(s)\}_{s \in [0, t]} : \omega(0) = 0\}$ , then  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P}^0)$  constitutes a filtered probability space. Hereafter, we often suppress the  $\omega$  in the argument of random variables (processes) for the notational convenience.

The two risky assets,  $S_1$  and  $S_2$ , follow an Itô process with an ambiguous correlation between the asset returns. For  $i = 1, 2$ ,

$$dS_i(t) = \mu_i S_i(t) dt + \sigma_i S_i(t) dB_i(t), \quad (2.2)$$

where  $\mu_i$  and  $\sigma_i$  are known constants, and  $B(t) = [B_1(t) \ B_2(t)]'$  is a two-dimensional correlated Brownian motion with  $dB_1(t)dB_2(t) = \rho(t)dt$ . However, the correlation coefficient  $\rho(t)$  is uncertain and possibly stochastic, in the sense that we only know  $\rho(t) \in [\underline{\rho}, \bar{\rho}] \subset [-1, 1]$  and the corresponding variance-covariance matrix of the two assets returns is positive definite for any choice of  $\rho(t)$ .

**2.1. The probabilistic setup.** The concept of the ambiguous correlation can be mathematically defined through a set of priors using the notion as of [10] and [11]. Let  $\Theta_t(\omega) : \Omega \rightarrow [\underline{\rho}, \bar{\rho}]$  be the set of feasible correlation at time  $t$  along the path  $\omega$ , i.e., all correlation processes with values lying in the interval  $[\underline{\rho}, \bar{\rho}]$ . Analogously to [11], we impose that  $\Theta_t$  satisfies the regularity conditions which are recapped in Appendix A. Denote  $\Theta^{SDE}$  as the set of all parameter processes that ensure a unique strong solution to the stochastic differential equations (SDEs) (2.2). Then the set of admissible correlation processes is given by

$$\Theta := \{\rho \in \Theta^{SDE} : \rho(t, \omega) \in \Theta_t(\omega), \forall (t, \omega) \in [0, T] \times \Omega\}.$$

Finally, the set of priors  $\mathcal{P}^\Theta$  is the set of probability measures  $\mathbb{P}_\rho$  on  $(\Omega, \mathcal{F}_T)$  induced by  $\mathbb{P}^0$ :

$$\mathcal{P}^\Theta = \{\mathbb{P}_\rho : \rho \in \Theta\}, \quad \mathbb{P}_\rho(A) = \mathbb{P}^0(\{\omega : S(\omega; \rho) \in A\}) \text{ for } A \in \mathcal{F}_T,$$

where  $S(\omega; \rho) = [S_1(\omega; \rho) \ S_2(\omega; \rho)]'$  is the unique strong solution to SDEs (2.2) given  $\rho$ .

**Remark:** The priors defined here are generally nonequivalent because many measures in  $\mathcal{P}^\Theta$  are mutually singular. Therefore, the analysis is very different from the existing literature on ambiguity in drift, such as those in [6] and [29].

**Remark:** As pointed out in [10, 11], such a specification for the set of priors is essentially equivalent to the terminology of  $G$ -framework, introduced by Peng [24, 25, 26]. Appendix B presents the terminology of the  $G$ -framework related to our formulation.

To construct the utility under the worst-case scenarios, we need to define “expectation” and “conditional expectation” through the set  $\mathcal{P}^\Theta$ . For any random variable  $\xi$  on  $(\Omega, \mathcal{F}_T)$ , if  $\sup_{\mathbb{P}_\rho \in \mathcal{P}^\Theta} \mathbb{E}^{\mathbb{P}_\rho} \xi < \infty$ , then we define the following nonlinear expectation of  $\xi$  as

$$\hat{\mathbb{E}}\xi := \sup_{\mathbb{P}_\rho \in \mathcal{P}^\Theta} \mathbb{E}^{\mathbb{P}_\rho} \xi. \quad (2.3)$$

The corresponding definition of the “conditional expectation” is nontrivial and requires sophisticated mathematical arguments. We refer interested readers to [11], where the framework is applicable to ours. The brief idea goes as follows. If  $\rho = \{\rho(s)\}$  is a conceivable scenario ex ante, then  $\{\rho(s, {}^t\omega)\}_{s \in [t, T]}$  is seen by the individual ex ante as a conceivable continuation from time  $t$  along the path  $\omega$ . We assume that it is also a conceivable scenario ex post conditionally on  $(t, \omega)$ . Then such a conditional scenario implies a process conditionally on  $(t, \omega)$  from SDEs (2.2). This implied process together with  $\mathbb{P}^0$  induce a probability measure  $\mathbb{P}_t^\omega$  (analogous to the induction of  $\mathbb{P}_\rho$ ), where  $\rho$  is suppressed. For each  $\rho \in \Theta$ ,  $\mathbb{P}_t^\omega$  is defined for every  $t$  and  $\omega$ , and it is a version of the regular  $\mathcal{F}_t$ -conditional probability of  $\mathbb{P}_\rho$ . We denote by  $\mathcal{P}_t^\omega = \{\mathbb{P}_t^\omega : \rho \in \Theta\}$  the set of all such conditionals or, equivalently, the set of priors conditional on  $(t, \omega)$ . The nonlinear conditional expectation is then defined as

$$\hat{\mathbb{E}}[\xi | \mathcal{F}_t](\omega) = \sup_{\mathbb{P}_t^\omega \in \mathcal{P}_t^\omega} \mathbb{E}^{\mathbb{P}_t^\omega} \xi \text{ for every } \xi \in UC_b(\Omega) \text{ and } (t, \omega) \in [0, T] \times \Omega,$$

where  $UC_b(\Omega)$  is the set of all bounded and uniformly continuous functions on  $\Omega$ . Let  $\hat{L}^2(\Omega)$  be the completion of  $UC_b(\Omega)$  under the norm  $\|\xi\|_2 := (\hat{\mathbb{E}}[|\xi|^2])^{1/2}$ . Then  $\hat{L}^2(\Omega)$  is a subset of the set of measurable random variables  $\xi$  for which  $\hat{\mathbb{E}}[|\xi|^2] = \sup_{\mathbb{P}_\rho \in \mathcal{P}^\Theta} \mathbb{E}^{\mathbb{P}_\rho}[|\xi|^2] < \infty$ . [11] has shown that the mapping  $\hat{\mathbb{E}}[\cdot | \mathcal{F}_t]$  on  $UC_b(\Omega)$  defined above can be continuously extended to a mapping  $\hat{\mathbb{E}}[\cdot | \mathcal{F}_t] : \hat{L}^2(\Omega) \rightarrow \hat{L}^2(t, \Omega)$ , and

$$\hat{\mathbb{E}}[\hat{\mathbb{E}}[\xi | \mathcal{F}_s] | \mathcal{F}_t] = \hat{\mathbb{E}}[\xi | \mathcal{F}_s] \text{ for all } \xi \in \hat{L}^2(\Omega) \text{ and } 0 \leq s \leq t \leq T.$$

With the terminology of  $G$ -framework, the process  $B$  in (2.2) can be referred to as a  $G$ -Brownian motion under  $\mathcal{P}^\Theta$ . Therefore, we study the problem of our interest on the nonlinear expectation space  $(\Omega, L_{ip}(\Omega), \hat{\mathbb{E}})$ , where  $L_{ip}(\Omega)$  is defined in (B.2), and make use of the theory of  $G$ -expectation and the related stochastic calculus of Itô's type, developed by [24, 25, 26, 20]. For the process on  $(\Omega, L_{ip}(\Omega), \hat{\mathbb{E}})$ , we first denote by  $M^{2,0}(0, T)$  the collection of processes  $\eta$  of the form

$$\eta(t, \omega) = \sum_{i=0}^{N-1} \xi_i(\omega) \mathbf{1}_{[t_i, t_{i+1})}(t), \quad (2.4)$$

where  $\xi_i(\omega) \in \hat{L}^2(t_i, \Omega)$ ,  $i = 0, 1, \dots, N-1$ ,  $0 = t_0 < \dots < t_N = T$ . Then we denote by  $M^2(0, T)$  the completion of  $M^{2,0}(0, T)$  under the norm

$$\|\eta\|_{M^2} \equiv \left( \hat{\mathbb{E}} \left[ \int_0^T |\eta(t)|^2 dt \right] \right)^{\frac{1}{2}} = \left( \sum_{i=0}^{N-1} \hat{\mathbb{E}}[|\xi_i(\omega)|^2](t_{i+1} - t_i) \right)^{\frac{1}{2}}.$$

In this paper, we consider stochastic processes in  $M^2(0, T)$ .

**2.2. Merton problem formulation.** Let  $u_i(t)$  be the money amount invested in asset  $i$  and  $N_i(t)$  be the number of asset  $i$  in the portfolio of the investor at time  $t \in [0, T]$ . The wealth of the investor at time  $t$  is then defined as  $X(t) = \sum_{i=0}^2 u_i(t) = \sum_{i=0}^2 N_i(t)S_i(t)$ .

**DEFINITION.** For each  $t \in [0, T]$ , the portfolio  $u = [u_1 \ u_2]'$  is said to be admissible strategy on  $[t, T]$  if  $u(t) \in M^2(t, T)$  and  $u : [t, T] \times \Omega \rightarrow \mathcal{U}$  is an  $\{\mathcal{F}_s\}_{s \in [t, T]}$ -adapted process, where  $\mathcal{U}$  is a subspace of  $\mathbb{R}^2$ . We denote by  $\Pi[t, T]$  the set of admissible strategies on  $[t, T]$ .

In this paper, we consider self-financing trading strategy. Hence, for each  $t \in [0, T]$  and  $u(t) \in \Pi[t, T]$ , the law of motion for the wealth process on  $[t, T]$  is given by

$$dX(s) = [rX(s) + \beta' u(s)]dt + [\sigma_1 u_1(s) \ \sigma_2 u_2(s)]dB(s), \quad X(t) = x, \quad (2.5)$$

where

$$\beta = \mu - r\mathbf{1} \quad (2.6)$$

is the excess return vector,  $\mu = [\mu_1 \ \mu_2]'$ , and  $\mathbf{1} = [1 \ 1]'$ . Since the drift and diffusion coefficients in (2.5) are Lipschitz functions with respect to  $X$  and  $u$ , there exists a unique solution of  $X \in M^2(0, T)$  of (2.5) as proved in [26].

Suppose that the investor has a terminal utility function  $U(x)$  on  $\mathbb{R}$ , which is strictly increasing and strictly concave, i.e.  $U'(x) > 0$  and  $U''(x) < 0$ . Following the work of [10] in the context of uncertain volatility of one risky asset, we define the worst-case utility function as

$$\hat{U}^{t,x,u} := -\hat{\mathbb{E}}[-U(X(T))|\mathcal{F}_t] = \inf_{\mathbb{P}_\rho \in \mathcal{P}^\Theta} \mathbb{E}^{\mathbb{P}_\rho}[U(X(T))|\mathcal{F}_t]$$

Then our portfolio optimization problem with uncertain correlation is to find  $u(\cdot) \in \Pi[t, T]$  such that maximizes the ‘‘expected utility’’ in the worst-case scenario  $\hat{U}^{t,x,u}$ .

We define the (robust) value function (with an uncertain correlation) as

$$V(t, x) := \sup_{u(\cdot) \in \Pi[t, T]} \hat{U}^{t,x,u} = \sup_{u(\cdot) \in \Pi[t, T]} \inf_{\mathbb{P}_\rho \in \mathcal{P}^\Theta} \mathbb{E}^{\mathbb{P}_\rho}[U(X(T))|X(t) = x]. \quad (2.7)$$

To simplify the notations, we replace  $u(\cdot) \in \Pi[t, T]$  with  $u \in \Pi(t)$  and  $\mathbb{P}_\rho \in \mathcal{P}^\Theta$  with  $\rho \in [\underline{\rho}, \bar{\rho}]$ .

**Remark:** An alternative formulation is based on the ambiguous covariance of returns:  $\text{Cov}(\frac{dS_1}{S_1}, \frac{dS_2}{S_2}) = \eta dt$ , such that  $\eta \in [\underline{\eta}, \bar{\eta}]$ . In practice, it is more difficult to specify the bounds for covariance than those for a correlation coefficient that has the natural bounds of -1 and 1. These two formulations are equivalent to each other if the

volatilities are constants, as in the BS economy. The BS economy is a very interesting case for us because we show later that the zeroth-order approximation of the solution to the Merton problem under the FMRSV model is the same as that under the BS model.

**2.3. Analytical solution.** We derive the analytical solution of (2.7) under the BS model (2.2) and offer a financial interpretation of it. Closed-form explicit solutions for two popular utility functions are provided as illustrations at the end of this section.

**THEOREM 2.1.** *If  $\mathcal{U}$ , the domain of  $u$ , is compact, then the value function  $V(t, x)$  is the unique<sup>1</sup> deterministic continuous viscosity solution of the following Hamilton–Jacobi–Bellman–Isaacs (HJBI) equation:*

$$V_t + \sup_{u \in \Pi(t)} \inf_{\rho \in [\underline{\rho}, \bar{\rho}]} \left\{ \frac{1}{2} (u_1^2 \sigma_1^2 + u_2^2 \sigma_2^2 + 2\rho \sigma_1 \sigma_2 u_1 u_2) V_{xx} + [rx + u' \beta] V_x \right\} = 0 \quad (2.8)$$

with terminal condition  $V(T, x) = U(x)$  and  $\beta = \mu - r\mathbf{1}$ . In addition, if  $V_{xx} < 0$ , then the HJBI equation reduces to

$$V_t + \sup_{u \in \Pi(t)} \left\{ \frac{1}{2} (u_1^2 \sigma_1^2 + u_2^2 \sigma_2^2 + 2\rho^* \sigma_1 \sigma_2 u_1 u_2) V_{xx} + [rx + u' \beta] V_x \right\} = 0 \quad (2.9)$$

with terminal condition  $V(T, x) = U(x)$ , where

$$\rho^* = \bar{\rho} \mathbb{1}_{\{u_1 u_2 > 0\}} + \underline{\rho} \mathbb{1}_{\{u_1 u_2 < 0\}}. \quad (2.10)$$

*Proof.* This theorem is a direct application of the main results in [20, 21], including the dynamic programming principle for a more general class of stochastic control problems. The proofs that  $V(t, x)$  exists as a viscosity solution of the HJBI equation (2.8), and that  $V(t, x)$  is a deterministic and continuous function are documented in [21], while the uniqueness of the viscosity solution can be proved similarly as in Section 5 in [3]. Equation (2.9) is obvious because  $\rho$  only appears in the coefficient of  $V_{xx}$  and  $V_{xx} < 0$ . Hence, the quantity  $\rho \sigma_1 \sigma_2 u_1 u_2 V_{xx}$  attains its minimum at  $\rho = \bar{\rho}$  for  $u_1 u_2 > 0$  and  $\rho = \underline{\rho}$  for  $u_1 u_2 < 0$ .  $\square$

We recall that the vector of excess returns  $\beta$  is given by (2.6), we introduce the covariance matrices

$$\Sigma^* = \begin{pmatrix} \sigma_1^2 & \rho^* \sigma_1 \sigma_2 \\ \rho^* \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}, \quad \bar{\Sigma} = \begin{pmatrix} \sigma_1^2 & \bar{\rho} \sigma_1 \sigma_2 \\ \bar{\rho} \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}, \quad \underline{\Sigma} = \begin{pmatrix} \sigma_1^2 & \underline{\rho} \sigma_1 \sigma_2 \\ \underline{\rho} \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}, \quad (2.11)$$

and we define the corresponding variance risk ratios

$$[\bar{\Upsilon}_1 \quad \bar{\Upsilon}_2]' = \bar{\Sigma}^{-1} \beta, \quad [\underline{\Upsilon}_1 \quad \underline{\Upsilon}_2]' = \underline{\Sigma}^{-1} \beta. \quad (2.12)$$

In the following result, we characterize the optimal feedback strategy of problem (2.8). The financial interpretation is given after the proof.

**THEOREM 2.2.** *We assume that  $V_x > 0$  and  $V_{xx} < 0$ , i.e.,  $V(t, x)$  is a utility function for all  $t \leq T$ , and we define the (positive) risk-tolerance function by  $R(t, x) = -\frac{V_x}{V_{xx}}$  (see, for instance, [19]). Let  $f(t, x, u) = \frac{1}{2} u' \Sigma^* u V_{xx} + u' \beta V_x$  and  $u^*$  is defined as the following:*

<sup>1</sup>Analogously to [3], the uniqueness of the viscosity solution in this paper is proved in a space of continuous functions with a growth condition  $\mathcal{G} := \{\varphi \in C([0, T] \times \mathbb{R}^n) : \exists \bar{A} > 0 \text{ s.t. } \lim_{|x| \rightarrow \infty} \varphi(t, x) \exp\{-\bar{A}[\log(|x|^2 + 1)]^2/4\} = 0 \text{ uniformly in } t \in [0, T]\}$  with  $n$  being the number of state variables.

1. If  $\bar{\Upsilon}_1 \bar{\Upsilon}_2 > 0$ ,  $\underline{\Upsilon}_1 \underline{\Upsilon}_2 \geq 0$ , then  $u^* = \bar{\Sigma}^{-1} \beta R$ .
2. If  $\bar{\Upsilon}_1 \bar{\Upsilon}_2 \leq 0$ ,  $\underline{\Upsilon}_1 \underline{\Upsilon}_2 < 0$ , then  $u^* = \underline{\Sigma}^{-1} \beta R$ .
3. If  $\bar{\Upsilon}_1 \bar{\Upsilon}_2 \leq 0$ ,  $\underline{\Upsilon}_1 \underline{\Upsilon}_2 \geq 0$ , then  $u^* = \begin{cases} u_1 = 0, u_2 = \sigma_2^{-2} \beta_2 R & \text{if } \sigma_2^{-2} \beta_2^2 \geq \sigma_1^{-2} \beta_1^2, \\ u_2 = 0, u_1 = \sigma_1^{-2} \beta_1 R & \text{otherwise.} \end{cases}$
4. If  $\bar{\Upsilon}_1 \bar{\Upsilon}_2 > 0$ ,  $\underline{\Upsilon}_1 \underline{\Upsilon}_2 < 0$ , then  $u^* = \begin{cases} \bar{\Sigma}^{-1} \beta R & \text{if } \beta' \bar{\Sigma}^{-1} \beta \geq \beta' \underline{\Sigma}^{-1} \beta, \\ \underline{\Sigma}^{-1} \beta R & \text{otherwise,} \end{cases}$

Then  $f(t, x, u^*) = \sup_u f(t, x, u)$ . Consequently, if  $u^* \in \Pi(t)$  and (2.8) admits a solution  $V$ , then  $(u^*, \rho^*)$  is the optimum of problem (2.8), where  $\rho^*$  is defined in (2.10). Moreover, if  $\mathcal{U}$  is compact, then  $u^*$  is the (robust) optimal strategy of the optimization problem (2.7).

*Proof.* Let  $\Pi_{\pm}(t) = \{u \in \Pi(t) | u_1 u_2 \gtrless 0\}$ ,  $\Pi_0(t) = \{u \in \Pi(t) | u_1 u_2 = 0\}$ , and  $\bar{\Pi}_{\pm}(t) = \Pi_{\pm}(t) \cup \Pi_0(t)$ . Then, it is clear that  $\bar{\Pi}_+(t) \cup \bar{\Pi}_-(t) = \Pi(t)$  and  $\bar{\Pi}_+(t) \cap \bar{\Pi}_-(t) = \Pi_0(t)$ . As  $V_{xx} < 0$ , we only need to solve the HJB equation (2.9) by Theorem 2.1. Using the definitions of  $\Pi_{\pm}(t)$  and  $\Pi_0(t)$ , the HJB equation can be written as

$$V_t + \max_{u \in \{\bar{u}, \underline{u}\}} \left[ \sup_{\bar{u} \in \bar{\Pi}_+(t)} \left\{ \bar{u}' \bar{\Sigma} \bar{u} \frac{V_{xx}}{2} + \bar{u}' \beta V_x \right\}, \sup_{\underline{u} \in \bar{\Pi}_-(t)} \left\{ \underline{u}' \underline{\Sigma} \underline{u} \frac{V_{xx}}{2} + \underline{u}' \beta V_x \right\} \right] = -rxV_x. \quad (2.13)$$

As  $V_x > 0$  and  $V_{xx} < 0$  are assumed, both problems of

$$\sup_{\bar{u} \in \bar{\Pi}_+(t)} \left\{ \bar{u}' \bar{\Sigma} \bar{u} \frac{V_{xx}}{2} + \bar{u}' \beta V_x \right\} \quad \text{and} \quad \sup_{\underline{u} \in \bar{\Pi}_-(t)} \left\{ \underline{u}' \underline{\Sigma} \underline{u} \frac{V_{xx}}{2} + \underline{u}' \beta V_x \right\}$$

are quadratic maximization problems. If we relax the supports of  $\bar{u}$  and  $\underline{u}$  to the whole of  $\Pi(t)$ , then these two problems attain their maximum values at  $\bar{u}^* = \bar{\Sigma}^{-1} \beta R$  and  $\underline{u}^* = \underline{\Sigma}^{-1} \beta R$ , respectively, where  $R = -V_x/V_{xx} > 0$ . Consider the following cases.

- $\bar{\Upsilon}_1 \bar{\Upsilon}_2 > 0$  and  $\underline{\Upsilon}_1 \underline{\Upsilon}_2 \geq 0$ : it is obvious that  $\bar{u}^* \in \Pi_+(t)$  and  $\underline{u}^* \notin \Pi_-(t)$ . The latter implies that  $\underline{u}^* \in \Pi_0(t)$ . Notice that  $\underline{u}' \underline{\Sigma} \underline{u} = \underline{u}' \bar{\Sigma} \underline{u}$  when  $u \in \Pi_0(t)$ . Hence,  $\underline{u}^* \in \bar{\Pi}_+(t)$  and  $u^* = \bar{u}^*$ .
- $\bar{\Upsilon}_1 \bar{\Upsilon}_2 \leq 0$  and  $\underline{\Upsilon}_1 \underline{\Upsilon}_2 < 0$ : following a similar argument to the previous case, we deduce that  $u^* = \underline{u}^*$ .
- $\bar{\Upsilon}_1 \bar{\Upsilon}_2 \leq 0$  and  $\underline{\Upsilon}_1 \underline{\Upsilon}_2 \geq 0$ : it is easy to see that  $\bar{u}^* \notin \Pi_+(t)$  and  $\underline{u}^* \notin \Pi_-(t)$  so that  $u^* \in \Pi_0(t)$ . In such a situation, either  $u_1 = 0$  or  $u_2 = 0$ . If  $u_1 = 0$ , then  $\tilde{u}_2^* = \arg \sup \{u_2^2 \sigma_2^2 V_{xx}/2 + u_2 \beta_2 V_x\} = \beta_2 \sigma_2^{-2} R$ ; otherwise, if  $u_2 = 0$ , then  $\tilde{u}_1^* = \beta_1 \sigma_1^{-2} R$ . Substituting these back into the HJB equation (2.13) yields

$$V_t - \frac{1}{2} \max_{u \in \{[0 \ \tilde{u}_2^*]', [\tilde{u}_1^* \ 0]'\}} (\sigma_2^{-2} \beta_2^2, \sigma_1^{-2} \beta_1^2) V_x^2 / V_{xx} + rxV_x = 0.$$

Hence,  $u^* = [0 \ \tilde{u}_2^*]'$  if  $\sigma_2^{-2} \beta_2^2 > \sigma_1^{-2} \beta_1^2$ , and  $u^* = [\tilde{u}_1^* \ 0]'$  otherwise.

- $\bar{\Upsilon}_1 \bar{\Upsilon}_2 > 0$  and  $\underline{\Upsilon}_1 \underline{\Upsilon}_2 < 0$ : then,  $\bar{u} \in \Pi_+(t)$  and  $\underline{u} \in \Pi_-(t)$ . Substituting the expressions of  $\bar{u}$  and  $\underline{u}$  into (2.13) yields

$$\frac{\partial V}{\partial t} - \frac{1}{2} \max_{u \in \{\bar{u}, \underline{u}\}} \left( \beta' \bar{\Sigma}^{-1} \beta, \beta' \underline{\Sigma}^{-1} \beta \right) \frac{V_x^2}{V_{xx}} + rxV_x = 0.$$

Hence, the result in case 4 follows.

The characterization of the optimal strategy follows from Theorem 2.1.  $\square$

**Remark:** Theorems 2.1 and 2.2 still hold true for bounded time-deterministic parameters:  $r(t)$ ,  $\mu_i(t)$ , and  $\sigma_i(t)$  for  $i = 1, 2$ . It is easy to check that their proofs also apply to bounded time-deterministic parameters.

The solution in Theorem 2.2 has an interesting and natural financial interpretation. When  $\bar{\Upsilon}_1\bar{\Upsilon}_2 > 0$  and  $\underline{\Upsilon}_1\underline{\Upsilon}_2 \geq 0$ , the market is in favor of directional trading. That is, both risky assets are either bought or sold simultaneously, because their projected variance risk ratios share the same sign regardless of the value of  $\rho$ , signaling the same investment direction for both. In such a situation, the worst-case scenario refers to  $\rho = \bar{\rho}$ , the highest correlation, so that risk-averse investors make their decisions as if there is a minimal diversification effect. In other words, when the market situation is extremely good in the sense that all stocks have high variance risk ratios, investors will buy all of them. However, when the situation turns to extremely bad, investors will sell them all at once.

When  $\bar{\Upsilon}_1\bar{\Upsilon}_2 \leq 0$  and  $\underline{\Upsilon}_1\underline{\Upsilon}_2 < 0$ , the market is in favor of spread trading on risky assets; that is, buying one and selling another, regardless of the value of  $\rho$ . In such a situation, the worst-case scenario corresponds to  $\rho = \underline{\rho}$ , in which the hedging benefit from the spread trading is minimal. In other words, if the market information enables investors to clearly distinguish between good and bad stocks, then they will consider spread trading even though the correlation is uncertain.

When  $\bar{\Upsilon}_1\bar{\Upsilon}_2 \leq 0$  and  $\underline{\Upsilon}_1\underline{\Upsilon}_2 \geq 0$ , it is optimal to invest in either one of the risky assets but not both, because the directional trading is not optimal for a high correlation and spread trading is not optimal for a low correlation. As there are situations where directional and spread trading are not preferred, a risk-averse investor only invests in one of the two risky assets. The optimal decision for selecting a risky asset is to pick the one with the highest squared Sharpe ratio  $(\sigma_j^{-1}\beta_j)^2$  for  $j = 1, 2$ . This situation is particularly interesting as it explains why some investors refuse to diversify their portfolios by investing in more risky assets. The uncertain dependence structure among risky assets and the unclear market situation make risk-averse investors limit the number of risky assets in their portfolios.

When  $\bar{\Upsilon}_1\bar{\Upsilon}_2 > 0$  and  $\underline{\Upsilon}_1\underline{\Upsilon}_2 < 0$ , both directional and spread trading are good strategies. To pick the best one, risk-averse investors examine the squared Sharpe ratio in the corresponding worst-case scenario:  $\beta'\bar{\Sigma}^{-1}\beta$  or  $\beta'\underline{\Sigma}^{-1}\beta$ .

An interesting extreme case occurs when the confidence interval of  $\rho$  ( $[\underline{\rho}, \bar{\rho}]$ ) happens to be very close to the natural bounds of  $[-1, 1]$ . In other words, the investor has no confidence on the correlation estimate, which is common because of its difficulty in practice; see the discussion in Section 1. Let  $\bar{\rho} = 1 - \bar{\epsilon}$  and  $\underline{\rho} = -1 + \underline{\epsilon}$ , where  $\bar{\epsilon}, \underline{\epsilon} \in (0, 2)$ . It is easy to verify that

$$\begin{aligned}\bar{\Upsilon}_1\bar{\Upsilon}_2 &= \left[ \bar{\epsilon}\sigma_1\sigma_2\beta_1\beta_2 - \frac{1-\bar{\epsilon}}{\bar{\epsilon}}(\sigma_1\beta_2 - \sigma_2\beta_1)^2 \right] / [\sigma_1\sigma_2(2-\bar{\epsilon})], \\ \underline{\Upsilon}_1\underline{\Upsilon}_2 &= \left[ \underline{\epsilon}\sigma_1\sigma_2\beta_1\beta_2 + \frac{1-\underline{\epsilon}}{\underline{\epsilon}}(\sigma_1\beta_2 + \sigma_2\beta_1)^2 \right] / [\sigma_1\sigma_2(2-\underline{\epsilon})],\end{aligned}$$

where  $\beta_i = \mu_i - r$ ,  $i = 1, 2$ . When the positive  $\bar{\epsilon}$  and  $\underline{\epsilon}$  are close to zero, we have  $\bar{\Upsilon}_1\bar{\Upsilon}_2 \leq 0$  and  $\underline{\Upsilon}_1\underline{\Upsilon}_2 \geq 0$ . In such an extreme case, our solution suggests that the investor is optimal to invest in only either one of the risky assets, or a sparse portfolio results. This observation partially explains why a sparse portfolio is stable and popular in practice. Experienced investors will select favorable stocks to their portfolios instead of investing in all available stocks in the market even though the classical finance theory advocates so-called diversification.

**THEOREM 2.3.** *If the utility function is the power (or constant relative risk aversion) utility  $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$  for some constant risk aversion coefficient  $\gamma > 0$ ,  $\gamma \neq 1$ , then problem (2.8) has the solution pair  $(V(t, x), u^*(t, x))$ , such that  $u^*(t, x)$  is presented in Theorem 2.2 with  $R(t, x) = \frac{x}{\gamma}$ , and  $V(t, x)$  has the following explicit form:*

$$V(t, x) = K(t) \frac{x^{1-\gamma}}{1-\gamma}, \quad (2.14)$$

where the optimal investment proportion vector  $\frac{u^*(t, x)}{x}$  is independent of  $t$  and  $x$ , and

$$K(t) = \exp \left[ (\gamma - 1) \left( \frac{1}{2} \beta' \frac{u^*}{x} + r \right) (T - t) \right].$$

In addition, if  $r(t)$ ,  $\mu_i(t)$ , and  $\sigma_i(t)$  are bounded time-deterministic functions for  $t \in [0, T]$  and  $i = 1, 2$ , then  $u^*(t, x)$  is presented in Theorem 2.2 with all constants replaced by time-deterministic functions and  $R(t, x) = \frac{x}{\gamma}$ , and  $V(t, x)$  takes the same form as (2.14) with  $\frac{u^*(t, x)}{x}$  being independent of  $x$  and

$$K(t) = \exp \left[ (\gamma - 1) \int_t^T \left( \frac{1}{2} \beta(\tau)' \left( \frac{u^*}{x} \right) (\tau) + r(\tau) \right) d\tau \right].$$

*Proof.* Consider the solution of (2.14) for  $V(t, x)$  with a positive  $K(t)$  so that  $V_x > 0$  and  $V_{xx} < 0$ . For such a solution form, the optimal strategy  $u^*(t, x)$  is given by Theorem 2.2, where  $u^*(t, x) \propto V_x/V_{xx} = -x/\gamma$  for all cases. By substituting  $u^*(t, x)$  into the HJB equation (2.9), we obtain

$$V_t + \frac{1}{2} \beta' u^* V_x + r x V_x = 0,$$

where  $\frac{u^*}{x}$  is independent of  $x$ . Yet,  $V_t = \frac{\dot{K}}{K} V$  and  $V_x = (1 - \gamma) \frac{V}{x}$ . Substituting these into the partial differential equation (PDE) further reduces it to

$$\dot{K}(t) = K(t) \cdot (\gamma - 1) \left( \frac{1}{2} \beta' \frac{u^*}{x}(t) + r \right), \quad K(T) = 1.$$

Hence,  $K(t) = \exp \left( \int_t^T (\gamma - 1) \left( \frac{1}{2} \beta(\tau)' \frac{u^*}{x}(\tau) + r(\tau) \right) d\tau \right)$ .  $\square$

**THEOREM 2.4.** *If the utility function is the exponential utility  $U(x) = -\frac{1}{c} e^{-cx}$  for some constant  $c > 0$ , and  $r(t)$ ,  $\mu_i(t)$ , and  $\sigma_i(t)$  are bounded time-deterministic functions for  $i = 1, 2$ , then Problem (2.8) has the solution pair  $(V(t, x), u^*(t))$  such that  $u^*(t)$  is presented in Theorem 2.2 with  $R(t, x) = \frac{1}{ck(t)}$ , and  $V(t, x)$  has the following explicit form:*

$$V(t, x) = -\frac{1}{c} \exp [-c(C_1(t) + k(t)x)], \quad (2.15)$$

where

$$C_1(t) = \frac{1}{2} \int_t^T \beta(\tau)' u^*(\tau) d\tau, \quad k(t) = \exp \left( \int_t^T r(\tau) d\tau \right),$$

and the vector of optimal investment amount  $u^*(t)$  is independent of  $x$ .

*Proof.* The proof is very similar to that of Theorem 2.3 and hence is omitted.  $\square$

**Remark:** The optimal strategy  $u^*$  in Theorem 2.4 or the optimal weight vector  $u^*/x$  in Theorem 2.3 is deterministic which depends on bounded time-deterministic parameters. For practical use, it is reasonable to assume that  $u^*$  (resp.,  $u^*/x$ ) has values in a compact set for the case of exponential (resp., power) utilities. The uniqueness of the viscosity solution to (2.8) is guaranteed for both cases. For the case of power utilities, the main idea of proving Theorem 2.1 is to transform  $u$  to  $u/x$ . We refer the reader to [27] for the lengthy details which we omit here. Hence, the strategies  $u^*$  in Theorems 2.3 and 2.4 are the (robust) optimal trading strategies of (2.7).

Theorems 2.3 and 2.4 present two examples where  $V_x > 0$ ,  $V_{xx} < 0$ , and  $(V(t, x), u^*(t))$  admits a closed-form explicit solution pair. Therefore, the conditions in Theorem 2.2 are not unrealistic.

**3. Merton problem under stochastic volatility models with an ambiguous correlation.** We proceed to investigate SV models with an ambiguous correlation. Consider an economy with a risk-free asset  $S_0(t)$  as in (2.1) and two risky assets,  $S_1$  and  $S_2$ , that follow an Itô process with stochastic volatilities (factors) and returns driven by  $Y_1$  and  $Y_2$ , and an ambiguous correlation between the asset returns. For  $i = 1, 2$ ,

$$dS_i(t) = \mu_i(Y_i(t))S_i(t)dt + \sigma_i(Y_i(t))S_i(t)dB_i(t), \quad (3.1)$$

$$dY_i(t) = m_i(Y_i(t))dt + \alpha_i(Y_i(t))dB_{i+2}(t), \quad (3.2)$$

where  $\mu_i, \sigma_i, m_i, \alpha_i$  are bounded Lipschitz functions. Here, we make a particular choice of correlation structure between the Brownian motions. Specifically, we assume that covariance matrix of the  $B_i$ 's is of the form

$$\begin{pmatrix} 1 & \rho & \rho_1 & 0 \\ \rho & 1 & 0 & \rho_2 \\ \rho_1 & 0 & 1 & 0 \\ 0 & \rho_2 & 0 & 1 \end{pmatrix}. \quad (3.3)$$

In other words, the nontradable factors  $Y_1$  and  $Y_2$  are independent, they are only correlated to the Brownian motion of the tradable stock they, respectively, drive, and the Brownian motions driving the stocks contain the ambiguous correlation  $\rho$  as before. We will see that this choice makes the calibration procedure relatively simple as well as the FMRSV asymptotics presented below. Note that positive definiteness implies  $(1 - \rho_1^2)(1 - \rho_2^2) - \rho^2 > 0$  and therefore the uncertainty bounds

$$[\underline{\rho}, \bar{\rho}] \subset \left[ -\sqrt{(1 - \rho_1^2)(1 - \rho_2^2)}, \sqrt{(1 - \rho_1^2)(1 - \rho_2^2)} \right].$$

In fact, more general correlation structures where the two stochastic volatility factors are correlated can also be considered but the computation and formulas are more involved. In the next subsection, we first formulate the problem and derive some general statements for it under the general SV model (3.1)–(3.2), which contains the FMRSV model as its special case. We then apply SV asymptotic techniques to solve the problem for the FMRSV model in the subsequent sections.

**3.1. Problem formulation and some general statements.** Under the SV model (3.1)–(3.2), the problem of interest is formulated similarly to that under the BS model in Sections 2.1 and 2.2, since there are only two more “certain” stochastic processes compared to (2.2). The main difference of the probabilistic setup is that  $\Theta^{SDE}$  is now a set of all parameter processes that ensure a unique strong solution to SDEs (3.1)–(3.2). The wealth process with stochastic volatilities reads

$$dX(t) = [rX(t) + \beta(Y(t))'u(t)]dt + u_1(t)\sigma_1(Y_1(t))dB_1(t) + u_2(t)\sigma_2(Y_2(t))dB_2(t), \quad (3.4)$$

where  $Y(t) = [Y_1(t) \ Y_2(t)]'$  is defined in (3.2) and  $[B_1(t) \ B_2(t)]'$  is referred to the  $G$ -Brownian motion introduced in Appendix B. Strictly speaking, we should generalize (B.1) and introduce a four-dimensional  $G$ -Brownian motion with an admissible set of covariance structure obtained by the Cholesky decomposition of the covariance matrix (3.3), where  $\rho_1$  and  $\rho_2$  are fixed parameters and  $\rho \in [\underline{\rho}, \bar{\rho}]$ . We omit the details since the minimization with respect to  $\rho$  involved in the HJBI equation derived in the next theorem only depends on the first two dimensions.

The (robust) value function of portfolio optimization problem under the SV model with uncertain correlation takes the form

$$V(t, x, y) = \sup_{u \in \Pi(t)} \inf_{\rho \in [\underline{\rho}, \bar{\rho}]} \mathbb{E}^{\mathbb{P}^\rho} [U(X(T)) | X(t) = x, Y(t) = y], \quad (3.5)$$

where  $y = [y_1 \ y_2]'$ ,  $\Pi(t) := \Pi[t, T]$  is defined in Section 2.2 so that we have  $u(t) = u(t, X(t), Y(t))$  as a feedback control, and recall that  $U(x)$  defined on  $\mathbb{R}$  is strictly increasing and strictly concave, i.e.,  $U'(x) > 0$  and  $U''(x) < 0$ .

**THEOREM 3.1.** *If  $\mathcal{U}$ , the domain of  $u$ , is compact, then the value function  $V$  is the unique deterministic continuous viscosity solution of the following HJBI equation*

$$V_t + \mathcal{L}V + \sup_{u \in \Pi(t)} \inf_{\rho \in [\underline{\rho}, \bar{\rho}]} \left\{ \frac{1}{2} u' \Sigma_y u V_{xx} + u' [\beta V_x + \mathcal{M}_y V_x] \right\} + rx V_x = 0 \quad (3.6)$$

with the terminal condition  $V(T, x, y) = U(x)$ , where  $\mathcal{L} = \mathcal{L}^1 + \mathcal{L}^2$ ,

$$\begin{aligned} \mathcal{L}^i &= \frac{1}{2} \alpha_i^2(y_i) \frac{\partial^2}{\partial y_i^2} + m_i(y_i) \frac{\partial}{\partial y_i}, \quad \mathcal{M}_y = \begin{pmatrix} \rho_1 \alpha_1(y_1) \sigma_1(y_1) \partial_{y_1} \\ \rho_2 \alpha_2(y_2) \sigma_2(y_2) \partial_{y_2} \end{pmatrix}, \\ \Sigma_y &= \begin{pmatrix} \sigma_1^2(y_1) & \rho \sigma_1(y_1) \sigma_2(y_2) \\ \rho \sigma_1(y_1) \sigma_2(y_2) & \sigma_2^2(y_2) \end{pmatrix}, \quad \beta = \begin{pmatrix} \mu_1(y_1) - r \\ \mu_2(y_2) - r \end{pmatrix}. \end{aligned} \quad (3.7)$$

In addition, if  $V_{xx} < 0$ , the governing equation in (3.6) can be written as

$$V_t + \mathcal{L}V + \sup_{u \in \Pi(t)} \left\{ \frac{1}{2} u' \Sigma_y^* u V_{xx} + u' [\beta V_x + \mathcal{M}_y V_x] \right\} + rx V_x = 0, \quad (3.8)$$

where  $\Sigma_y^* = \Sigma_y|_{\rho=\rho^*}$  and  $\rho^* = \bar{\rho} \mathbb{1}_{\{u_1 u_2 > 0\}} + \underline{\rho} \mathbb{1}_{\{u_1 u_2 < 0\}}$ .

*Proof.* The proof is similar to that of Theorem 2.1 and relies on the main results in [3, 20, 21].  $\square$

To solve the HJBI equation (3.6), consider the auxiliary HJB equation

$$V_t + \mathcal{L}V + \sup_{\bar{u} \in \Pi(t)} \left\{ \frac{1}{2} \bar{u}' \bar{\Sigma}_y \bar{u} V_{xx} + \bar{u}' [\beta V_x + \mathcal{M}_y V_x] \right\} + rx V_x = 0 \quad (3.9)$$

with terminal condition  $V(T, x, y) = U(x)$ , where  $\bar{\Sigma}_y = \Sigma_y|_{\rho=\bar{\rho}}$  and  $|\bar{\rho}| < 1$  is known.

LEMMA 3.2. *If the utility function satisfies  $U_x > 0, U_{xx} < 0$ , and  $\frac{U_x^2}{U \cdot U_{xx}} \equiv c$  for some constant  $c$ , then the HJB equation (3.9) has the solution pair*

$$\begin{aligned} V(t, x, y) &= U(k(t)x) \cdot \bar{v}(t, y), \\ \bar{u}(t, x, y) &= -\bar{\Sigma}_y^{-1} \left( \beta + \frac{\mathcal{M}_y \bar{v}(t, y)}{\bar{v}(t, y)} \right) \frac{V_x}{V_{xx}}, \end{aligned} \quad (3.10)$$

where  $k(t) = \exp(\int_t^T r(\tau) d\tau)$ , and  $\bar{v}$  is the solution of the PDE

$$\bar{v}_t + \mathcal{L}\bar{v} - \frac{c}{2} \left( \beta + \frac{\mathcal{M}_y \bar{v}}{\bar{v}} \right)' \bar{\Sigma}_y^{-1} \left( \beta + \frac{\mathcal{M}_y \bar{v}}{\bar{v}} \right) \bar{v} = 0, \quad \bar{v}(T, y) = 1. \quad (3.11)$$

*Proof.* Consider the solution form of  $V$  in (3.10). Then, we have

$$\frac{1}{2} \bar{u}' \bar{\Sigma}_y \bar{u} V_{xx} + \bar{u}' [\beta V_x + \mathcal{M}_y V_x] = \frac{1}{2} \bar{u}' \bar{\Sigma}_y \bar{u} V_{xx} + \bar{u}' \left[ \beta + \frac{\mathcal{M}_y \bar{v}}{\bar{v}} \right] V_x,$$

so that the maximization problem in (3.9) attains the maximum value at the  $\bar{u}$  defined in (3.10) because  $V_{xx} < 0$  for a positive  $\bar{v}$ .

Substituting the expression of  $V$  and  $\bar{u}$  into the HJB equation (3.9) yields

$$(\dot{k} + rk) \cdot x U_x \bar{v} + \left[ \bar{v}_t + \mathcal{L}\bar{v} - \frac{c}{2} \left( \beta + \frac{\mathcal{M}_y \bar{v}}{\bar{v}} \right)' \bar{\Sigma}_y^{-1} \left( \beta + \frac{\mathcal{M}_y \bar{v}}{\bar{v}} \right) \bar{v} \right] = 0.$$

Hence, the result follows.  $\square$

Lemma 3.2 essentially characterizes the solution of the utility maximization problem with SV for any given known correlation coefficient  $\bar{\rho}$  once the utility function satisfies the condition of the lemma. Note that both the power and exponential utilities satisfy  $U_x > 0, U_{xx} < 0$ , and  $\frac{U_x^2}{U \cdot U_{xx}} \equiv c$  for some constant  $c$ . In fact, Lemma 3.2 still holds true for a portfolio of  $n$  risky assets for  $n \geq 1$ . However, the key observation related to our problem is that the positivity of the product  $\bar{u}_1 \bar{u}_2$  is independent of the wealth level  $x$ , where  $\bar{u} = [\bar{u}_1 \ \bar{u}_2]'$ .

To characterize the solution of the HJBI equation (3.8), we rewrite it as

$$V_t + \mathcal{L}V + \sup_{u \in \{u^+, u^-\}} \{H_+(u^+), H_-(u^-)\} + rxV_x = 0, \quad (3.12)$$

where

$$\begin{aligned} H_{\pm}(u) &= \sup_{u \in \Pi_{\pm}(t)} \left\{ \frac{1}{2} u' \Sigma_y^{\pm} u V_{xx} + u' [\beta V_x + \mathcal{M}_y V_x] \right\}, \\ \Sigma_y^+ &= \Sigma_y |_{\rho=\bar{\rho}}, \quad \Sigma_y^- = \Sigma_y |_{\rho=-\bar{\rho}}, \quad \Pi_0(t) = \{u \in \Pi(t) | u_1 u_2 = 0\}, \\ \Pi_{\pm}(t) &= \{u \in \Pi(t) | u_1 u_2 \gtrless 0\}, \quad \bar{\Pi}_{\pm}(t) = \Pi_{\pm}(t) \cup \Pi_0(t). \end{aligned} \quad (3.13)$$

THEOREM 3.3. *If the utility function satisfies  $U_x > 0, U_{xx} < 0$ , and  $\frac{U_x^2}{U \cdot U_{xx}} \equiv c$  for some constant  $c$ , then the HJB equation (3.8) has the solution pair*

$$\begin{aligned} V(t, x, y) &= U \left( e^{\int_t^T r(\tau) d\tau} x \right) \cdot v(t, y), \\ u^*(t, x, y) &= \hat{\Sigma}_y^{-1} \xi(t, y) R(t, x), \end{aligned} \quad (3.14)$$

where  $R(t, x) = -V_x/V_{xx}$  is the risk-tolerance function,

$$\xi(t, y) = \beta + \frac{\mathcal{M}_y v(t, y)}{v(t, y)}, \quad (3.15)$$

and  $v(t, y)$  is the solution of the PDE

$$v_t + \mathcal{L}v - \frac{c}{2}\xi(t, y)' \widehat{\Sigma}_{t, y}^{-1} \xi(t, y) v = 0, \quad v(T, y) = 1, \quad (3.16)$$

in which  $\Sigma_y^\pm$  is defined in (3.13),

$$\begin{aligned} \widehat{\Sigma}_{t, y}^{-1} &= (\Sigma_y^+)^{-1} \mathbb{1}_{\{\widehat{\Omega}_+\}} + (\Sigma_y^-)^{-1} \mathbb{1}_{\{\widehat{\Omega}_-\}} + \Lambda_y^1 \mathbb{1}_{\{\Omega_1\}} + \Lambda_y^2 \mathbb{1}_{\{\Omega_2\}}, \\ \Lambda_y^1 &= \begin{pmatrix} \sigma_1^{-2}(y_1) & 0 \\ 0 & 0 \end{pmatrix}, \quad \Lambda_y^2 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_2^{-2}(y_2) \end{pmatrix}, \\ \Omega_+ &= \{(t, y) \in \mathbb{R}^+ \times \mathbb{R}^2 \mid \Upsilon_1^+(t, y) \Upsilon_2^+(t, y) > 0, \Upsilon_1^-(t, y) \Upsilon_2^-(t, y) \geq 0\}, \\ \Omega_- &= \{(t, y) \in \mathbb{R}^+ \times \mathbb{R}^2 \mid \Upsilon_1^+(t, y) \Upsilon_2^+(t, y) \leq 0, \Upsilon_1^-(t, y) \Upsilon_2^-(t, y) < 0\}, \\ \widehat{\Omega} &= \{(t, y) \in \mathbb{R}^+ \times \mathbb{R}^2 \mid \Upsilon_1^+(t, y) \Upsilon_2^+(t, y) > 0, \Upsilon_1^-(t, y) \Upsilon_2^-(t, y) < 0\}, \\ \Omega_0 &= \{(t, y) \in \mathbb{R}^+ \times \mathbb{R}^2 \mid \Upsilon_1^+(t, y) \Upsilon_2^+(t, y) \leq 0, \Upsilon_1^-(t, y) \Upsilon_2^-(t, y) \geq 0\}, \\ \widehat{\Omega}_\pm &= \Omega_\pm \cup \{(t, y) \in \widehat{\Omega} \mid \xi(t, y)' (\Sigma_y^+)^{-1} \xi(t, y) \geq \xi(t, y)' (\Sigma_y^-)^{-1} \xi(t, y)\}, \\ \Omega_1 &= \{(t, y) \in \Omega_0 \mid \xi(t, y)' \Lambda_y^1 \xi(t, y) > \xi(t, y)' \Lambda_y^2 \xi(t, y)\}, \\ \Omega_2 &= \{(t, y) \in \Omega_0 \mid \xi(t, y)' \Lambda_y^1 \xi(t, y) \leq \xi(t, y)' \Lambda_y^2 \xi(t, y)\}, \\ \Upsilon^\pm(t, y) &= [\Upsilon_1^\pm(t, y) \quad \Upsilon_2^\pm(t, y)]' = (\Sigma_y^\pm)^{-1} \xi(t, y). \end{aligned} \quad (3.18)$$

*Proof.* Let  $G_\pm(u^\pm)$  be the maximization problem that relaxes the  $\overline{\Pi}_\pm(t)$  in  $H_\pm(u)$  in (3.13) to  $\Pi(t)$ , so that

$$G_\pm(u) = \sup_{u \in \Pi(t)} \left\{ \frac{1}{2} u' \Sigma_y^\pm u V_{xx} + u' [\beta V_x + \mathcal{M}_y V_x] \right\}.$$

Substituting the solution form of  $V$  in (3.14) into  $G_\pm(\tilde{u}^\pm)$  deduces that

$$\tilde{u}^\pm = \Upsilon^\pm(t, y) R(t, x).$$

If  $\tilde{u}^\pm \in \Pi_\pm(t)$ , then  $u^\pm = \tilde{u}^\pm$ , where  $u^\pm$  is the maximizer of the problem  $H_\pm(u^\pm)$ ; otherwise,  $u^\pm \in \Pi_0(t)$ . In the latter situation, we know that  $u_1^\pm u_2^\pm = 0$ , so either  $u_1^\pm = 0$  or  $u_2^\pm = 0$ . If  $u_2^\pm = 0$ , then the maximization problem  $H_\pm(u^\pm)$  reduces to

$$\begin{aligned} H_\pm(u^\pm) &= \sup_{u_1^\pm \in \mathbb{R}} \left\{ \frac{1}{2} (u_1^\pm)^2 \sigma_1^2(y_1) V_{xx} + u_1^\pm [\beta_1 + \rho_1 \alpha_1(y_1) v_{y_1}/v] V_x \right\} \\ &\Rightarrow u^\pm = \Lambda_y^1 \xi(t, y) R(t, x). \end{aligned}$$

Similarly, if  $u_1^\pm = 0$ , then  $u^\pm = \Lambda_y^2 \xi(t, y) R(t, x)$ .

Consider the maximization problem  $\sup_{u \in \{u^+, u^-\}} (H_+(u^+), H_-(u^-))$  in the HJBI equation (3.12). There are four possible cases for the maximizer  $u^*$ .

1. If  $\tilde{u}^+ \in \Pi_+(t)$  and  $\tilde{u}^- \notin \Pi_-(t)$ , then  $\tilde{u}^- \in \Pi_0(t) \in \overline{\Pi}_+(t)$  and it implies that  $u^* = u^+ = \tilde{u}^+$ .

2. If  $\tilde{u}^+ \notin \Pi_+(t)$  and  $\tilde{u}^- \in \Pi_-(t)$ , then  $\tilde{u}^+ \in \Pi_0(t) \in \overline{\Pi_-}(t)$  and it implies that  $u^* = u^- = \tilde{u}^-$ .
3. If  $\tilde{u}^+ \in \Pi_+(t)$  and  $\tilde{u}^- \in \Pi_-(t)$ , then  $u^+ = \tilde{u}^+$  and  $u^- = \tilde{u}^-$ , so that

$$\begin{aligned} u^* &= \arg \max(H_+(u^+), H_-(u^-)) \\ &= \left(-\frac{V_x}{V_{xx}}\right) \max[\xi(t, y)'(\Sigma_y^+)^{-1}\xi(t, y), \xi(t, y)'(\Sigma_y^-)^{-1}\xi(t, y)]. \end{aligned}$$

In other words,  $u^* = \tilde{u}^+$  if  $\xi(t, y)'(\Sigma_y^+)^{-1}\xi(t, y) > \xi(t, y)'(\Sigma_y^-)^{-1}\xi(t, y)$ , and  $u^* = \tilde{u}^-$  otherwise.

4. If  $\tilde{u}^+ \notin \Pi_+(t)$  and  $\tilde{u}^- \notin \Pi_-(t)$ , then  $u^+, u^- \in \Pi_0(t)$  so that  $u^* \in \Pi_0(t)$ . From the previous analysis, we know that

$$u^* = \left(-\frac{V_x}{V_{xx}}\right) \max[\xi(t, y)'\Lambda_1\xi(t, y), \xi(t, y)'\Lambda_2\xi(t, y)].$$

From all four cases, it is clear that the sign of  $u_1^*u_2^*$  is independent of the wealth level  $x$ , but depends on the signs of  $\tilde{u}_1^+\tilde{u}_2^+$  and  $\tilde{u}_1^-\tilde{u}_2^-$ . The expression of  $\tilde{u}^\pm$  shows that

$$\text{sign}(\tilde{u}_1^\pm\tilde{u}_2^\pm) = \text{sign}(\Upsilon_1^\pm\Upsilon_2^\pm).$$

Hence, we have the following four sets corresponding to the four cases:

$$\begin{aligned} \Omega^+ &= \{(t, y) \in \mathbb{R}^+ \times \mathbb{R}^2 \mid \tilde{u}^+ \in \Pi_+(t), \tilde{u}^- \notin \Pi_-(t)\}, \\ \Omega^- &= \{(t, y) \in \mathbb{R}^+ \times \mathbb{R}^2 \mid \tilde{u}^+ \notin \Pi_+(t), \tilde{u}^- \in \Pi_-(t)\}, \\ \widehat{\Omega} &= \{(t, y) \in \mathbb{R}^+ \times \mathbb{R}^2 \mid \tilde{u}^+ \in \Pi_+(t), \tilde{u}^- \in \Pi_-(t)\}, \\ \Omega_0 &= \{(t, y) \in \mathbb{R}^+ \times \mathbb{R}^2 \mid \tilde{u}^+ \notin \Pi_+(t), \tilde{u}^- \notin \Pi_-(t)\}, \end{aligned}$$

where  $\Omega^+, \Omega^-, \widehat{\Omega}$ , and  $\Omega_0$  are alternatively described through  $\Upsilon^\pm$  in (3.18). In addition, the optimal feedback strategy  $u^*$  can be written as in (3.14).

It is easy to verify that  $(V, u^*)$  in (3.14) satisfies the HJBI equation (3.12), or equivalently (3.8). The verification procedure resembles the proof of Lemma 3.2.  $\square$

Once the utility function satisfies the condition of Theorem 3.3, it characterizes the solution pair  $(V, u^*)$  of the optimal portfolio problem with SV and an uncertain correlation. Similarly to the BS model with an ambiguous correlation, risk-averse investors make their investment decisions based on market situations reflected by the variance risk ratio, but the variance risk ratio should be adjusted for the additional risk associated with SV in the SV economy. Mathematically, the adjustment terms are  $\frac{1}{v}(\Sigma_y^\pm)^{-1}\mathcal{M}_y v$  because  $\Upsilon^+ = \overline{\Upsilon} - \frac{1}{v}(\Sigma_y^+)^{-1}\mathcal{M}_y v$  and  $\Upsilon^- = \underline{\Upsilon} - \frac{1}{v}(\Sigma_y^-)^{-1}\mathcal{M}_y v$ . Therefore, the financial interpretation derived from the BS model still holds, except that the determination of market situations relies on the volatility-adjusted variance risk ratios  $\Upsilon^\pm$  in the SV economy.

Theorem 3.3 also reduces the optimization problem (3.5) to solving the nonlinear PDE (3.16) for  $v$ , in which both  $\xi(t, y)$  and  $\widehat{\Sigma}_{t, y}^{-1}$  are functions (functionals) of  $v$  itself. It is very difficult to derive the solution of (3.16) given its highly nonlinear nature. Instead, we consider an approximation obtained in the regime of FMRSV.

**3.2. FMRSV.** In addition to (3.2), the FMRSV model with an ambiguous correlation that we consider here assumes that  $\mu_i(y)$  and  $\sigma_i(y)$  are bounded functions, while  $m_i(y) = \frac{1}{\epsilon}(\theta_i - y)$  and  $\alpha_i(y) = \frac{1}{\sqrt{\epsilon}}\nu_i$  for some constants  $\theta_i, \nu_i, i = 1, 2$ , and  $0 < \epsilon \ll 1$ . In other words, the two SV processes, chosen here as Ornstein–Uhlenbeck have the same order of fast mean-reversion rate. Hence,

$$dS_i(t) = \mu_i(Y_i(t))S_i(t) dt + \sigma_i(Y_i(t))S_i(t) dB_i(t),$$

$$dY_i(t) = \frac{1}{\epsilon}(\theta_i - Y_i(t)) dt + \frac{1}{\sqrt{\epsilon}}\nu_i dB_{i+2}(t) \text{ for } i = 1, 2. \quad (3.19)$$

This model is discussed in full details in [17]. We are interested in the problem (3.5) where the volatility process  $Y(t) = [Y_1(t) \ Y_2(t)]'$  follows the dynamics (3.19).

**3.2.1. The full feedback strategy.** The SV models (3.1) with (3.2) and (3.19) both assume that the volatility process  $Y(t) = [Y_1(t) \ Y_2(t)]'$  is observed or can be filtered from historical data (see [16]). In such a situation, it is reasonable to look for the optimal strategy  $u^*(t) = u^*(t, X(t), Y(t))$ , which is a function that depends on the realized wealth level,  $X(t) = x$ , and the volatility process,  $Y(t) = y = [y_1 \ y_2]'$ . We call this optimal strategy the full feedback strategy because it incorporates the full information generated by the processes of assets and volatilities. However, market practitioners may prefer to extract information from forward-looking volatility surfaces rather than filtering out the realized volatilities. In this latter consideration, the investor may be interested in the partial feedback strategy that is independent of  $Y$ , so that  $u^*(t) = u^*(t, X(t))$ . This type of strategy is also considered in [19] and references therein. We study the former consideration in this subsection and the latter in the next subsection.

According to the FMRSV model in (3.1) and (3.19), we have  $\mathcal{L} = \mathcal{L}_0/\epsilon$  and  $\mathcal{M}_y = \mathcal{M}_y^0/\sqrt{\epsilon}$ . Hence, the PDE (3.16) reads

$$v_t + \frac{1}{\epsilon}\mathcal{L}_0 v - \frac{c}{2}\xi^\epsilon(t, y)' \widehat{\Sigma}_{t, y}^{-1} \xi^\epsilon(t, y) v = 0, \quad v(T, y) = 1, \quad (3.20)$$

where  $0 < \epsilon \ll 1$ ,

$$\begin{aligned} \mathcal{L}_0 &= \mathcal{L}_1^0 + \mathcal{L}_1^0, \quad \mathcal{L}_i^0 = \frac{1}{2}\nu_i^2 \frac{\partial^2}{\partial y_i^2} + (\theta_i - y_i) \frac{\partial}{\partial y_i}, \\ \xi^\epsilon(t, y) &= \beta + \frac{\mathcal{M}_y^0 v}{v\sqrt{\epsilon}}, \quad \mathcal{M}_y^0 = \begin{pmatrix} \mathcal{M}_{y_1}^0 \\ \mathcal{M}_{y_2}^0 \end{pmatrix} = \begin{pmatrix} \rho_1 \nu_1 \sigma_1 (y_1) \partial_{y_1} \\ \rho_2 \nu_2 \sigma_2 (y_2) \partial_{y_2} \end{pmatrix}. \end{aligned} \quad (3.21)$$

Consider the asymptotic expansion for  $v$  denoted here by  $v^\epsilon$ :

$$v^\epsilon(t, y) = v^{(0)}(t, y) + \sqrt{\epsilon}v^{(1)}(t, y) + \epsilon v^{(2)}(t, y) + \epsilon^{\frac{3}{2}}v^{(3)}(t, y) + \dots, \quad (3.22)$$

where  $v^{(i)}$  depends on  $\xi^\epsilon$ , which is also a function of  $v^\epsilon$  itself. This looped feature resembles the free boundary problems for pricing American options, induced by PDE methods. The American option pricing problem involves two regions, exercise and continuation, separated by the optimal exercise boundary. The price function and the boundary also have an inseparable looped feature. We refer the reader to [28] for an asymptotic approach to American option pricing problem. However, our problem has multiple regions, which are characterized by (3.17). Fortunately, the optimal switching decision only depends on  $\xi$ . Therefore, analogously to [28], we also need an asymptotic expansion for  $\xi^\epsilon$

$$\xi^\epsilon(t, y) = \xi^{(0)}(t, y) + \sqrt{\epsilon}\xi^{(1)}(t, y) + \dots. \quad (3.23)$$

We aim to derive the first-order approximation for  $v^\epsilon$  so that the objective function is close to the optimal one with an error of  $o(\sqrt{\epsilon})$ . After substituting the expansion (3.22) of  $v$  into (3.20), we collect terms according to the order of  $\epsilon$ . Specifically, the highest-order term is of  $\mathcal{O}(\epsilon^{-1})$ :

$$\mathcal{L}_0 v^{(0)} - \frac{c}{2} \left( \mathcal{M}_y^0 v^{(0)} \right)' \left( \widehat{\Sigma}_y^{(0)} \right)^{-1} \left( \mathcal{M}_y^0 v^{(0)} \right) / v^{(0)} = 0,$$

where  $(\widehat{\Sigma}_y^{(0)})^{-1} = \widehat{\Sigma}_{t,y}^{-1}|_{\xi^\epsilon = \xi^{(0)}}$ . As  $\mathcal{L}_0$  and  $\mathcal{M}_y^0$  only involves differentiations with respect to  $y_1$  and  $y_2$ , this equation derived from the  $\mathcal{O}(\epsilon^{-1})$  terms is satisfied if  $v^{(0)}$  is independent of  $y$ . Therefore, we seek a leading-order term  $v^{(0)}$  which only depends on  $t$ . This choice is also consistent with the target expansion for  $\xi^\epsilon$  in (3.23) because  $\xi^\epsilon$  would blow up as  $\epsilon \rightarrow 0$  if  $v^{(0)}$  depends on  $y$ ; see (3.21). Hence,  $\xi^{(0)} = \beta$ .

With  $v^{(0)} = v^{(0)}(t)$ , the equation derived from  $\mathcal{O}(\epsilon^{-1/2})$  is  $\mathcal{L}_0 v^{(1)} = 0$ . As for  $v^{(0)}$ , we seek  $v^{(1)} = v^{(1)}(t)$ , a function that is independent of  $y$  so that this equation is satisfied. To collect the  $\mathcal{O}(1)$  terms, we need to investigate the expansion for  $\widehat{\Sigma}_{t,y}^{-1}$  or  $\beta' \widehat{\Sigma}_{t,y}^{-1} \beta$ , which depends on  $\xi^\epsilon$ . The following lemmas are useful.

**LEMMA 3.4.** *Suppose that a perturbed function  $\omega^\epsilon$  has the following expansion,*

$$\omega^\epsilon = \omega^{(0)} + \sqrt{\epsilon} \omega^{(1)} + \epsilon \omega^{(2)} + \dots.$$

For any integer  $p \geq 2$ ,

$$|\mathbb{1}_{\{\omega^\epsilon > 0\}} - \mathbb{1}_{\{\omega^{(0)} > 0\}}| = \mathbb{1}_{\{\omega^\epsilon > 0, \omega^{(0)} \leq 0\}} + \mathbb{1}_{\{\omega^{(0)} > 0, \omega^\epsilon \leq 0\}} \leq 2 \left| \frac{\omega^{(1)}}{\omega^{(0)}} \right|^p \epsilon^{\frac{p}{2}} + o(\epsilon^{\frac{p}{2}}).$$

*Proof.* Let  $I_1 = \mathbb{1}_{\{\omega^\epsilon > 0, \omega^{(0)} \leq 0\}}$  and  $I_2 = \mathbb{1}_{\{\omega^{(0)} > 0, \omega^\epsilon \leq 0\}}$ ,

$$\begin{aligned} I_1 &= \frac{\omega^\epsilon - \omega^{(0)} + \omega^{(0)}}{\omega^\epsilon} I_1 \leq \frac{\omega^\epsilon - \omega^{(0)}}{\omega^\epsilon} I_1 \leq \left( \frac{\omega^\epsilon - \omega^{(0)}}{\omega^\epsilon} \right)^2 I_1 \\ &\leq \dots \leq \left( \frac{\omega^\epsilon - \omega^{(0)}}{\omega^\epsilon} \right)^p I_1 \leq \left| \frac{\omega^{(1)}}{\omega^{(0)}} \right|^p \epsilon^{\frac{p}{2}} + o(\epsilon^{\frac{p}{2}}). \end{aligned}$$

Similarly, by replacing  $\omega^\epsilon$  with  $\omega^{(0)}$  in the denominator, we can prove that  $I_2$  has the same upper bound.  $\square$

**Remark:** By the same principle as in the proof of Lemma 3.4, we find that

$$\begin{aligned} &\max(|\mathbb{1}_{\{\omega^\epsilon < 0\}} - \mathbb{1}_{\{\omega^{(0)} < 0\}}|, |\mathbb{1}_{\{\omega^\epsilon \geq 0\}} - \mathbb{1}_{\{\omega^{(0)} \geq 0\}}|, |\mathbb{1}_{\{\omega^\epsilon \leq 0\}} - \mathbb{1}_{\{\omega^{(0)} \leq 0\}}|) \\ &< 2 \left| \frac{\omega^{(1)}}{\omega^{(0)}} \right|^p \epsilon^{\frac{p}{2}} + o(\epsilon^{\frac{p}{2}}). \end{aligned}$$

Lemma 3.4 implies the following lemma, which gives us the error bound when  $\beta' \widehat{\Sigma}_{t,y}^{-1} \beta$  is approximated by  $\beta' (\widehat{\Sigma}_y^{(0)})^{-1} \beta$ .

**LEMMA 3.5.** *For any integer  $p \geq 2$ , there exists a constant  $C$  and a function  $f(t, y)$  such that*

$$|\beta' \widehat{\Sigma}_{t,y}^{-1} \beta - \beta' (\widehat{\Sigma}_y^{(0)})^{-1} \beta| \leq C \epsilon^{\frac{p}{2}} f^p(t, y) + o(\epsilon^{\frac{p}{2}}).$$

*Proof.* We find that

$$\begin{aligned} &\beta' \widehat{\Sigma}_{t,y}^{-1} \beta - \beta' (\widehat{\Sigma}_y^{(0)})^{-1} \beta \\ &= \beta' (\Sigma_y^+)^{-1} \beta (\mathbb{1}_{\{\widehat{\Omega}_+\}} - \mathbb{1}_{\{\widehat{\Omega}_+^{(0)}\}}) + \beta' (\Sigma_y^-)^{-1} \beta (\mathbb{1}_{\{\widehat{\Omega}_-\}} - \mathbb{1}_{\{\widehat{\Omega}_-^{(0)}\}}) \\ &\quad + \beta' \Lambda_y^1 \beta (\mathbb{1}_{\{\Omega_1\}} - \mathbb{1}_{\{\Omega_1^{(0)}\}}) + \beta' \Lambda_y^2 \beta (\mathbb{1}_{\{\Omega_2\}} - \mathbb{1}_{\{\Omega_2^{(0)}\}}), \end{aligned}$$

where  $\Omega_i^{(0)} = \Omega_i|_{\xi=\beta}$ ,  $\widehat{\Omega}_{\pm}^{(0)} = \widehat{\Omega}_{\pm}|_{\xi=\beta}$ . Then,  $|\beta' \widehat{\Sigma}_{t,y}^{-1} \beta - \beta' (\widehat{\Sigma}_y^{(0)})^{-1} \beta|$  is bounded above by

$$C \left[ \mathbb{1}_{\{\widehat{\Omega}_+ \setminus \widehat{\Omega}_+^{(0)}\}} + \mathbb{1}_{\{\widehat{\Omega}_- \setminus \widehat{\Omega}_-^{(0)}\}} + \mathbb{1}_{\{\Omega_1 \setminus \Omega_1^{(0)}\}} + \mathbb{1}_{\{\Omega_2 \setminus \Omega_2^{(0)}\}} \right. \\ \left. + \mathbb{1}_{\{\widehat{\Omega}_+^{(0)} \setminus \widehat{\Omega}_+\}} + \mathbb{1}_{\{\widehat{\Omega}_-^{(0)} \setminus \widehat{\Omega}_-\}} + \mathbb{1}_{\{\Omega_1^{(0)} \setminus \Omega_1\}} + \mathbb{1}_{\{\Omega_2^{(0)} \setminus \Omega_2\}} \right]$$

for some constant  $C$ , because  $\beta' (\Sigma_y^+)^{-1} \beta$ ,  $\beta' (\Sigma_y^-)^{-1} \beta$ ,  $\beta' \Lambda_y^1 \beta$ ,  $\beta' \Lambda_y^2 \beta$  are bounded due to the boundedness of  $\sigma_i(y_i)$ ,  $\mu_i(y_i)$ . The analyses for all eight indicator functions above are similar, and thus we illustrate the analysis with the example of  $\mathbb{1}_{\{\widehat{\Omega}_+ \setminus \widehat{\Omega}_+^{(0)}\}}$ .

Noting the definition of  $\Omega$ 's in Theorem 3.3, we have

$$\mathbb{1}_{\{\widehat{\Omega}_+ \setminus \widehat{\Omega}_+^{(0)}\}} \leq \mathbb{1}_{\{\Upsilon_1^+ \Upsilon_2^+ > 0, \Upsilon_1^{(0)+} \Upsilon_2^{(0)+} \leq 0\}} + \mathbb{1}_{\{\Upsilon_1^- \Upsilon_2^- \geq 0, \Upsilon_1^{(0)-} \Upsilon_2^{(0)-} < 0\}} \\ + \mathbb{1}_{\{\Upsilon_1^- \Upsilon_2^- < 0, \Upsilon_1^{(0)-} \Upsilon_2^{(0)-} \geq 0\}} + \mathbb{1}_{\{\xi' [(\Sigma_y^+)^{-1} - (\Sigma_y^-)^{-1}] \xi > 0, \beta' [(\Sigma_y^+)^{-1} - (\Sigma_y^-)^{-1}] \beta \leq 0\}}.$$

Recall that

$$\Upsilon^{\pm} = (\Sigma_y^{\pm})^{-1} \left( \beta + \sqrt{\epsilon} \frac{\mathcal{M}_y^0 v^{(2)}}{v^{(0)}} \right) + o(\sqrt{\epsilon}) =: \begin{pmatrix} \Upsilon_1^{(0)\pm} \\ \Upsilon_2^{(0)\pm} \end{pmatrix} + \sqrt{\epsilon} \begin{pmatrix} \Upsilon_1^{(1)\pm} \\ \Upsilon_2^{(1)\pm} \end{pmatrix} + o(\sqrt{\epsilon}).$$

Hence,

$$\Upsilon_1^{\pm} \Upsilon_2^{\pm} = \Upsilon_1^{(0)\pm} \Upsilon_2^{(0)\pm} + \sqrt{\epsilon} (\Upsilon_1^{(0)\pm} \Upsilon_2^{(1)\pm} + \Upsilon_1^{(1)\pm} \Upsilon_2^{(0)\pm}) + o(\sqrt{\epsilon}).$$

We let  $\check{\Sigma}$  be the matrix, which could be  $(\Sigma_y^+)^{-1} - (\Sigma_y^-)^{-1}$  or  $(\Lambda_y^1)^{-1} - (\Lambda_y^2)^{-1}$ , to simplify the notation. We expand

$$\xi' \check{\Sigma} \xi = \beta' \check{\Sigma} \beta + 2\sqrt{\epsilon} \beta' \check{\Sigma} \frac{\mathcal{M}_y^0 v^{(2)}}{v^{(0)}} + o(\sqrt{\epsilon}).$$

Direct application of Lemma 3.4 completes the proof and  $C$  and  $f(t, y)$  are identified as

$$C = 16 \sup_y (\beta' (\Sigma_y^+)^{-1} \beta, \beta' (\Sigma_y^-)^{-1} \beta, \beta' \Lambda_y^1 \beta, \beta' \Lambda_y^2 \beta),$$

which is a constant due to the boundedness of  $\mu_i$  and  $\sigma_i$ , and

$$f(t, y)^p = (\Upsilon_1^{(0)+} \Upsilon_2^{(1)+} + \Upsilon_1^{(1)+} \Upsilon_2^{(0)+})^p + (\Upsilon_1^{(0)-} \Upsilon_2^{(1)-} + \Upsilon_1^{(1)-} \Upsilon_2^{(0)-})^p \\ + \left( 2\beta' ((\Sigma_y^+)^{-1} - (\Sigma_y^-)^{-1}) \frac{\mathcal{M}_y^0 v^{(2)}}{v^{(0)}} \right)^p + \left( 2\beta' ((\Lambda_y^1)^{-1} - (\Lambda_y^2)^{-1}) \frac{\mathcal{M}_y^0 v^{(2)}}{v^{(0)}} \right)^p.$$

□

The implication of Lemma 3.5 is that we are allowed to use  $\beta' (\widehat{\Sigma}_y^{(0)})^{-1} \beta$  in place of  $\beta' \widehat{\Sigma}_{t,y}^{-1} \beta$  when collecting the terms in all orders of  $\epsilon$  in (3.20). Consequently, if the optimal  $\rho$  is chosen to be the same as that in the zeroth-order case, which is deterministic, the order of accuracy of the objective function will not be affected up to the first order of the asymptotic solution.

Collecting  $\mathcal{O}(1)$  terms, we have

$$\mathcal{L}_0 v^{(2)} + v_t^{(0)} - \frac{c}{2} \beta' (\widehat{\Sigma}_y^{(0)})^{-1} \beta v^{(0)} = 0. \quad (3.24)$$

This is a Poisson equation for  $v^{(2)}$ , whose solvability condition (see Chapter 3 in [17]) requires that the source term be centered:

$$\left\langle v_t^{(0)} - \frac{c}{2} \beta' (\widehat{\Sigma}_y^{(0)})^{-1} \beta v^{(0)} \right\rangle = v_t^{(0)} - \frac{c}{2} \left\langle \beta' (\widehat{\Sigma}_y^{(0)})^{-1} \beta \right\rangle v^{(0)} = 0, \quad v^{(0)}(T) = 1, \quad (3.25)$$

where  $\langle \cdot \rangle$  is the average with respect to the invariant distribution of  $Y$ :

$$\langle g(y_1, y_2) \rangle = \int \int g(y_1, y_2) \frac{1}{\pi \nu_1 \nu_2} e^{-\frac{2(y_1 - \theta_1)^2}{\nu_1^2} - \frac{2(y_2 - \theta_2)^2}{\nu_2^2}} dy_1 dy_2,$$

since  $Y_1$  and  $Y_2$  are independent Ornstein–Uhlenbeck processes. In addition, this implies that

$$\mathcal{L}_0 v^{(2)} = \frac{c}{2} \left( \beta' (\widehat{\Sigma}_y^{(0)})^{-1} \beta - \left\langle \beta' (\widehat{\Sigma}_y^{(0)})^{-1} \beta \right\rangle \right) v^{(0)}. \quad (3.26)$$

Let  $\theta(y)$  be a solution of the Poisson equation  $\mathcal{L}_0 \theta(y) = \beta' (\widehat{\Sigma}_y^{(0)})^{-1} \beta - \langle \beta' (\widehat{\Sigma}_y^{(0)})^{-1} \beta \rangle$  defined up to an additive constant in  $y$ . By (3.26), we express  $v^{(2)}$  as follows:

$$v^{(2)}(t, y) = \frac{c}{2} [\theta(y) + \chi(t)] v^{(0)}(t),$$

where  $\chi(t)$  is a function independent of  $y$ . From (3.25), it is obvious that

$$v^{(0)}(t) = \exp \left( -\frac{c}{2} \left\langle \beta' (\widehat{\Sigma}_y^{(0)})^{-1} \beta \right\rangle (T - t) \right).$$

Collecting the  $\mathcal{O}(\sqrt{\epsilon})$  terms in (3.20), with the use of Lemma 3.5 ( $p \geq 2$ ), gives

$$\mathcal{L}_0 v^{(3)} + v_t^{(1)} - \frac{c}{2} \left[ \beta' (\widehat{\Sigma}_y^{(0)})^{-1} \beta v^{(1)} + 2\beta' (\widehat{\Sigma}_y^{(0)})^{-1} \mathcal{M}_y^0 v^{(2)} \right] = 0.$$

The solvability condition for  $v^{(3)}$  and the substitution of  $v^{(2)}$  together yield

$$v_t^{(1)} - \frac{c}{2} \left[ \left\langle \beta' (\widehat{\Sigma}_y^{(0)})^{-1} \beta \right\rangle v^{(1)} + c \left\langle \beta' (\widehat{\Sigma}_y^{(0)})^{-1} \mathcal{M}_y^0 \theta(y) \right\rangle v^{(0)} \right] = 0$$

with the terminal condition  $v^{(1)}(T) = 0$ . It is easy to verify that

$$v^{(1)}(t) = -\frac{c^2 v^{(0)}(t)}{2} \left\langle \beta' (\widehat{\Sigma}_y^{(0)})^{-1} \mathcal{M}_y^0 \theta(y) \right\rangle (T - t).$$

By substituting the expansion of  $v(t, y)$  into (3.15), we have

$$\xi^\epsilon(t, y) = \beta + \sqrt{\epsilon} \frac{c}{2} \mathcal{M}_y^0 \theta(y) + \dots$$

Therefore, the first-order approximation of the optimal trading strategy reads

$$u_\epsilon^*(t, x, y) \simeq -\frac{U_x(k(t)x)}{k(t)U_{xx}(k(t)x)} \widehat{\Sigma}_y^{-1} \left[ \beta + \sqrt{\epsilon} \frac{c}{2} \mathcal{M}_y^0 \theta(y) \right],$$

where  $k(t) = \exp\left\{\int_t^T r(\tau)d\tau\right\}$ , and the first-order approximation of the optimal value function is given by

$$\begin{aligned} V^\epsilon(t, x, y) &= U(k(t)x)v^\epsilon(t, y) \\ &\simeq U(k(t)x) \left[ 1 - \sqrt{\epsilon} \frac{c^2(T-t)}{2} \left\langle \beta'(\widehat{\Sigma}_y^{(0)})^{-1} \mathcal{M}_y^0 \theta(y) \right\rangle \right] e^{-\frac{\epsilon}{2} \langle \beta'(\widehat{\Sigma}_y^{(0)})^{-1} \beta \rangle (T-t)}. \end{aligned} \quad (3.27)$$

Thus, we have derived the first-order SV approximation for optimal portfolio selection with an uncertain correlation. An accuracy result for this approximation amounts to proving that pointwise in  $(t, y)$ , there exists a constant  $C$  independent of  $\epsilon \leq 1$  such that  $|v^\epsilon(t, y) - (v^{(0)}(t) + \sqrt{\epsilon}v^{(1)}(t))| \leq C\epsilon$ . Such a result is challenging due the nonlinear nature of the PDE (3.20) satisfied by  $v^\epsilon$ . A proof is provided in [19] (Section 6.3) using a distortion transformation which linearizes the equation and then applying classical asymptotic techniques for linear PDEs. Unfortunately, this transformation works only in the case with one SV factor. In fact, as already observed in [19] in the case of a single stock, the zeroth-order term of the strategy produces the value function up to the first-order approximation. We develop this point in the following section providing an alternative way to obtain accuracy among a smaller set of admissible strategies, namely, those converging as  $\epsilon \rightarrow 0$ .

This is treated in full detail in [14] in the context of slowly varying volatility and in [15] in the case of fast mean-reverting volatility which we refer to for a precise definition of asymptotic optimality of the zeroth-order strategy among a subclass of admissible strategies. The proofs are quite lengthy and should carry out without additional difficulties to our situation. In fact, since we essentially consider here power utilities, the associated separation of variables makes the argument much simpler in the sense that the necessary regularity conditions of the zeroth-order value function with respect to the variable  $x$  are explicitly imposed on the utility function. As noted in these references, this asymptotic optimality does not imply optimality in the full class of admissible strategies. Obtaining such a result, in the classical Merton problem in varying environment or in the context of uncertain correlation and varying environment as in the present paper, would imply working with viscosity solutions and their expansions. This is ongoing research and certainly beyond the scope of this paper.

**3.2.2. Using the moving Merton strategy.** In this subsection, we demonstrate that using the ‘‘moving Merton’’ zeroth-order strategy  $(u^{(0)}, \rho^{(0)})$ ,

$$u^{(0)}(t, x, y) = -\frac{U_x(k(t)x)}{k(t)U_{xx}(k(t)x)} (\widehat{\Sigma}_y^{(0)})^{-1} \beta, \quad \rho^{(0)} = \bar{\rho} \mathbb{1}_{\{\widehat{\Omega}_+^{(0)}\}} + \underline{\rho} \mathbb{1}_{\{\widehat{\Omega}_-^{(0)}\}},$$

where  $(\widehat{\Sigma}_y^{(0)})^{-1} = \widehat{\Sigma}_{t,y}^{-1}|_{\xi^\epsilon = \xi^{(0)}}$  and  $\widehat{\Omega}_\pm^{(0)} = \widehat{\Omega}_\pm|_{\xi = \beta}$ , can produce the first-order optimal value function (3.27); therefore, the corrections to the strategy result in the value function only at the  $V^{(2)}$  term (order  $\epsilon$ ).

Using the ‘‘moving Merton’’ strategy  $(u^{(0)}, \rho^{(0)})$ , which moves with the volatility factor  $Y(t)$ , the wealth process  $X(t)$  follows

$$\begin{aligned} dX(t) &= [rX(t) + \beta' u^{(0)}]dt + u_1^{(0)} \sigma_1(Y_1(t)) dB_1(t) + u_2^{(0)} \sigma_2(Y_2(t)) dB_2(t), \\ dY_i(t) &= \frac{1}{\epsilon} (\theta_i - Y_i(t))dt + \frac{1}{\sqrt{\epsilon}} \nu_i dB_{i+2}(t), \quad i = 1, 2, \end{aligned}$$

where  $\mathbb{E}[dW_1(t)dW_2(t)] = \rho^{(0)}dt$ . Then, the corresponding value function is given by

$$\tilde{V}(t, x, y_1, y_2) = \mathbb{E}^{\mathbb{P}_{\rho^{(0)}}}[U(X(T))|X(t) = x, Y(t) = y],$$

which solves the linear PDE

$$\begin{aligned} \tilde{V}_t + \mathcal{L}\tilde{V} + \frac{\tilde{R}^2(t, x)}{2}\beta'(\hat{\Sigma}_y^{(0)})^{-1}\tilde{\Sigma}_y(\hat{\Sigma}_y^{(0)})^{-1}\beta\tilde{V}_{xx} \\ + \tilde{R}(t, x)\beta'(\hat{\Sigma}_y^{(0)})^{-1}(\beta + \mathcal{M}_y)\tilde{V}_x + rx\tilde{V}_x = 0 \end{aligned}$$

with the terminal condition  $\tilde{V}(T, x, y) = U(x)$ , where  $\tilde{R}(t, x) = -\tilde{V}_x/\tilde{V}_{xx}$  and  $\tilde{\Sigma}_y = \Sigma_y|_{\rho=\rho^{(0)}}$ . It is easy to verify that  $(\hat{\Sigma}_y^{(0)})^{-1}\tilde{\Sigma}_y(\hat{\Sigma}_y^{(0)})^{-1} = (\hat{\Sigma}_y^{(0)})^{-1}$ . In addition, if the utility function satisfies  $\frac{U_x^2}{U_{xx}} = c$ , we can express  $\tilde{V}(t, x, y) = U(k(t)x)\tilde{v}(t, y)$ , where  $k(t) = \exp\left(\int_t^T r(\tau)d\tau\right)$ , and  $\tilde{v}$  satisfies the linear PDE

$$\tilde{v}_t + \frac{1}{\epsilon}\mathcal{L}_0\tilde{v} - \frac{c}{2}\beta'(\hat{\Sigma}_y^{(0)})^{-1}\beta\tilde{v} - \frac{c}{\sqrt{\epsilon}}\beta'(\hat{\Sigma}_y^{(0)})^{-1}\mathcal{M}_y^0\tilde{v} = 0, \quad \tilde{v}(T, y) = 1, \quad (3.28)$$

where  $\mathcal{L}_0$  and  $\mathcal{M}_y^0$  are defined in (3.21).

Now, we consider the expansion for  $\tilde{v}$  denoted here by  $\tilde{v}^\epsilon$ :

$$\tilde{v}^\epsilon(t, y) = \tilde{v}^{(0)}(t, y) + \sqrt{\epsilon}\tilde{v}^{(1)}(t, y) + \epsilon\tilde{v}^{(2)}(t, y) + \epsilon^{\frac{3}{2}}\tilde{v}^{(3)}(t, y) + \dots,$$

and show that  $\tilde{v}^{(0)} \equiv v^{(0)}$ ,  $\tilde{v}^{(1)} \equiv v^{(1)}$ , such that  $\tilde{V}^\epsilon$  coincides with  $V^\epsilon$  up to and including the order of  $\sqrt{\epsilon}$ .

Inserting the expansion into (3.28) and collecting the order  $\epsilon^{-1}$  terms gives  $\mathcal{L}_0\tilde{v}^{(0)} = 0$ . Hence, we choose  $\tilde{v}^{(0)} = \tilde{v}^{(0)}(t)$  independent of  $y$ , which satisfies this equation. At the order of  $\epsilon^{-\frac{1}{2}}$ , we have  $\mathcal{L}_0\tilde{v}^{(1)} = 0$  and again we choose  $\tilde{v}^{(1)} = \tilde{v}^{(1)}(t)$  independent of  $y$ .

Collecting the  $\mathcal{O}(1)$  terms in (3.28) gives

$$\mathcal{L}_0\tilde{v}^{(2)} + \tilde{v}_t^{(2)} - \frac{c}{2}\beta'(\hat{\Sigma}_y^{(0)})^{-1}\beta\tilde{v}^{(2)} = 0,$$

which is the same as (3.24). Using the same arguments, we conclude that  $\tilde{v}^{(2)} \equiv v^{(2)}$  and  $\tilde{v}^{(2)} \equiv v^{(2)}$ . Next, collecting the  $\mathcal{O}(\sqrt{\epsilon})$  terms in (3.28) gives

$$\mathcal{L}_0\tilde{v}^{(3)} + \tilde{v}_t^{(3)} - \frac{c}{2}\left[\beta'(\hat{\Sigma}_y^{(0)})^{-1}\beta\tilde{v}^{(3)} + 2\beta'(\hat{\Sigma}_y^{(0)})^{-1}\mathcal{M}_y^0\tilde{v}^{(2)}\right] = 0, \quad \tilde{v}^{(3)}(T) = 0.$$

The solvability condition for  $\tilde{v}^{(3)}$  and the expression of  $\tilde{v}^{(2)}$  yields

$$\tilde{v}_t^{(3)} - \frac{c}{2}\left[\left\langle\beta'(\hat{\Sigma}_y^{(0)})^{-1}\beta\right\rangle\tilde{v}^{(3)} + c\left\langle\beta'(\hat{\Sigma}_y^{(0)})^{-1}\mathcal{M}_y^0\theta(y)\right\rangle\tilde{v}^{(2)}\right] = 0, \quad \tilde{v}^{(3)}(T) = 0.$$

Thus, we have  $\tilde{v}^{(1)} \equiv v^{(1)}$ .

To sum up, using the ‘‘moving Merton’’ strategy  $(u^{(0)}, \rho^{(0)})$  can recover the optimal value function up to the order of  $\sqrt{\epsilon}$ . A rigorous proof of accuracy of this result follows using the asymptotic techniques for linear PDEs developed in [17] in the context of FMRSV models and the notion of asymptotic optimality among a subclass of admissible strategies introduced in [14] and [15]. We omit the lengthy details here.

However, in order to implement this strategy, one needs to know the coefficients  $(\mu_i(y_i), \sigma_i(y_i))$  and the varying levels of volatilities  $(y_1, y_2)$ . Practitioners may prefer to use strategies which do not require a high-frequency filtering of volatility levels. For that, we propose in the following subsection a partial feedback strategy that is independent of the SV factors but instead relies on averaged coefficients which can be calibrated to implied volatilities.

**3.3. The partial feedback (practical) strategy.** Next, we investigate the SV asymptotic strategy with partial feedback,  $u^*(t) = u^*(t, X(t))$ , which is independent of  $Y$ . We show that this partial feedback strategy can incorporate effective parameters calibrated to the implied volatility surfaces. Hence, the trading strategy is of the forward-looking type. To distinguish between the full and partial feedback strategies, we relabel the latter as  $w(t, X(t))$ . In what follows, we assume that  $\mu_i(y_i) \equiv \mu_i$  in (3.1) are constants for  $i = 1, 2$ . They can be thought as sample means of historical returns. We do not require the estimation of the SV factors and we refer to [16] for a filtering approach to stochastic drifts in asset returns.

To simplify matters, we revise our model setting from ambiguous correlation to ambiguous covariance:  $\eta \in [\bar{\eta}, \underline{\eta}]$ , where  $\eta = \rho\sigma_1(y_1)\sigma_2(y_2)$ . Note that the FMRSV model assumes that  $\sigma_1(\cdot)$  and  $\sigma_2(\cdot)$  are bounded functions and are bounded away from zero. As we show below that the zeroth-order approximation is indeed the optimal strategy in the BS economy in which  $\sigma_i^2$  is replaced by  $\langle \sigma_i(y_i)^2 \rangle$ , this revised setting produces the same zeroth-order approximation as the original setting but shortens the proof.

Similarly to (3.5), the (robust) value function of portfolio optimization problem under the FMRSV model with ambiguous covariance and a partial feedback strategy is formulated as

$$\mathcal{V}(t, x, y) = \sup_{w \in \Pi_p(t)} \inf_{\eta \in [\underline{\eta}, \bar{\eta}]} \mathbb{E}^{\mathbb{P}^\eta} [U(X(T)) | X(t) = x, Y(t) = y], \quad (3.29)$$

where  $X$  is driven by (3.4) with  $\mu_i(y_i) \equiv \mu_i$  and (3.19); the set of priors is defined similarly as in Section 2.1, and  $\Pi_p(t) = \{w \in M^2(t, T) : w = w(t, X(t))\}$ .

Based on the  $G$ -HJB framework in [20], if  $\mathcal{U}$ , the domain of  $u$ , is compact, then the value function  $\mathcal{V}$  is the unique deterministic continuous viscosity function of the following HJBI equation analogously to Theorem 3.1:

$$\mathcal{V}_t + \frac{1}{\epsilon} \mathcal{L}_0 \mathcal{V} + \sup_{w \in \Pi_p(t)} \inf_{\eta \in [\underline{\eta}, \bar{\eta}]} \left\{ \frac{1}{2} w' \Gamma_y w \mathcal{V}_{xx} + w' \left[ \beta \mathcal{V}_x + \frac{1}{\sqrt{\epsilon}} \mathcal{M}_y^0 \mathcal{V}_x \right] \right\} + r x \mathcal{V}_x = 0 \quad (3.30)$$

with the terminal condition  $\mathcal{V}(T, x, y) = U(x)$ , where  $\mathcal{L}_0$  and  $\mathcal{M}_y^0$  are defined in (3.21) and

$$\Gamma_y = \begin{pmatrix} \sigma_1^2(y_1) & \eta \\ \eta & \sigma_2^2(y_2) \end{pmatrix}, \quad \beta = \begin{pmatrix} \mu_1 - r \\ \mu_2 - r \end{pmatrix}.$$

Here,  $\Gamma_y$  is essentially the same as  $\Sigma_y$  in the previous subsection. By minimizing the linear expression involving  $\eta$ , (3.30) becomes

$$\mathcal{V}_t + \frac{1}{\epsilon} \mathcal{L}_0 \mathcal{V} + \sup_{w \in \Pi_p(t)} \left\{ \frac{1}{2} w' \Gamma_{t,y}^* w \mathcal{V}_{xx} + w' \left[ \beta \mathcal{V}_x + \frac{1}{\sqrt{\epsilon}} \mathcal{M}_y^0 \mathcal{V}_x \right] \right\} + r x \mathcal{V}_x = 0, \quad (3.31)$$

where  $\Gamma_{t,y}^* = \Gamma_y |_{\eta=\eta^*}$  and  $\eta^* = \bar{\eta} \mathbb{1}_{\{w_1 w_2 > 0\}} + \underline{\eta} \mathbb{1}_{\{w_1 w_2 < 0\}}$ .

As we are looking for a partial feedback strategy, Theorem 3.3 no longer holds because the function  $v$ , which appears in both the objective function and the strategy in Theorem 3.3, depends on  $y$ . Thus, we cannot reduce problem (3.29) into a single nonlinear PDE for  $v$ . Rather, consider the pair of asymptotic expansions

$$\mathcal{V}^\epsilon(t, x, y) = \mathcal{V}^{(0)}(t, x, y) + \sqrt{\epsilon} \mathcal{V}^{(1)}(t, x, y) + \epsilon \mathcal{V}^{(2)}(t, x, y) + \epsilon^{\frac{3}{2}} \mathcal{V}^{(3)}(t, x, y) + \dots$$

and

$$w(t, x) = w^{(0)}(t, x) + \sqrt{\epsilon}w^{(1)}(t, x) + \dots$$

Similarly, we derive the first-order approximation for  $\mathcal{V}$  and  $w$ . After substituting these two expansions into (3.31), we collect terms according to the power of  $\epsilon$ . The  $\mathcal{O}(\epsilon^{-1})$  term is  $\mathcal{L}_0\mathcal{V}^{(0)} = 0$ . This equation is satisfied by  $\mathcal{V}^{(0)}(t, x)$ , which is independent of  $y$ . With this choice, we explore the equation  $\mathcal{O}(\epsilon^{-1/2})$ :  $\mathcal{L}_0\mathcal{V}^{(1)} = 0$ . Again, we choose  $\mathcal{V}^{(1)} = \mathcal{V}^{(1)}(t, x)$ .

From the  $\mathcal{O}(1)$  terms, we have

$$\sup_{w^{(0)} \in \Pi_p(t)} \left\{ \mathcal{L}_0\mathcal{V}^{(2)} + \mathcal{V}_t^{(0)} + \frac{1}{2}w^{(0)'} \Gamma_y^{(0)} w^{(0)} \mathcal{V}_{xx}^{(0)} + w^{(0)'} \beta \mathcal{V}_x^{(0)} + rx \mathcal{V}_x^{(0)} \right\} = 0 \quad (3.32)$$

with terminal condition  $\mathcal{V}^{(0)}(T, x) = U(x)$ , where  $\Gamma_y^{(0)} = \Gamma_y^*|_{w=w^{(0)}}$ . To make the quantity in the above bracket independent of  $y$  (and thus ensure the  $y$ -independence of  $w^{(0)}$ ), we choose  $\mathcal{V}^{(2)}$  to be a solution of the Poisson equation

$$\begin{aligned} \mathcal{L}_0\mathcal{V}^{(2)} &= -\frac{1}{2}w^{(0)'} \Gamma_y^{(0)} w^{(0)} \mathcal{V}_{xx}^{(0)} + \frac{1}{2}w^{(0)'} \langle \Gamma_y^{(0)} \rangle w^{(0)} \mathcal{V}_{xx}^{(0)} \\ &= -\frac{1}{2} \left[ w_1^{(0)2} (\sigma_1^2(y_1) - \langle \sigma_1^2(y_1) \rangle) + w_2^{(0)2} (\sigma_2^2(y_2) - \langle \sigma_2^2(y_2) \rangle) \right] \mathcal{V}_{xx}^{(0)}. \end{aligned} \quad (3.33)$$

Since the source of the above Poisson equation for  $\mathcal{V}^{(2)}$  is centered, a solution exists. Then, (3.32) becomes

$$\mathcal{V}_t^{(0)} + \sup_{w^{(0)} \in \Pi_p(t)} \left\{ \frac{1}{2}w^{(0)'} \langle \Gamma_y^{(0)} \rangle w^{(0)} \mathcal{V}_{xx}^{(0)} + w^{(0)'} \beta \mathcal{V}_x^{(0)} \right\} + rx \mathcal{V}_x^{(0)} = 0. \quad (3.34)$$

The following proposition summarizes the solution for  $\mathcal{V}^{(0)}$ .

**THEOREM 3.6.** *If the utility function satisfies  $U_x > 0, U_{xx} < 0$ , and  $\frac{U_x^2}{U U_{xx}} \equiv c$  for some constant  $c$ , then the HJB equation (3.34) has the solution pair*

$$\begin{aligned} \mathcal{V}^{(0)}(t, x) &= U(k(t)x) \exp \left\{ -\frac{c(T-t)}{2} \beta' \langle \Gamma_y^{(0)} \rangle^{-1} \beta \right\}, \\ w^{(0)}(t, x) &= -\langle \Gamma_y^{(0)} \rangle^{-1} \beta \frac{\mathcal{V}_x^{(0)}}{\mathcal{V}_{xx}^{(0)}}, \end{aligned}$$

where  $k(t) = e^{\int_t^T r(\tau) d\tau}$ ,  $\Psi^{(0)\pm} = [\Psi_1^{(0)\pm} \ \Psi_2^{(0)\pm}]'$  :=  $\langle \Gamma_y^\pm \rangle^{-1} \beta$ , and

$$\begin{aligned} \Gamma_y^+ &= \begin{pmatrix} \sigma_1^2(y_1) & \bar{\eta} \\ \bar{\eta} & \sigma_2^2(y_2) \end{pmatrix}, \Gamma_y^- = \begin{pmatrix} \sigma_1^2(y_1) & \eta \\ \eta & \sigma_2^2(y_2) \end{pmatrix}, \\ \Delta_y^1 &= \begin{pmatrix} \langle \sigma_1^2(y_1) \rangle^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \Delta_y^2 = \begin{pmatrix} 0 & 0 \\ 0 & \langle \sigma_2^2(y_2) \rangle^{-1} \end{pmatrix}, \\ \langle \Gamma_y^{(0)} \rangle^{-1} &= \langle \Gamma_y^+ \rangle^{-1} \mathbb{1}_{\{\widehat{\Xi}_+^{(0)}\}} + \langle \Gamma_y^- \rangle^{-1} \mathbb{1}_{\{\widehat{\Xi}_-^{(0)}\}} + \Delta_y^1 \mathbb{1}_{\{\Xi_1^{(0)}\}} + \Delta_y^2 \mathbb{1}_{\{\Xi_2^{(0)}\}}, \\ \Xi_+^{(0)} &= \{\Psi_1^{(0)+} \Psi_2^{(0)+} > 0, \Psi_1^{(0)-} \Psi_2^{(0)-} \geq 0\}, \\ \Xi_-^{(0)} &= \{\Psi_1^{(0)+} \Psi_2^{(0)+} \leq 0, \Psi_1^{(0)-} \Psi_2^{(0)-} < 0\}, \\ \widehat{\Xi}^{(0)} &= \{\Psi_1^{(0)+} \Psi_2^{(0)+} > 0, \Psi_1^{(0)-} \Psi_2^{(0)-} < 0\}, \\ \Xi_0^{(0)} &= \{\Psi_1^{(0)+} \Psi_2^{(0)+} \leq 0, \Psi_1^{(0)-} \Psi_2^{(0)-} \geq 0\}, \\ \widehat{\Xi}_\pm^{(0)} &= \Xi_\pm^{(0)} \cup \{\widehat{\Xi}^{(0)} | \beta' \langle \Gamma_y^+ \rangle^{-1} \beta \geq \beta' \langle \Gamma_y^- \rangle^{-1} \beta\}, \\ \Xi_1^{(0)} &= \{\Xi_0^{(0)} | \beta' \Delta_y^1 \beta > \beta' \Delta_y^2 \beta\}, \Xi_2^{(0)} = \{\Xi_0^{(0)} | \beta' \Delta_y^1 \beta \leq \beta' \Delta_y^2 \beta\}. \end{aligned} \quad (3.35)$$

*Proof.* As (3.34) is similar to (3.8), this proof is almost the same as that of Theorem 3.3, and hence is omitted.  $\square$

It is interesting to link the zeroth-order practical solution to the solution obtained in Section 2 for the BS model with constant volatilities. If we write  $\bar{\eta} = \bar{\rho} \sigma_1 \sigma_2$  and  $\eta = \rho \sigma_1 \sigma_2$ , then the consideration of  $\eta \in [\eta, \bar{\eta}]$  is equivalent to that of  $\rho \in [\bar{\rho}, \rho]$ . It can be shown that Theorem 3.6 still holds true for the model setting with an ambiguous correlation  $\rho$ , and the proof only needs some minor (but tedious) modifications.

**COROLLARY 3.7.** *If the utility function satisfies  $U_x > 0, U_{xx} < 0$ , and  $\frac{U_x^2}{U_{xx}} \equiv c$  for some constant  $c$ , and  $\mu(y_i) \equiv \mu_i$  are constants for  $i = 1, 2$ , then the solution pair  $(\mathcal{V}^{(0)}(t, x), w^{(0)}(t, x))$  is the one obtained in Section 2 in the case of constant coefficients, except that  $\sigma_i^2$  is replaced by  $\langle \sigma_i^2(y_i) \rangle$  for  $i = 1, 2$  and the belief of  $\rho \in [\rho, \bar{\rho}]$  is replaced by  $\rho \in [\underline{\rho}, \bar{\rho}]$ , where  $\underline{\rho} = \underline{\rho} \frac{\langle \sigma_1(y_1) \sigma_2(y_2) \rangle}{\sqrt{\langle \sigma_1^2(y_1) \rangle \langle \sigma_2^2(y_2) \rangle}}$ ,  $\bar{\rho} = \bar{\rho} \frac{\langle \sigma_1(y_1) \sigma_2(y_2) \rangle}{\sqrt{\langle \sigma_1^2(y_1) \rangle \langle \sigma_2^2(y_2) \rangle}}$  so that  $|\underline{\rho}| \leq |\rho|$  and  $|\bar{\rho}| \leq |\bar{\rho}|$ .*

*Proof.* For each case in Theorem 2.2, we can solve for the value function of the form in Theorem 3.6. Note that the boundaries defined in Theorem 2.2 separate the whole space into  $\widehat{\Xi}_\pm^{(0)}$ ,  $\Xi_1^{(0)}$ , and  $\Xi_2^{(0)}$ . Thus, it is trivial to check that the solution pair  $(\mathcal{V}^{(0)}(t, x), w^{(0)}(t, x))$  is equivalent to the solution discussed in Section 2, except that  $\sigma_i^2$  is replaced by  $\langle \sigma_i^2(y_i) \rangle$  for  $i = 1, 2$ ,  $\bar{\Sigma}$  is replaced by  $\langle \Gamma_y^+ \rangle$ , and  $\underline{\Sigma}$  is replaced by  $\langle \Gamma_y^- \rangle$ . In addition, we recognize that

$$\langle \Gamma_y^+ \rangle = \begin{pmatrix} \frac{\langle \sigma_1^2(y_1) \rangle}{\bar{\rho} \sqrt{\langle \sigma_1^2(y_1) \rangle \langle \sigma_2^2(y_2) \rangle}} & \bar{\rho} \sqrt{\langle \sigma_1^2(y_1) \rangle \langle \sigma_2^2(y_2) \rangle} \\ \bar{\rho} \sqrt{\langle \sigma_1^2(y_1) \rangle \langle \sigma_2^2(y_2) \rangle} & \langle \sigma_2^2(y_2) \rangle \end{pmatrix},$$

which is equivalent to the  $\bar{\Sigma}$  of the case in which  $\sigma_i^2$  is replaced by  $\langle \sigma_i^2(y_i) \rangle$  for  $i = 1, 2$  and  $\bar{\rho}$  is replaced by  $\bar{\rho}$ . By Hölder's inequality, we have  $|\bar{\rho}| \leq |\bar{\rho}|$ . A similar argument applies to  $\underline{\rho}$  in  $\underline{\Sigma}$ .  $\square$

Corollary 3.7 not only spells out the connection between the solution with constant coefficients and the zeroth-order approximation for the partial feedback strategy, but also confirms the potential use of the information contained in implied volatility surfaces because the averaged square volatilities (or effective square volatilities),

$\langle \sigma_i^2(y_i) \rangle$ , can be calibrated from the implied volatility smile of the risky asset  $i$  for  $i = 1, 2$ . Alternatively, these average square volatilities can be calibrated from past returns. The calibration procedure is detailed in [17]. In addition, if  $\underline{\rho} < 0 < \bar{\rho}$ , then  $[\underline{\varrho}, \bar{\varrho}] \subset [\underline{\rho}, \bar{\rho}]$ . Therefore, the uncertainty region becomes narrower after incorporating implied volatility information. Therefore, the zeroth-order approximation of the practical solution alone already outperforms the solution with constant volatilities once the risky assets follow the FMRSV model and the volatility surfaces are available. The reason is that the zeroth-order approximation has less uncertainty in the correlation when  $[\underline{\varrho}, \bar{\varrho}] \subset [\underline{\rho}, \bar{\rho}]$ .

To collect the  $\mathcal{O}(\sqrt{\epsilon})$  terms, we establish a lemma, analogously to Lemma 3.5, to examine the accuracy of  $w^{(0)'} \Gamma_y^{(0)} w^{(0)}$ .

LEMMA 3.8. *For any integer  $p \geq 2$ , there is a constant  $C$  and a function  $g(t, y)$ , such that*

$$|w^{(0)'} \Gamma_{t,y}^* w^{(0)} - w^{(0)'} \Gamma_y^{(0)} w^{(0)}| \leq C \epsilon^{\frac{p}{2}} g^p(t, y) + o(\epsilon^{\frac{p}{2}}).$$

*Proof.* We write

$$\begin{aligned} & w^{(0)'} \Gamma_{t,y}^* w^{(0)} - w^{(0)'} \Gamma_y^{(0)} w^{(0)} \\ &= w^{(0)'} \Gamma_y^+ w^{(0)} (\mathbb{1}_{\{w_1 w_2 > 0\}} - \mathbb{1}_{\{w_1^{(0)} w_2^{(0)} > 0\}}) + w^{(0)'} \Gamma_y^- w^{(0)} (\mathbb{1}_{\{w_1 w_2 < 0\}} - \mathbb{1}_{\{w_1^{(0)} w_2^{(0)} < 0\}}). \end{aligned}$$

Because  $w^{(0)'} \Gamma_y^+ w^{(0)}$  and  $w^{(0)'} \Gamma_y^- w^{(0)}$  are bounded, we have the upper bound for  $|w^{(0)'} \Gamma_{t,y}^* w^{(0)} - w^{(0)'} \Gamma_y^{(0)} w^{(0)}|$  given by

$$C \left[ \mathbb{1}_{\{w_1 w_2 > 0, w_1^{(0)} w_2^{(0)} \leq 0\}} + \mathbb{1}_{\{w_1 w_2 \leq 0, w_1^{(0)} w_2^{(0)} > 0\}} + \mathbb{1}_{\{w_1 w_2 < 0, w_1^{(0)} w_2^{(0)} \geq 0\}} + \mathbb{1}_{\{w_1 w_2 \geq 0, w_1^{(0)} w_2^{(0)} < 0\}} \right].$$

Note that

$$w_1 w_2 = w_1^{(0)} w_2^{(0)} + \sqrt{\epsilon} (w_1^{(0)} w_2^{(1)} + w_1^{(1)} w_2^{(0)}) + o(\sqrt{\epsilon}).$$

From Lemma 3.4, we have the desirable upper bounds for the above four indicator functions, and thus the proof is complete.  $\square$

Collecting the  $\mathcal{O}(\sqrt{\epsilon})$  terms, we have

$$\begin{aligned} \sup_{w^{(1)} \in \Pi_p(t)} \left\{ \mathcal{L}_0 \mathcal{V}^{(3)} + \mathcal{V}_t^{(1)} + \frac{1}{2} w^{(0)'} \Gamma_y^{(0)} w^{(0)} \mathcal{V}_{xx}^{(1)} + w^{(0)'} \beta \mathcal{V}_x^{(1)} + r x \mathcal{V}_x^{(1)} \right. \\ \left. + w^{(0)'} \Gamma_y^{(0)} w^{(1)} \mathcal{V}_{xx}^{(0)} + w^{(1)'} \beta \mathcal{V}_x^{(0)} + w^{(0)'} \mathcal{M}_y^0 \mathcal{V}_x^{(2)} \right\} = 0 \end{aligned}$$

with the terminal condition  $\mathcal{V}^{(1)}(T, x, y) = 0$ . Similarly, we choose  $\mathcal{V}^{(3)}$ , such that  $\mathcal{V}^{(1)}$  is independent of  $Y$ . Then,

$$\mathcal{V}_t^{(1)} + \frac{1}{2} w^{(0)'} \langle \Gamma_y^{(0)} \rangle w^{(0)} \mathcal{V}_{xx}^{(1)} + w^{(0)'} \beta \mathcal{V}_x^{(1)} + r x \mathcal{V}_x^{(1)} + w^{(0)'} \langle \mathcal{M}_y^0 \rangle \mathcal{V}_x^{(2)} = 0,$$

where  $w^{(0)}$  was obtained from Theorem 3.6. It can be seen that this equation does not involve  $w^{(1)}$ . Thus,  $w^{(1)}$  does not contribute to  $\mathcal{V}^{(1)}$ . By (3.34), we express  $\mathcal{V}^{(2)}$  as

$$\mathcal{V}^{(2)} = -\frac{1}{2} \left[ w_1^{(0)2} \phi_1(y_1) + w_2^{(0)2} \phi_2(y_2) + K(t) \right] \mathcal{V}_{xx}^{(0)},$$

where  $\mathcal{L}_0^1 \phi_1(y_1) = \sigma_1^2(y_1) - \langle \sigma_1^2(y_1) \rangle$  and  $\mathcal{L}_0^2 \phi_2(y_2) = \sigma_2^2(y_2) - \langle \sigma_2^2(y_2) \rangle$ . Therefore, we have

$$w^{(0)'} \langle \mathcal{M}_y^0 \mathcal{V}_x^{(2)} \rangle = -c^2 \beta' \langle \Gamma_y^{(0)} \rangle^{-1} \begin{pmatrix} \Psi_1^{(0)2} \Phi_1 \\ \Psi_2^{(0)2} \Phi_2 \end{pmatrix} \mathcal{V}^{(0)} := -c^2 \mathcal{K} \mathcal{V}^{(0)},$$

where  $\Phi_i = -\frac{\rho_i}{2} \langle a_i(y_i) \sigma_i(y_i) \frac{\partial \phi_i}{\partial y_i} \rangle$ ,  $i = 1, 2$ , and  $\Phi^\epsilon = \sqrt{\epsilon} \Phi_i$  is a calibrated parameter from the volatility surface implied by the options on risky asset  $i$  for  $i = 1, 2$ . In addition,  $\Psi^{(0)} = [\Psi_1^{(0)} \quad \Psi_2^{(0)}]' = \langle \Gamma_y^{(0)} \rangle^{-1} \beta$ . By the assumed property of the utility function, we have  $\mathcal{V}^{(2)}(t, x, y) = U(k(t)x) \nu^{(2)}(t, y)$ . It is then easy to verify that

$$\mathcal{V}^{(1)}(t, x) = -\frac{c^2 \mathcal{V}^{(0)}(t, x)}{2} \mathcal{K}(T-t).$$

Therefore, the optimal practical strategy resembles the corresponding optimal strategy in the BS economy, as shown in Theorem 3.6. This strategy ensures the objective value function is close to its optimal value, up to  $\mathcal{O}(\sqrt{\epsilon})$ . Specifically,

$$\mathcal{V}(t, x) = U(k(t)x) \left[ 1 - \frac{c^2}{2} \mathcal{K}^\epsilon(T-t) \right] \exp \left\{ -\frac{c(T-t)}{2} \beta' \langle \Gamma_y^{(0)} \rangle^{-1} \beta \right\} + \mathcal{O}(\epsilon),$$

where  $\mathcal{K}^\epsilon = \beta' \langle \Gamma_y^{(0)} \rangle^{-1} \begin{pmatrix} \Psi_1^{(0)2} \Phi_1^\epsilon \\ \Psi_2^{(0)2} \Phi_2^\epsilon \end{pmatrix}$ , which can be calibrated using the skews of implied volatilities as developed in [17].

#### 4. Discussion and future works.

**4.1. The choice of correlation bounds.** When we assume  $\rho \in [\underline{\rho}, \bar{\rho}]$ , a natural question concerns how the bounds are obtained in practice. The same concern appears in all models with ambiguous parameters and the standard procedure may involve the statistical interval estimate of the parameter, which in our case is the correlation. Practically, one may use the confidence interval or the support of a posterior distribution from a Bayesian perspective as used in [10] for uncertain volatility. Here, we would like to point out that the interval  $[\underline{\rho}, \bar{\rho}]$  is narrower after incorporating the volatility information as shown in Corollary 3.7.

Another concern is the generalization to high-dimensional portfolios. This is certainly an interesting area for future research. The setting can be based on defining the set of all uncertain correlations. As the correlation matrix is positive definite, the uncertainty set can be constructed using the positivity of the principal minors of the matrix. We believe, however, that the technique developed in this paper can be generalized to such cases.

**4.2. Other SV models.** Although we use the FMRSV model and derive asymptotic solutions, Theorem 3.3 does provide a general framework for reducing the portfolio problem under general SV models to a nonlinear PDE problem. One could try a different SV model to see if the PDE can be solved analytically. We have put the Heston model on our research agenda.

**5. Conclusion.** This paper introduces the notion of ambiguous correlation to optimal portfolio management. We derive a closed-form solution to the Merton problem in the case of risky assets following a BS model, and an asymptotic solution for the FMRSV models. We show that the portfolio selection problem with an ambiguous

correlation and stochastic volatilities can be transformed into a highly nonlinear PDE problem. The PDE resembles an optimal switching problem.

The major economic finding is that the optimal investment decision that is robust to the correlation mainly depends on the variance risk ratios projected onto each risky asset using the most unfavorable correlation. This reflects an additional ambiguity aversion against the uncertain correlation. If the market has liquidly traded options on single assets, then the information contained in the options through the implied volatility surfaces can be used to calibrate an approximated model with reduced uncertainty band in the correlation risk.

**Appendix A. Regularity conditions on  $\{\Theta_t\}$ .** The following technical regularity conditions on  $\{\Theta_t\}$ , specified in [11], are imposed:

1. *Measurability*: The correspondence  $(t, \omega) \rightarrow \Theta_t(\omega)$  on  $[0, s] \times \Omega$  is  $\mathcal{B}([0, s]) \times \mathcal{F}_s$ -measurable for every  $0 < s \leq T$ .
2. *Uniform boundedness*: There is a compact subset  $\mathcal{S}$  in  $\mathbb{R}$  such that  $\Theta_t : \Omega \rightarrow \mathcal{S}$  for each  $t$ .
3. *Compact-convex*: Each  $\Theta_t$  is compact valued and convex valued.
4. *Uniform nondegeneracy*: There is  $\hat{a}$ , a  $2 \times 2$  real-valued positive definite matrix, such that for every  $t$  and  $\omega$ , if  $\rho \in \Theta_t(\omega)$ , then  $\Sigma \geq \hat{a}$ , where  $\Sigma$  is the variance-covariance matrix of the assets returns.
5. *Uniform continuity*: The process  $\{\Theta_t\}$  is uniformly continuous.
6. *Uniform interiority*: There exists  $\delta > 0$  such that  $ri^\delta \Theta_t(\omega) \neq \emptyset$  for all  $t$  and  $\omega$ , where  $ri^\delta \Theta_t(\omega)$  is the  $\delta$ -relative interior of  $\Theta_t(\omega)$ . (For any  $D \subset \mathbb{R}$  and  $\delta > 0$ ,  $ri^\delta D \equiv \{x \in D : (x + B_\delta(x)) \cap (\text{aff } D) \in D\}$ , where  $\text{aff } D$  is the affine hull of  $D$  and  $B_\delta(x)$  denotes the open ball of radius  $\delta$ .)
7. *Uniform affine hull*: The affine hulls of  $\Theta_{t'}(\omega')$  and  $\Theta_t(\omega)$  are the same for every  $(t', \omega')$  and  $(t, \omega)$  in  $[0, T] \times \Omega$ .

**Appendix B. Terminology of  $G$ -framework.** This appendix summarizes and applies a few main results about  $G$ -framework in [24, 25, 26, 20] to the problem of our interest. Let  $\mathcal{H}$  be a linear space of real-valued functions defined on  $\Omega$  and  $\mathcal{H}$  can be considered as the space of random variables. Then it can be verified that  $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$  defines a sublinear expectation space; see [26, 20] for more details of nonlinear expectation space and the related optimal control problem.

Let  $C_{Lip}(\mathbb{R}^2)$  be the space of real continuous functions defined on  $\mathbb{R}^2$  such that  $\phi \in C_{Lip}(\mathbb{R}^2)$  satisfies

$$|\phi(x) - \phi(y)| \leq C(1 + \|x\|^k + \|y\|^k)|x - y| \quad \forall x, y \in \mathbb{R}^2$$

for  $k$  and  $C$  which depend only on  $\phi$ . Peng [24, 25, 26] introduces the  $G$ -normal distribution and  $G$ -Brownian motion using nonlinear parabolic PDE.

**Definition of  $G$ -normal distribution.** For each  $\phi \in C_{Lip}(\mathbb{R}^2)$ , define

$$\varphi(t, x) := \hat{\mathbb{E}}[\phi(x + \sqrt{t}B)], \quad (t, x) \in [0, \infty) \times \mathbb{R}^2.$$

Then,  $B$  is  $G$ -normally distributed if and only if  $\varphi$  is the viscosity solution of the following (nonlinear)  $G$ -heat equation

$$\frac{\partial \varphi}{\partial t} - G_\Gamma(D^2 \varphi) = 0, \quad \varphi(0, x) = \phi(x), \quad G_\Gamma(A) := \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr}[\gamma \gamma^T A]$$

for  $A \in \mathbb{S}_2$  and  $\Gamma \subset \mathbb{R}^{2 \times 2}$  is bounded and closed. For the application of this paper, we take

$$\Gamma = \left\{ \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} : \rho \in [\underline{\rho}, \bar{\rho}] \right\}. \quad (\text{B.1})$$

A  $G$ -normal distribution is a nonlinear expectation  $P_1^G(\phi) := \varphi(1, 0)$  for  $\phi \in C_{Lip}(\mathbb{R}^2)$ . An equivalent definition of  $G$ -normal distribution can be found in [26].

Let

$$L_{ip}(\Omega) := \{\phi(\omega(t_1), \dots, \omega(t_n)) : n \geq 1, t_1, \dots, t_n \in [0, \infty), \phi \in C_{Lip}(\mathbb{R}^{2 \times n})\}, \quad (\text{B.2})$$

and  $\{\xi_n\}_{n \geq 1}$  be a sequence of identically distributed two-dimensional  $G$ -normally distributed random vectors in  $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$  such that  $\xi_{i+1}$  is independent of  $(\xi_1, \dots, \xi_i)$  for every  $i \geq 1$ . Here, “ $X$  and  $Y$  are identically distributed” means that  $\hat{\mathbb{E}}[\phi(X)] = \hat{\mathbb{E}}[\phi(Y)] \forall \phi \in C_{Lip}(\mathbb{R}^2)$ , and “ $Y$  is independent of  $X$ ” means that  $\hat{\mathbb{E}}[\phi(X, Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\phi(x, Y)]_{x=X}]$ ,  $\forall \phi \in C_{Lip}(\mathbb{R}^{2 \times 2})$ .

**Definitions of  $G$ -expectation, conditional  $G$ -expectation, and  $G$ -Brownian motion.** For each  $X = \varphi(\omega(t_1) - \omega(t_0), \omega(t_2) - \omega(t_1), \dots, \omega(t_m) - \omega(t_{m-1})) \in L_{ip}(\Omega)$  with  $0 \leq t_0 < \dots < t_m$ ,  $m \geq 1$ , the  $G$ -expectation of  $X$  is defined by

$$\hat{\mathbb{E}}[X] = \hat{\mathbb{E}}[\varphi(\sqrt{t_1 - t_0}\xi_1, \dots, \sqrt{t_m - t_{m-1}}\xi_m)].$$

The conditional  $G$ -expectation of  $X$  given  $\mathcal{F}_{t_i}$  is defined by

$$\begin{aligned} & \hat{\mathbb{E}}[\varphi(\omega(t_1) - \omega(t_0), \omega(t_2) - \omega(t_1), \dots, \omega(t_m) - \omega(t_{m-1})) | \mathcal{F}_{t_i}] \\ &= \tilde{\varphi}(\omega(t_1) - \omega(t_0), \omega(t_2) - \omega(t_1), \dots, \omega(t_i) - \omega(t_{i-1})), \end{aligned}$$

where  $\tilde{\varphi}(x_1, \dots, x_i) = \hat{\mathbb{E}}[\varphi(x_1, \dots, x_i, \omega(t_{i+1}) - \omega(t_i), \dots, \omega(t_m) - \omega(t_{m-1}))]$ .

$(\Omega, L_{ip}(\Omega), \hat{\mathbb{E}})$  is called a  $G$ -expectation space, and the corresponding canonical process  $\{B(t, \omega) = \omega(t)\}_{t \geq 0}$  is called a  $G$ -Brownian motion. Let  $\hat{L}^p(\Omega)$  be the completion of  $L_{ip}(\Omega)$  under the norm  $\|\xi\|_p = (\hat{\mathbb{E}}|\xi|^p)^{1/p}$  for  $p \geq 1$ . For each  $t \geq 0$ ,  $\hat{\mathbb{E}}[\cdot | \mathcal{F}_t]$  can be extended continuously to  $\hat{L}^p(\Omega)$  under the norm  $\|\cdot\|_p$ . The properties of  $\hat{\mathbb{E}}[\cdot | \mathcal{F}_t]$  can be found in [25].

**Definition of stochastic integral of  $G$ -Brownian motion.** Let  $B(t)$  be the two-dimensional  $G$ -Brownian motion defined above, and  $\eta(t, \omega) \in M^{2,0}(0, T)$  is a  $(d \times 2)$ -matrix process of the form  $\eta(t, \omega) = \sum_{i=0}^{N-1} \xi_i(\omega) \mathbf{1}_{[t_i, t_{i+1})}(t)$  for  $d \geq 1$ . We define

$$I(\eta) = \int_0^T \eta(s) dB(s) := \sum_{i=0}^{N-1} \xi_i(B(t_{i+1}) - B(t_i)).$$

Then we have for each  $\eta \in M^{2,0}(0, T)$ ,

$$\begin{aligned} & \hat{\mathbb{E}} \left[ \int_0^T \eta(s) dB(s) \right] = 0, \\ & \hat{\mathbb{E}} \left[ \left( \int_0^T \eta(s) dB(s) \right) \left( \int_0^T \eta(s) dB(s) \right)' \right] \leq \hat{\mathbb{E}} \left[ \int_0^T \eta(s) \begin{pmatrix} 1 & \rho(s) \\ \rho(s) & 1 \end{pmatrix} \eta(s)' ds \right]. \end{aligned}$$

Hence, the linear mapping  $I : M^{2,0}(0, T) \rightarrow \hat{L}^2(\mathcal{F}_T)$  is continuous and thus can be continuously extended to  $I : M^2(0, T) \rightarrow \hat{L}^2(\mathcal{F}_T)$ . Therefore, for  $\eta(t, \omega) \in M^2(0, T)$ , the corresponding stochastic integral of Itô's type is defined by

$$\int_0^T \eta(s) dB(s) := I(\eta).$$

**Acknowledgments.** We would like to thank the anonymous referees for their valuable comments. H.Y. Wong acknowledges the Direct Research Grant of the Chinese University of Hong Kong.

## REFERENCES

- [1] Y. AIT-SAHALIA, J. FAN AND D. XIU, *High frequency covariance estimates with noisy and asynchronous financial data*, J. Amer. Statist. Assoc., 105 (2010), pp. 1504-1517.
- [2] M. AVELLANEDA, A. LEVY AND A. PARAS, *Pricing and hedging derivatives securities in markets with uncertain volatility*, Appl. Math. Finance, 2 (1995), pp. 73-88.
- [3] R. BUCKDAHN AND J. LI, *Stochastic differential games and viscosity solutions of Hamilton–Jacobi–Bellman–Isaacs equations*, SIAM J. Control Optim., 47 (2008), pp. 444-475.
- [4] R. BUFF, *Uncertain Volatility Models: Theory and Applications*, Springer, Berlin, 2002.
- [5] A. BURASCHI, P. PORCHIA AND F. TROJANI, *Correlation risk and optimal portfolio choice*, J. Finance, 65 (2010), pp. 393-420.
- [6] Z. CHEN AND L. EPSTEIN, *Ambiguity, risk, and asset returns in continuous Time*, Econometrica, 70 (2002), pp. 1403-1443.
- [7] M.C. CHIU AND H.Y. WONG, *Mean-variance portfolio selection with correlation risk*, J. Comput. Appl. Math., 263 (2014), pp. 432-444.
- [8] J.C. DUAN AND W. ZHANG, *Forward-looking market risk premium*, Manag. Sci., 60 (2014), pp. 521-538.
- [9] L. DENIS AND C. MARTINI, *A theoretical framework for the pricing of contingent claims in the presence of model uncertainty*, Ann. Appl. Probab., 16 (2006), pp. 827-852.
- [10] L.G. EPSTEIN AND S. JI, *Ambiguous volatility and asset pricing in continuous time*, Rev. Financ. Stud., 26 (2013), pp. 1740-1786.
- [11] L.G. EPSTEIN AND S. JI, *Ambiguous volatility, possibility and utility in continuous time*, J. Math. Econom., 50 (2014), pp. 269-282.
- [12] L.G. EPSTEIN AND M. SCHNEIDER, *Ambiguity, information quality and asset pricing*, J. Finance 63 (2008), pp. 197-228.
- [13] J. FAN AND J. LV *Sure independence screening for ultra-high dimensional feature space*, J. R. Stat. Soc. Ser. B Stat. Methodol., 70 (2008), pp. 849-911.
- [14] J.-P. FOUQUE AND R. HU, *Asymptotic Optimal Strategy for Portfolio Optimization in a Slowly Varying Stochastic Environment*, preprint, arxiv: 1603.03538, 2016.
- [15] J.-P. FOUQUE AND R. HU, *Asymptotic Methods for Portfolio Optimization Problems in Multiscale Stochastic Environments*, manuscript.
- [16] J.-P. FOUQUE, A. PAPANICOLAOU AND R. SIRCAR, *Filtering and Portfolio Optimization with Stochastic Unobserved Drift in Asset Returns*, Commun. in Math. Sci., 13 (2015), pp. 935-953.
- [17] J.-P. FOUQUE, G. PAPANICOLAOU, R. SIRCAR AND K. SOLNA, *Multiscale Stochastic Volatility for Equity, Interest Rate and Credit Derivatives*, Cambridge University Press, 2011.
- [18] J.-P. FOUQUE AND B. REN, *Approximation for option prices under uncertain volatility*, SIAM J. Finan. Math., 5 (2014), pp. 360-383.
- [19] J.-P. FOUQUE, R. SIRCAR AND TH. ZARIPHPOULOU, *Portfolio optimization & stochastic volatility asymptotics*, Math. Finance, to appear.
- [20] M. HU, S. JI AND S. YANG, *A stochastic recursive optimal control problem under the G-expectation framework*, Appl. Math. Optim., 70 (2014), pp. 253-278.
- [21] M. HU AND S. JI, *Dynamic Programming Principle for Stochastic Recursive Optimal Control Problem under G-framework*, preprint, arxiv: 1410.3538, 2014.
- [22] S. HERRMANN, J. MUHLE-KARBE AND F. SEIFRIED, *Hedging with small uncertainty aversion*, Finance Stoch., to appear.
- [23] P.J. HUBER, *Robust Statistics*, Wiley, New York, 1981.

- [24] S. PENG, *G-Expectation, G-Brownian Motion and Related Stochastic Calculus of Itô's Type*. in Stochastic Analysis and Applications, F.E. Benth and K. Itô, Abel Symp. 2, Springer, Berlin, 2007, pp. 541-567.
- [25] S. PENG, *Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation*, Stochastic Process. Appl., 118 (2008), pp. 2223-2253.
- [26] S. PENG, *Nonlinear Expectations and Stochastic Calculus under Uncertainty*, preprint, arxiv: 1002.4546, 2010.
- [27] C.S. PUN, *Robust Stochastic Control and High-Dimensional Statistics with Applications in Finance*, Ph.D. thesis, The Chinese University of Hong Kong, Hong Kong, 2016.
- [28] C.S. PUN AND H.Y. WONG, *CEV asymptotics of American options*, J. Math. Anal. Appl., 403 (2013), pp. 451-463.
- [29] C.S. PUN AND H.Y. WONG, *Robust investment-reinsurance optimization with multiscale stochastic volatility*, Insurance: Math. Econom., 62 (2015), pp. 245-256.
- [30] R. TEVZADZE, T. TORONJADZE AND T. UZUNASHVILI, *Robust utility maximization for a diffusion market model with misspecified coefficients*, Finance Stoch., 17 (2013), pp. 535-563.