STABILITY IN A MODEL OF INTER-BANK LENDING *

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Abstract. We propose a simple model of the banking system and analyze stochastic stability of inter-bank lending. The monetary reserves of banks are modeled as a system of interacting Feller diffusions. The model is simple enough for mathematical analysis, yet captures how lending preferences of banks affect possible multiple bank failures. In our model we quantify the lending preference from one bank to another as a function of all the reserves and find an extreme example that only k out of n banks can survive after the multiple bank failures. This banking system induces a class of random graph processes in continuous time exhibiting some stability property. Our analysis reveals quantities which can be used as indicators by regulators to assess the systemic risk.

Key words. Feller diffusion, Inter-bank lending system, Multiple defaults, Interacting particle system algorithm, Stability, Exit system, Systemic Risk.

AMS subject classifications. 60H30, 91B26, 91B70

1. Introduction. In this paper we analyze a mechanism of financial crises in a simple model of the banking industry. Building a safe banking system is one of the central issues in view of financial risk management of the whole economy, and as a result, the regulation of the banking industry plays a crucial role. However, it is often difficult to quantify the connection between regulation and safety of the system. It is partly because risky phenomena are unexpected by definition and then we lack sufficient understanding of human economic behaviors. Toward a deeper understanding of safer systems our attempt is to study a mathematical mechanism of financial crisis through a continuous-time stochastic model.

Commercial banks compete each other carrying out various risky activities based on financial technologies and strategies. These banking activities bring about monetary flow. One bank reserves less money than other banks. When funding is not available enough, the bank borrows from other banks in overnight markets for one's immediate needs. If required collateral is too high, the bank fails. Contagion effect is often observed in financial crisis: the default bank affects other banks and subsequently induces multiple defaults. In such occasion the deficits of banks spread among banks along with the monetary flow, as if they were viruses of infectious diseases or of computer network.

There have been several attempts to quantify the contagious effects in the banking network from Epidemics, Physics and Engineering. In literature also reported are cascading effects, snowball effects and herding behaviors besides the contagious effects. These effects are measured as non-linear functions of bank capitalization, in general, and are coined as *systemic risk*.

Among others, Nier et al. [28] investigate with simulations how exposure size, connectivity and degree of banking network affect the risk of contagion under a model of random Erdös-Rényi graph with a one-time idiosyncratic shock. They also provide a nice literature review from Theoretical Economics, Empirical studies, and Network studies of inter-bank contagion. Adrian & Shin [2] discuss the leverage effect of

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resizing balance sheets in financial crises. Lorenz et al. [25] introduce a general framework of cascade and contagion process on network under several discrete-time models of social, epidemic and financial cascades which include the Voter model and the Susceptible-Infective-Susceptible model. Under those models it is discussed how the differences on microscopic level translate into significant differences on macroscopic level. These models analyze the banking activities under discrete-time models.

As information and financial technologies have been developed, banking practices become more intense and seamless. It is natural to think of an approximation of banking activities by continuous-time models instead of discrete-time models. Here is a naive question: what is systemic risk of the banking system in continuous time?

In order to approach this question, we shall focus on the monetary flow in a continuous-time Markov model, and study likelihoods of multiple default events and long-time behavior. In Section 2.1 we propose a simple model of monetary flow, in terms of Feller diffusion process which originally comes from the study of Population Dynamics. In Section 2.2 the behavior of total monetary reserve is studied.

Note that there are other mathematical models for multiple defaults in continuous time, for example, see the Handbook on Systemic Risk [13] for an overview, Fouque and Sun [14] on systemic risk, Garnier, Papanicolaou and Yan [16] for some large deviation results under a bistable potential, and Giesecke and Kim [17] for a detailed empirical study based on the proportional hazard intensity model.

The diffusion model presented here is simple enough for mathematical analysis, yet captures how the individual growth rate and the lending preferences of banks affect possible multiple defaults, as it is shown in Propositions 2.2-2.3 with some examples in Section 2.3. In Section 2.4 we focus on some quantities that control the probability of defaults in the system. For the financial regulator's purposes these quantities may be monitored from instantaneous monetary flows among the banks and can be seen as health indicators of the financial system. Namely, given a level of financial crisis (or the number of simultaneous defaults), the financial regulator can determine from these quantities whether the system is close or not to the risky level. With this indicator the probability distributions of multiple default times and the number of default banks are approximated in Sections 2.5-2.6. Since it is often difficult to obtain explicit probability formulae, we utilize Interacting Particle System Algorithm to estimate the default probabilities.

Our simple model has some drawbacks, mainly the lack of *assets and liabilities accounting* and of *incentives for lending and borrowing*. These are discussed further at the end of Section 2.1. In particular, here we have not considered the strategies of the banks (in a game approach) and we focus on how the system behaves for various given lending preference structures. Introducing a game between the banks, each bank trying to optimize its action with some penalty, is out of the scope of the present paper but certainly one of the topics of our ongoing research (see for instance Carmona, Fouque and Sun [6] for an attempt in that direction). Also, recently, Rogers & Veraart [29] examine a network model of banking system and show that the banks have incentives to bailout other banks minimizing the default cost of the system. Similarly, Eisert & Eufinger [10] explain the incentives of banks to have large interbank exposures.

In Sections 2.7-2.8 we shall see that the diffusion model has the stochastic stability property under some condition on the lending preference (Proposition 2.4 and Corollary 2.2). Consequently, a stochastically stable random graph structure is derived from the diffusion model. Here the stochastic stability property is different from the financial stability. That is, a stochastically stable model does not necessarily imply safety of its finical system. For simplicity default banks are assumed to remain in the system except in Section 2.9 where we extend the original model to consider sub-systems in which default banks are not allowed to participate in the market anymore.

We conclude in Section 3 and the proof of Proposition 2.4 is given in the Appendix Section 4.

2. Model of banking system.

2.1. Feller diffusion. On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ let us consider a banking system $X := (X(t) := (X_1(t), \ldots, X_n(t)), 0 \le t < \infty)$ of $n\,(\ge 2)$ banks where bank i has a monetary reserve $X_i(t)$ at time t for each $i = 1, \ldots, n$ with the dynamics of Feller's diffusion: for $0 \le t < \infty$,

(2.1)
$$X_i(t) = X_i(0) + \int_0^t \Big[\delta_i + \sum_{j=1}^n (X_j(u) - X_i(u)) \cdot p_{i,j}(X(u))\Big] d\,u + 2\int_0^t \sqrt{X_i(u)} d\,W_i(u)\,,$$

where we normalized the volatility. If one is interested in the effect of size of volatility, the simple change of time $t \to \sigma^2 t$ can restore the volatility parameter.

Assumption (A): Throughout this paper we assume that the process $X(\cdot)$ starts from $\mathrm{x} := (X_1(0), \ldots, X_n(0)) \in [0, \infty)^n$, $W := ((W_1(t), \ldots, W_n(t)), 0 \le t < \infty)$ is the standard n-dimensional Brownian motion, δ_i is a nonnegative constant for $i = 1, \ldots, n$, and that the function $p_{i,j} : [0,\infty)^n \to [0,1]$ is a bounded α-Hölder continuous on compact sets in $(0,\infty)^n$ for some $\alpha \in (0,1]$, $1 \le i, j \le n$.

Since each function $(p_{i,j}(\cdot))_{1\le i,j\le n}$ is bounded, the drifts of $X(\cdot)$ in (2.1) grow at most linearly in absolute value and hence it satisfies the linear growth condition. On the boundary, say $X_i(\cdot) = 0$ for some $1 \le i \le n$, the drift coefficient of $X_i(\cdot)$ in (2.1) satisfies $\delta_i + \sum_{j=1}^n X_j(u) \cdot p_{i,j}(X(u)) \ge \delta_i \ge 0$, since $p_{i,j}(\cdot) \ge 0$ in $[0,\infty)^n$. Thus on the boundary $\cup_{k=1}^n \{x \in [0,\infty) \,|\, x_k = 0\}$, the drift coefficient is bounded below by $\min_{1\le i\le n} \delta_i \ge 0$.

As a direct consequence of Theorem 1.2 of Bass & Perkins [4] on degenerate diffusions, we obtain the existence and uniqueness of the weak solution to (2.1) in the positive polyhedral domain $[0,\infty)^n$.

Proposition 2.1 (Theorem 1.2 of Bass & Perkins [4]). *Under the assumption on* $(\mathrm{x}, (\delta_i), (p_{i,j}(\cdot))_{1\le i,j\le n})$, *a weak solution* $(X, W), (\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{0\le t<\infty}, \mathbb{P}_{\mathrm{x}})$ *to (2.1) exists and is unique in the sense of probability distribution. This solution satisfies*

$$\mathbb{P}_{\mathrm{x}}(X(t) \in [0,\infty) \quad \textit{for} \;\; 0 \le t < \infty) = 1\,; \quad \mathrm{x} \in [0,\infty)\,.$$

Note that this model (2.1) is closely related to the interacting diffusions studied in Shiga & Shimizu [31] and Example 2 of Cox, Greven & Shiga [8] which deal with the unique solution in the strong sense under some conditions on $(p_{i,j}(\cdot))$. Here we relax the condition on $(p_{i,j}(\cdot))_{1\le i,j\le n}$ and consider the weak solution of (2.1), instead of the strong solution. Following [4], we may relax the condition on $(p_{i,j}(\cdot))_{1\le i,j\le n}$ even more, so that the drifts themselves satisfy the Hölder continuity. Furthermore, by the

Girsanov change of measure, we may deal with more general, possibly discontinuous drifts. Here we take our assumption (A) for the sake of simplicity.

The system (2.1) appears as one of the simplest *continuous-time* models for a banking system. Each bank i reserves money with a drift $\delta_i\,(\geq 0)$ from its own banking activity (re-investing and lending) plus a drift from the inter-bank short-time (over-night) lending: if j has more reserve than i, i.e., $X_j(t) > X_i(t)$, money flows from j to i proportional to the difference $X_j(t) - X_i(t)$ with rate $p_{i,j}(X(t)) \geq 0$ at time $t > 0$. If $p_{i,j}(\cdot) = 0 = p_{j,i}(\cdot)$, then there is no monetary flow between i and j. This rate function $p_{i,j}(\cdot) \geq 0$ represents *lending preference* from j to i for $1 \leq i, j \leq n$. The random Brownian shock $W_i(\cdot)$ affects each monetary reserve $X_i(\cdot)$ where the variance of the shock is proportional to its size for $1 \leq i \leq n$. In our future study we will consider a wider class of models with general drifts, volatility structure and some jump components, however, we start here our analysis from the simple model (2.1).

We interpret the event $X_i(t) = 0$ as *bankruptcy* of i at time $t \geq 0$ for $1 \leq i \leq n$. Up to Section 2.8 we also assume that there is some external support (financial bailouts) for the default bank from the other banks and the other business sectors, so that the monetary reserve process $X_i(\cdot)$ has the nonnegative drift rate $\delta_i + \sum_{j=1}^{n} X_j(t) \cdot p_{i,j}(X(t)) \geq \delta_i \geq 0$, at the time $\{t \geq 0 \,|\, X_i(t) = 0\}$ of its default for $i = 1, \ldots, n$. Another interpretation of this non-negative drift is modeling of the births of new small banks at the time of defaults of the old banks. In practice, the default banks may not participate in the financial market anymore. Later in Section 2.9 we extend our considerations to the case that each default bank is removed from the system (2.1) but the total monetary reserve in the system grows at the same rate over time on average.

It should be emphasized that this simple model has the following shortcomings.

Asset and Liabilities : When banks borrow from each other in the model, they incur no obligation to repay their debts. The model does not include any notion of assets or liabilities, and therefore, does not describe the conventional notion of bankruptcy resulting from not being able to meet payment obligations. Our definition of bankruptcy is when a bank's monetary reserve reaches zero, but since banks derive no benefit from having funds, there is in fact no harm within the model to hitting a fund balance of zero. Taking debt obligations into account would require to deal with coupled infinite-dimensional diffusions due to the forward maturity components. This is outside of the scope of this paper but certainly the topic of future research.

Incentives for lending and borrowing : Banks do not derive any benefit from having more funds or from lending to other banks and the income rate δ_i is not influenced by the fund flows. Introducing a game aspect in the model is also a topic of our ongoing research (Carmona, Fouque and Sun [6]). As many simplistic models in Applied Mathematics, our model in this paper exhibits some features of collective behavior of coupled diffusions which can be interpreted in terms of inter-bank lending and from which quantities of interest for measuring stability of the system can be identified.

2.2. Total monetary reserve process. Summing $X_i(\cdot)$ in (2.1) over $i = 1, \ldots, n$, we obtain the dynamics of *total monetary reserve* $\mathcal{X}(\cdot) := \sum_{i=1}^n X_i(\cdot)$:

$$\begin{aligned}\mathcal{X}(t) \,=\, \mathcal{X}(0) + \int_0^t \big(\sum_{i=1}^n \delta_i + \sum_{i,j=1}^n (X_j(s) - X_i(s)) p_{i,j}(X(s))\big)\mathrm{d}s \\ + 2\int_0^t \sum_{i=1}^n \sqrt{X_i(s)}\mathrm{d}W_i(s)\,. \end{aligned} \tag{2.2}$$

for $0 \le t < \infty$. By P. Lévy's theorem (e.g., see Theorem 3.4.2 of Karatzas & Shreve [23]) we can write the stochastic integral part as $2\int_0^t \sqrt{\mathfrak{X}(u)} d\,\beta(u)$ by possibly extending the probability space and introducing another Brownian motion $\beta(\cdot)$, so that $\int_0^\cdot \sqrt{\mathcal{X}(s)}\mathrm{d}\beta(s) \,=\, \int_0^\cdot \sum_{i=1}^n \sqrt{X_i(s)}\mathrm{d}W_i(s)$. Thus we obtain

$$\mathcal{X}(t) \,=\, \mathcal{X}(0) + \int_0^t \big(\sum_{i=1}^n \delta_i + \sum_{i,j=1}^n (X_j(s) - X_i(s)) p_{i,j}(X(s))\big)\mathrm{d}s + 2\int_0^t \sqrt{\mathcal{X}(s)}\mathrm{d}\beta(s)$$

for $0 \le t < \infty$.

Note that the drift of $\mathcal{X}(\cdot)$ is not a simple function of $\mathcal{X}(\cdot)$ in general, however, if we assume additionally the symmetry $p_{i,j}(\cdot) = p_{j,i}(\cdot)$ of the lending preference for $1 \le i, j \le n$, then it can be verified that

$$\sum_{i=1}^n \sum_{j=1}^n (x_j - x_i) \cdot p_{i,j}(x) = \sum_{i<j} (x_j - x_i) \cdot p_{i,j}(x) + \sum_{j<i} (x_j - x_i) \cdot p_{i,j}(x) = 0\,;$$

for $x = (x_1, \ldots, x_n) \in (\mathbb{R}_+)^n$, and hence the $\mathfrak{X}(\cdot)$ follows a squared Bessel process of dimension $\delta_* := \sum_{i=1}^n \delta_i$:

$$\mathfrak{X}(t) = \mathfrak{X}(0) + \delta_* t + 2\int_0^t \sqrt{\mathfrak{X}(u)}\, d\,\beta(u)\,; \quad 0 \le t < \infty\,. \tag{2.3}$$

The constant δ_* represents the total *growth rate* of the banking system. It follows from the properties of the squared Bessel process of dimension δ that

- If $\delta_* \ge 2$, the total reserve $\mathfrak{X}(\cdot)$ never reaches zero:
$$\mathbb{P}_{\mathrm{x}}(\mathfrak{X}(t) > 0\,, \text{ for all } t \in [0, \infty)) = 1\,; \quad \mathrm{x} \in (0, \infty)^n\,. \tag{2.4}$$
- If $\delta_* = 2$, the system survives as in (2.4), and grows forever
$$\mathbb{P}_{\mathrm{x}}(\limsup_{t\to\infty} \mathfrak{X}(t) = \infty) = 1\,; \quad \mathrm{x} \in (0, \infty)^n \tag{2.5}$$
but it faces a big financial crisis of shrinking the total reserves to almost nothing at some point in the future almost surely:
$$\mathbb{P}_{\mathrm{x}}(\inf_{0\le s<\infty} \mathfrak{X}(s) = 0) = 1\,; \quad \mathrm{x} \in [0, \infty)^n\,. \tag{2.6}$$
- If $0 < \delta_* < 2$, the property (2.4) does not hold; the property (2.5) still holds; $\mathfrak{X}(\cdot)$ attains zero almost surely; and the point $\{0\}$ is instantaneously reflecting.
- If $\delta_* = 0$, the process $\mathcal{X}(\cdot)$ in (2.3) attains zero in a finite time and stops thereafter almost surely.

Thus under the additional assumption of the symmetry $p_{i,j}(\cdot) = p_{j,i}(\cdot)$, $1 \le i, j \le n$, we can simply describe different possible behaviors of the total reserve by the different choices of the growth rate $\delta_* = \sum_{i=1}^n \delta_i$ in the model (2.1).

2.3. Multiple defaults. In the previous section we saw the growth of the total monetary reserve process. Now we shall study how the lending preference affects each monetary reserve of $i = 1, \dots, n$. If the inter-bank lending is frozen to some extent, or equivalently, the lending preference $(p_{i,j}(\cdot))_{1 \le i,j \le n}$ is restricted to some range close to zero, some banks are broke together at the same time almost surely, that is, *multiple default* occurs.

PROPOSITION 2.2. *In addition to the model assumption (A) in Section 2.1, let us assume that for some* $k \in \{1, \dots, n\}$ *indexes* $(\ell_1, \dots, \ell_k) \subset \{1, \dots, n\}$ *the lending preference* $(p_{i,j}(\cdot))$ *and the growth rate* $(\delta_{\ell_1}, \dots, \delta_{\ell_k})$ *satisfy*

$$\sup_{x \in [0,\infty)^n} |x_{\ell_i} - x_j| \cdot p_{\ell_i,j}(x) < 2c_0 := \frac{1}{k(n-1)} \big(2 - \sum_{i=1}^{k} \delta_{\ell_i}\big) \tag{2.7}$$

for $1 \le i \le k$, $1 \le j \le n$. *Then, the* k *banks* $(\ell_1, \dots, \ell_k)$ *are broke simultaneously at some time* $t \in (0, \infty)$ *almost surely, that is,*

$$\mathbb{P}_{\mathrm{x}}\big(X_{\ell_1}(t) = X_{\ell_2}(t) = \cdots = X_{\ell_k}(t) = 0 \ \ \textit{for some } t \in (0,\infty)\big) = 1 \,. \tag{2.8}$$

Proof. The proof is based on an application of the comparison theorem of IKEDA & WATANABE [22] with the squared Bessel process. The condition (2.7) implies

$$\begin{aligned} \overline{\delta} \ &:= \sum_{i=1}^{k} \delta_{\ell_i} + \sup_{x \in [0,\infty)^n} \Big| \sum_{i=1}^{k} \sum_{j=1}^{n} (x_j - x_{\ell_i}) \cdot p_{\ell_i,j}(x) \Big| \\ &< \sum_{i=1}^{k} \delta_{\ell_i} + 2c_0 k(n-1) \ = \ 2 \,, \end{aligned} \tag{2.9}$$

In a similar way as we have derived (2.3), by extending the probability space and introducing another Brownian motion $\beta_k(\cdot)$, we see that the sum $\mathcal{X}_k(\cdot) := \sum_{i=1}^{k} X_{\ell_i}(\cdot)$ of the monetary reserve of the chosen banks indexed by $(\ell_1, \dots, \ell_k)$ satisfies

$$\mathrm{d}\mathcal{X}_k(t) = \sum_{i=1}^{k} \Big[\delta_{\ell_i} + \sum_{j=1}^{n} (X_j(u) - X_{\ell_i}(t)) \cdot p_{\ell_i,j}(X(t))\Big] d\,t + 2\sqrt{\mathcal{X}_k(t)} d\,\beta_k(t) \,; \quad t \ge 0 \,.$$

It follows from an application of the comparison theorem in [22] (see also Lemma 2.1 of [20]) that the sum $\mathcal{X}_k(\cdot)$ is less than or equal to the squared Bessel process $\widetilde{\mathcal{X}}_k(\cdot)$ of dimension $\overline{\delta}(< 2)$ defined by

$$\widetilde{\mathcal{X}}_k(t) := \mathcal{X}_k(0) + \overline{\delta} t + 2 \int_0^t \sqrt{\widetilde{\mathcal{X}}_k(u)} d\,\beta_k(u) \,, \tag{2.10}$$

with the same initial value $\widetilde{\mathcal{X}}_k(0) = \mathcal{X}_k(0)$. Since $\widetilde{\mathcal{X}}_k(\cdot)$ attains the origin infinitely often almost surely, so does the sum $\mathcal{X}_k(\cdot)$. Thus we obtain (2.8). □

Note that even in the case $\delta_* = \sum_{i=1}^{n} \delta_i \ge 2$ when the total reserve grows as in (2.5), if the condition (2.7) is satisfied, multiple defaults can occur almost surely. The condition (2.7) on the preference $(p_{i,j}(\cdot))_{1 \le i,j \le n}$ restricts the inter-banking monetary

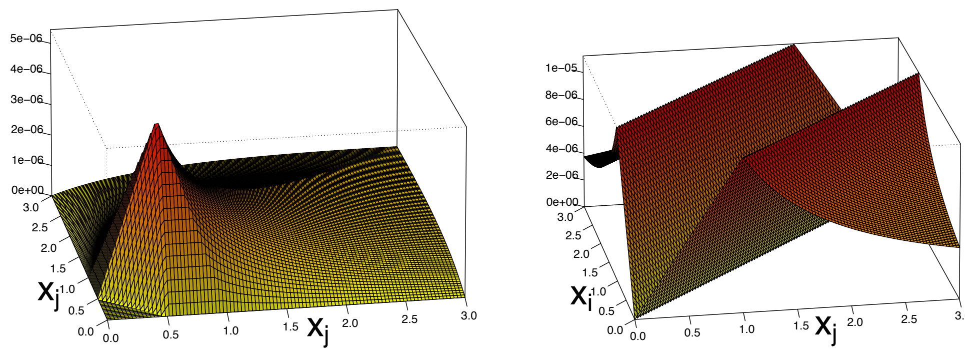

FIG. 2.1. *Examples of lending preferences* $p_{i,j}(\cdot)$ *that satisfy* (2.7) *in* x_i*-*x_j *axis. Examples of* (2.11) (*left*) *and* (2.12) (*right*).

flow in a stringent way, so that every bank can get bankrupt together with other $(k-1)$ banks at the same time for every $k = 1, \ldots, n-1$ due to the lack of support from the other banks that have more reserves. Note also that there are many possible choices of the preference $(p_{i,j}(\cdot))_{1 \le i,j \le n}$ that satisfy (2.7). Here are some examples.

- No monetary flow case $p_{i,j}(\cdot) \equiv 0$, $1 \le i, j \le n$ satisfies (2.7), if $\sum_{i=1}^{k} \delta_{\ell_i} < 2$.
- Given a constant $c_1 (< c_0)$, where c_0 is defined in (2.7), we define
(2.11)
$$\frac{p_{i,j}(x)}{c_1} := \begin{cases} 2(x_i \wedge x_j)/(x_i + x_j)^2 & \text{if } x_i + x_j \ge 1\,, \\ 2(x_i \wedge x_j) & \text{if } x_i \wedge x_j \ge 1/2\,, 1/2 \le x_i + x_j \le 1\,, \\ 2(x_i + x_j) - 1 & \text{if } x_i \wedge x_j \le 1/2\,, 1/2 \le x_i + x_j \le 1\,, \\ 0 & \text{if } x_i + x_j \le 1/2\,, \end{cases}$$
for $1 \le i, j \le n$, $x = (x_1, \ldots, x_n) \in [0, \infty)^n$. It can be checked that (2.11) satisfies the model assumption (A) and condition (2.7). This is the case that $p_{i,j}(\cdot) = 0$ on the boundary $\{x \in [0,\infty) \,|\, x_i x_j = 0\}$ for every $1 \le i, j \le n$. With $n = 100$, $k = 10$, $\delta = 2$, $c_1 = c_0/2$, the resulting function is shown in Figure 2.1. □
- Similarly, given a nonnegative function $h : [0, \infty) \to [0, 1]$ which is α-Hölder continuous on compact sets in $(0, \infty)$ for some $\alpha \in (0, 1]$, we can take
$$p_{i,j}(x) = h(|x_i - x_j|)\,; \quad x = (x_1, \ldots, x_n) \in [0, \infty)^n\,,\ 1 \le i, j \le n\,. \tag{2.12}$$
The condition (2.7) holds if we choose $c_1 < c_0$ and $h(s) = c_1 / s$ for $s \ge 1$ and $h(s) = c_1 s$ for $s \le 1$. The resulting function is shown in Figure 2.1. □

Conversely, to avoid the multiple defaults of k banks indexed by $\{\ell_1, \ldots, \ell_k\}$, we obtain some condition on the lending preference $(p_{i,j}(\cdot))_{1 \le i,j \le n}$ in the following proposition.

PROPOSITION 2.3. *In addition to the model assumption (A) in Section 2.1, let us assume that for some* $k \in \{1, \ldots, n\}$ *indexes* $(\ell_1, \ldots, \ell_k) \subset \{1, \ldots, n\}$ *the lending* *preference* $(p_{i,j}(\cdot))$ *and the growth rate* $(\delta_{\ell_1}, \ldots, \delta_{\ell_k})$ *satisfy*

$$\inf_{x\in[0,\infty)^n} \sum_{j=1}^{n}\sum_{i=1}^{k}(x_j - x_{\ell_i})\cdot p_{\ell_i,j}(x) \ge 2c_0 k \tag{2.13}$$

for $1 \le i \le k,\ 1 \le j \le n$ *, where* c_0 *is in (2.7). Then the* k *banks* $(\ell_1, \ldots, \ell_k)$ *are broke simultaneously at some time* $t \in (0, \infty)$ *almost surely, that is,*

$$\mathbb{P}_{\mathrm{x}}\big(X_{\ell_1}(t) = X_{\ell_2}(t) = \cdots = X_{\ell_k}(t) = 0 \ \text{ for some } t \in (0,\infty)\big) = 0\,. \tag{2.14}$$

The meaning of the condition (2.13) is that if the monetary reserve X_{ℓ_i} of ℓ_i is small, there are enough monetary flow from the other banks to ℓ_i for $i = 1, \ldots, k$. We may relax the condition (2.13) for (2.14). For example, if the lending preference $(p_{i,j}(\cdot))_{1\le i,j\le n}$ is strictly positive in the sense that $\min_{1\le i<j\le n}\inf_{x\in(0,\infty)^n} p_{i,j}(x) > 0$, then (2.7) is not satisfied. In particular, it contains the case $p_{i,j}(\cdot) = 1\,/\,n$. In this case, as $n \to \infty$, we can consider formally a mean-field limit approximation:

$$d\,X_i(t) = (\delta_i + \mathbf{m} - X_i(t))d\,t + 2\sqrt{X_i(t)}d\,W_i(t)\,,$$

where $\mathbf{m} := \lim_{n\to\infty}\sum_{i=1}^{n} X_i(0)\,/\,n$ is the empirical mean of initial distribution; the bank i will not default, if $\mathbf{m} + \delta_i > 1$.

2.4. Financial Health Indicators. Suppose now that the financial regulator of the banking system is interested in monitoring the system, especially a subgroup $\{\ell_1, \ldots, \ell_k\}$ of k banks. From the viewpoint of the financial regulator Proposition 2.3 suggests that the quantity

$$\mathfrak{I}(t, \ell_1, \ldots, \ell_k) := \sum_{j=1}^{n}\sum_{i=1}^{k}(X_j(t) - X_{\ell_i}(t)))\,p_{\ell_i,j}(X(t)) \tag{2.15}$$

is a health barometer for the subgroup $\{\ell_1, \ldots, \ell_k\}$ of banks at time $t \ge 0$. This is the sum of net instantaneous monetary flows between the subgroup and the rest of the system. If this quantity $\mathfrak{I}(\cdot)$ is bounded below by $2c_0k$ as in (2.13), then the subgroup $\{\ell_1, \ldots, \ell_k\}$ of banks do not have multiple defaults in the sense of (2.14).

On the other hand, if this quantity $\mathfrak{I}(\cdot)$ is small, then the subgroup may fall in a risky situation. In fact, Proposition 2.2 suggests that another quantity

$$\mathfrak{J}(t, \ell_1, \ldots, \ell_k) := \sup_{1\le i\le k, 1\le j\le n} |X_{\ell_i}(t) - X_j(t)|\,p_{\ell_i,j}(X(t)) \tag{2.16}$$

is a risk indicator of the subgroup $\{\ell_1, \ldots, \ell_k\}$ at time $t \ge 0$. If the quantity $\mathfrak{J}(\cdot)$ is less than $2c_0$ in (2.7), then the subgroup $\{\ell_1, \ldots, \ell_k\}$ may face multiple defaults as in (2.8). Therefore, the financial regulator may use these quantities $\mathfrak{I}(\cdot)$ and $\mathfrak{J}(\cdot)$ in order to assess the health condition of the banking system.

It is noteworthy that these indicators are delicate quantities. If the financial regulator observes only $(X_1(\cdot), \ldots, X_n(\cdot))$ but cannot directly observe $\mathfrak{I}(\cdot;\ell_1, \ldots, \ell_k)$ nor $\mathfrak{J}(\cdot;\ell_1, \ldots, \ell_k)$ (that is the preferences $p_{i,j}$), then one needs to estimate them by filtering techniques. If in addition, each bank follows the strategy optimizing its action, the problem of determining the financial health of the system becomes even more difficult. These issues are outside the scope of the present paper and are topics of ongoing research.

2.5. Multiple Default Time Probability Estimations. The sum $\mathcal{X}_k(\cdot) = \sum_{i=1}^k X_{\ell_i}(\cdot)$ of arbitrary choice $(\ell_1, \ldots, \ell_k)$ of k banks is compared with the squared Bessel process $\widetilde{\mathcal{X}}_k(\cdot)$ of dimension $\overline{\delta}$ defined in (2.9) in the proof of Proposition 2.2. With the condition (2.7) still enforced, let us consider the multiple default time τ_k as well as the first passage time $\widetilde{\tau}_k$ of $\widetilde{\mathcal{X}}_k(\cdot)$:

$$\tau_k := \inf\{t \ge 0 : \mathcal{X}_k(t) = 0\}\,, \quad \widetilde{\tau}_k := \inf\{t \ge 0 : \widetilde{\mathcal{X}}_k(t) = 0\}\,.$$

Applying the comparison theorem with the results of [11] and [19] on the squared Bessel process again, we can estimate the tail probability distribution

$$\begin{aligned}\mathbb{P}_{\mathrm{x}}(\tau_k > t) \le \mathbb{P}_{\mathrm{x}}(\widetilde{\tau}_k > t) &= \int_t^\infty \frac{1}{s\Gamma(\overline{\delta})}\Big(\frac{(\mathcal{X}_k(0))^2}{2s}\Big)^{\delta_1} \exp\Big(-\frac{(\mathcal{X}_k(0))^2}{2s}\Big)\, d\,s \\ &=: \gamma\Big(\frac{(\mathcal{X}_k(0))^2}{2t}\,;\overline{\delta}\Big)\,; \quad t \ge 0\,,\ \mathrm{x} \in [0,\infty)^n\,,\end{aligned} \tag{2.17}$$

where $\gamma(x\,;s) := \int_0^x u^{s-1}e^{-u}d\,u$ is the lower incomplete gamma function for $s > 0$, $x \ge 0$. Thus the tail probability (2.17) is of the order of $t^{-\overline{\delta}}$ as $t \to \infty$. In this way $\overline{\delta}$ in (2.9) is interpreted as the indicator of possible maximum lending activities, that is, the more inter-bank lending activities (large $\overline{\delta}$), the more likelihood of survival. The estimate (2.17) is only an upper bound. To obtain a rough lower bound, let us define the lower bound

$$\underline{\delta} := \sum_{i=1}^k \delta_{\ell_i} + \inf_{x\in[0,\infty)^n} \sum_{i=1}^k \sum_{j=1}^n (x_j - x_i)\cdot p_{\ell_i,j}(x)\,.$$

Again by the comparison theorem, if $\underline{\delta} > 0$,

$$\mathbb{P}_{\mathrm{x}}(\tau_k \ge t)\ \ge\ \gamma\Big(\frac{(\mathcal{X}_k(0))^2}{2t}\,;\underline{\delta}\Big); \quad t \ge 0\,,\ \mathrm{x} \in [0,\infty)^n\,. \tag{2.18}$$

Thus (2.17) and (2.18) are the lower and upper estimates of the default time probability for some appropriate lending preference $(p_{ij}(\cdot))_{1\le i,j\le n}$.

If there is no lending of money among the banks, that is, $p_{ij}(\cdot) \equiv 0$, then $\underline{\delta} = \sum_{i=1}^k \delta_{\ell_i}$, gives the exact probability

$$\mathbb{P}_{\mathrm{x}}(\tau_k \ge t) = \gamma\Big(\frac{(\mathcal{X}_k(0))^2}{2t}; \sum_{i=1}^k \delta_{\ell_i}\Big)\,; \quad t \ge 0\,,\ \mathrm{x} \in [0,\infty)^n\,. \tag{2.19}$$

This is a benchmark to compare the different lending preferences. In general, it is hard to obtain an explicit formula for the first multiple default time distribution for the arbitrary choice $(\ell_1, \ldots, \ell_k)$ of k banks. We shall discuss a numerical procedure in Section 2.6.

2.6. Number of defaults estimated by Interacting Particle System Algorithm. In practice, it is interesting to see what is the probability that many defaults occur in a given time. Let us denote by $N_0(t)$ the number of defaults before time $t \ge 0$, i.e.,

$$N_0(t) := \sum_{i=1}^n \chi(\min_{0\le s\le t} X_i(s) = 0)\,; \quad 0 \le t < \infty\,,$$

where $\chi(A)$ is the indicator function which takes one if A is true and zero otherwise.

If there is *no monetary flow* $p_{ij}(\cdot) \equiv 0$, $1 \le i, j \le n$ between banks, and hence the reserve processes $(X_1(\cdot), \ldots, X_n(\cdot))$ are independent, we can compute this probability exactly as we do in (2.19). By considering all possible choice $(\ell_1, \ldots, \ell_k) \subset \{1, \ldots, n\}$ of default bank names and $(n-k)$ non-default banks until time t, we obtain

$$\mathbb{P}_{\mathrm{x}}(N_0(t) = k)$$
$$= \sum_{1 \le \ell_1 < \cdots < \ell_k \le n} \Big[\prod_{j=1}^{k} \Big(1 - \gamma\Big(\frac{(X_{\ell_j}(0))^2}{2t} ; \delta_{\ell_j} \Big) \Big) \Big] \cdot \Big[\prod_{i \notin \{\ell_1, \ldots, \ell_k\}} \gamma\Big(\frac{(X_i(0))^2}{2t} ; \delta_i \Big) \Big] ,$$

for $0 \le t < \infty$, where $\gamma(\cdot;\cdot)$ is defined in (2.17). Even in this independent, simplest case, the computation of these probabilities becomes delicate for larger n and k, since the number of summands becomes large but each summand is small.

Unfortunately, it seems very hard to obtain an explicit theoretical answer, for any given lending preference $(p_{ij}(\cdot))_{1 \le i,j \le n}$. Instead, here we suggest a Monte Carlo scheme to compute the small probability, following the interacting particle method proposed by Carmona, Fouque & Vestal [5]. Let us define a small threshold $b > 0$ and the number

$$N_b(T) := \sum_{i=1}^{n} \chi\big(\min_{0 \le s \le T} X_i(s) \le b \big)$$

of banks whose reserves go below the level b before time $T > 0$. Let us estimate the probability mass function of the number of defaults:

$$\mathbb{P}_{\mathrm{x}}(N_b(T) = k) = \mathbb{E}\Big[\chi\big(\sum_{i=1}^{n} \chi\big(\min_{0 \le s \le T} X_i(s) \le b \big) = k \big) \Big] ; \quad k = 1, \ldots, n . \tag{2.20}$$

Interacting Particle System (IPS) Algorithm [5] Dividing the time interval $[0, T]$ into L equal subintervals $[\,(\ell-1)T\,/\,L\,, \ell\,T\,/\,L\,]$ with $\ell = 1 \ldots, L$, we simulate M random chains where each chain is $\{Y^{(j)}_{\ell} = (\widehat{X}^{(j)}(\ell T\,/\,L), \widehat{m}^{(j)}(\ell T\,/\,L))\}_{1 \le \ell \le L}$ for $j = 1, \ldots, M$. Here $\widehat{X}^{(j)}(\cdot)$ is the jth simulation of $X(\cdot)$ in the system (2.1) and $\widehat{m}^{(j)}$ is the jth simulation of the vector $m(\cdot) := (m_1(\cdot), \ldots, m_n(\cdot))$ of the running minimum $m_i(t) = \min_{0 \le s \le t} X_i(s)$, for $i = 1, \ldots, n$, $j = 1, \ldots, M$, $0 \le t \le T$. After initializing the chain, for each $\ell = 1, \ldots, L$, repeat the following selection and mutation stages alternatively:

- (Selection Stage). Sampling M new particles from $\{Y^{(j)}_{\ell}\}_{1 \le j \le M}$ with Gibbs weights

$$\Big[\sum_{j=1}^{M} \prod_{i=1}^{n} \gamma^{(j)}_{i,\ell} \Big]^{-1} \prod_{i=1}^{n} \gamma^{(j)}_{i,\ell}$$

where

$$\gamma^{(j)}_{i,\ell} := \Big[\frac{\min(\widehat{m}^{(j)}_i((\ell-1)T\,/\,L), \widehat{X}^{(j)}_i(\ell T\,/\,L))}{\widehat{m}^{(j)}_i((\ell-1)T\,/\,L)} \Big]^{-\alpha} ,$$

for each $i = 1, \ldots, n$, $j = 1, \ldots, M$ with some choice of $\alpha > 0$.

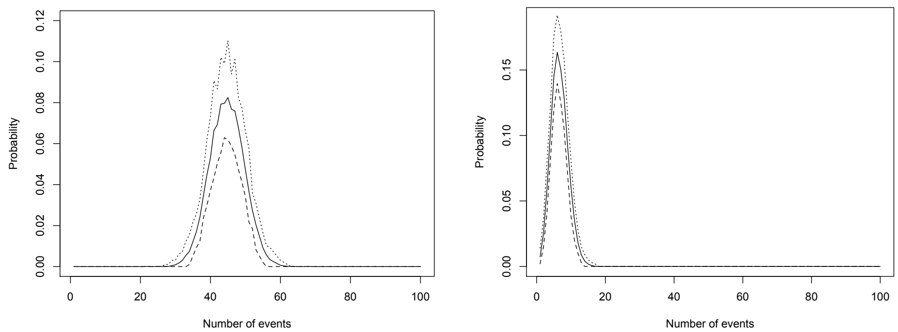

FIG. 2.2. *Estimated probability mass function (2.20), $b = 0.01$ (left), $b = 0.0001$ (right) by IPS method. The parameters are specified in Example 2.1. The solid (dashed, dotted, resp.) line indicates the estimated probability (estimated 5 percentile, 95 percentile of the simulation, resp.)*

- (Mutation Stage). Running Euler scheme with time mesh of size $\Delta t\,(\ll T\,/\,L)$ to get the new value $Y^{(j)}_{\ell+1}$, $j = 1, \ldots, M$, starting from the new particles sampled in the above.

The probability estimate $\widehat{\mathbb{P}}_{\mathrm{x}}(N_b(T) = k)$ of $\mathbb{P}_{\mathrm{x}}(N_b(T) = k)$ in (2.20) is given by

$$\widehat{\mathbb{P}}_{\mathrm{x}}(N_b(T) = k) = \frac{1}{M}\sum_{j=1}^{M}\Big(\chi\,(\widehat{N}^{(j)}_b = k)\prod_{i=1}^{n}\Big[\frac{\widehat{m}^{(j)}_i(T)}{\widehat{m}^{(j)}_i(0)}\Big]^{\alpha}\Big)\cdot\Big[\prod_{\ell=0}^{L-1}\Big(\frac{1}{M}\sum_{a=1}^{M}\prod_{i=1}^{n}\gamma^{(a)}_{i,\ell}\Big)\Big]\,,$$

for $k = 1, \ldots, n$, where $\widehat{N}^{(j)}_b(T)$ is the corresponding number to $N_b(T)$ in the jth simulation for $j = 1, \ldots, M$. □

EXAMPLE 2.1. *Here is an extreme example: with initial point* $\mathrm{x} = (1, \ldots, 1)$, *growth rate* $\delta = 2$ *and lending preference* $p_{i,j}(\cdot)$ *specified as in (2.11) with* $k = n-1$, *let us set* $T = 1$, $n = 100$, $M = 1000$ *(the number of simulation),* $L = 10$ *(the number of subintervals in* $[0, 1]$*),* $\alpha = 0.0001$, *and run (2.1) with the time mesh* $\Delta t = 0.001$ *for the Euler scheme in the Mutation Stage to compute the probability mass functions (2.20) for* $b = 0.01$ *and* $b = 0.0001$. *The results are shown in Figure 2.2 with the estimated* 5 *and* 95 *percentiles from one hundred repetitions.* □

The advantages of this IPS approach are its simplicity and small storage space for the implementation of the algorithm. We may choose various Gibbs weights with different choice of $(\gamma^{(j)}_{i,\ell})$ in the Selection Stage, for example, the number of defaults, instead of the running minimum. We may extend this algorithm for the probability distribution of the number of times when the k banks run out of their reserves together before time T:

$$\mathbb{P}_{\mathrm{x}}\Big[\sum_{1\le\ell_1<\cdots<\ell_k\le n}\chi\Big(\min_{0\le s\le T}\max(X_{\ell_1}(s), \ldots, X_{\ell_k}(s)) \le b\Big) = j\Big]\,; \quad j \ge 1\,,$$

by looking at maximum $\max(X_{\ell_1}(\cdot), \ldots, X_{\ell_k}(\cdot))$, instead of $X_i(\cdot)$ in (2.20). There is another kind of freedom in choosing the tuning parameter α in the above IPS algorithm, which is discussed in detail in [5]. For the computation of the rare event probabilities note that there are other approaches, for example, sequential important sampling (e.g., see [9] in a similar financial context) and forward flux sampling methods (e.g., see [1] and its references in Physics and Chemistry).

2.7. Stochastic stability. The squared Bessel process $\mathfrak{X}(\cdot)$ of dimension $\delta_* \in (0,2]$ in (2.3) is (null) recurrent in $(0,\infty)$, that is, for every Borel measurable subset A of $(0,\infty)$, the total reserve $\mathfrak{X}(\cdot)$ in (2.3) hits A, and hence hits A infinitely often almost surely by the strong Markov property. On the other hand, if $\delta_* = \sum_{i=1}^n \delta_i \in (2,\infty)$, then $\mathfrak{X}(\cdot)$ is transient. Thus, the whole growth rate δ_* characterizes the behavior of the total reserve $\mathfrak{X}(\cdot)$. Here we shall consider the asymptotic behavior of the system in (2.1).

Suppose that the bank i reserves the highest amount $X_i(t) = \max_{1\le j\le n} X_j(t)$ at some time $t \ge 0$ in the system (2.1). It follows from the dynamics (2.1) that the deviation $X_i(t) - \delta_i t$ has the non-positive drift

$$\sum_{j=1}^n (X_j(t) - X_i(t)) \cdot p_{i,j}(X(t)) \le 0\,, \quad \text{ if } \quad X_i(t) = \max_{1\le j<n} X_j(t)\,.$$

In other words, every bank cannot be too large relative to the size of other banks. This intuition can be formulated in the following way.

Let us consider the deviation $Y_i(\cdot) := X_i(\cdot) - n^{-1}\mathfrak{X}(\cdot) = X_i(\cdot) - n^{-1}\sum_{j=1}^n X_j(\cdot)$ from the average monetary reserve $n^{-1}\mathfrak{X}(\cdot)$ for $i = 1,\ldots,n$. Since the sum is $\sum_{i=1}^n Y_i(\cdot) = 0$, the n-dimensional process $Y(\cdot) := (Y_1(\cdot),\ldots,Y_n(\cdot))$ stays on the hyperplane $\Pi := \{y := (y_1,\ldots,y_n) \in \mathbb{R}^n \mid \sum_{i=1}^n y_i = 0\}$.

PROPOSITION 2.4. *In addition to the model assumption (A) in Section 2.1, let us assume that the individual growth rate is identical $\delta_i \equiv \delta > 0$, the lending preference is symmetric $p_{i,j}(\cdot) = p_{j,i}(\cdot)$ for $i,j = 1,\ldots,n$, and that there exist positive constants c_3 and c_4 such that the lending preference $(p_{i,j}(\cdot))_{1\le i,j\le n}$ of the banks satisfy*

$$\min_{1\le i,j\le n} \inf_{x\in[0,\infty)^n} \big\{p_{i,j}(x) \,:\, |x_i - x_j| > c_3\big\} \ge c_4 > 0. \tag{2.21}$$

Then, the Π-valued process $Y(\cdot)$ is stochastically stable, that is, there exists a unique invariant probability measure $\mu(\cdot)$ such that the Strong Law of Large Numbers

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(Y(t))dt = \int_\Pi f(y)\mu(dy) \quad a.s. \tag{2.22}$$

holds for every bounded measurable function $f : \Pi \to \mathbb{R}$.

For example, the condition (2.21) is satisfied in the case of constant lending preferences. Furthermore, it can be satisfied by the lending preferences of the form (2.12). Note that we may relax the identical growth-rate condition $\delta_i \equiv \delta$ and the symmetric lending preference condition $p_{i,j}(\cdot) = p_{j,i}(\cdot)$ for $i,j = 1,\ldots,n$ in Proposition 2.4 to another weaker condition, but then we should change the condition (2.21) to another complicated condition. For our purpose of simple presentation we demonstrate the proposition under the identical growth-rate condition.

The proof is given in Appendix 4. Since $X_i(\cdot) - X_j(\cdot) = Y_i(\cdot) - Y_j(\cdot)$ for $1 \le i,j \le n$, as a direct consequence of Proposition 2.4 we obtain the following corollary. We shall use this result later in Section 2.8.

COROLLARY 2.1. *Under the conditions in Proposition 2.4, the* $(n \times n)$ *matrix-valued process* $(X_i(\cdot) - X_j(\cdot))_{1 \le i,j \le n}$ *is stochastically stable.*

2.8. Network representation. Let us construct a random graph from (2.1) by considering each bank as a node (vertex) and the connection between two banks as a link in the graph. Here we consider the connection between i and j, in terms of the (absolute) monetary flow $|X_j(t) - X_i(t)| \cdot p_{i,j}(X(t))$ among $1 \le i < j \le n$. For each fixed $r \ge 0$, we *connect* i and j with indicator $\chi_{i,j;r}(t) = 1$, if the monetary flow is larger than r, otherwise $\chi_{i,j;r}(t) = 0$, i.e.,

$$\chi_{i,j;r}(t) := \chi(\,|X_j(\cdot) - X_i(\cdot))| \cdot p_{i,j}(X(\cdot)) \ge r\,)\,; \quad 1 \le i,j \le n\,,\ 0 \le t < \infty\,, \tag{2.23}$$

where $\chi(\cdot)$ is the indicator function. Thus we obtain the matrix-valued process

$$\chi_r := \{\chi_{i,j;r}(t)\,, 1 \le i,j \le n,\ 0 \le t < \infty\,\} \tag{2.24}$$

on the space of undirected graphs for each threshold $r \ge 0$. We may consider the directed graphs with the directions of monetary flows, if we replace the indicator by the sign of $(X_j(\cdot) - X_i(\cdot)) \cdot p_{i,j}(X(\cdot)) - r\,; \quad 1 \le i,j \le n$ in (2.23).

The theory of Random Graph has been developed with many applications. It started by the study of GILBERT [18], ERDÖS AND RENYI [12], In Economics and Finance the random graph has been used for multiple agent problem, for example, exchange market of many interacting agents (FÖLLMER [15]) and herd behavior (CONT & BOUCHAUD [7]). There are several statistics that describe the monetary flow in the random graph representation. The total number of links for i, the sum of $\chi_{i,j;r}(\cdot)$ in (2.24) over $j = 1, \ldots, n$, is called the *degree* of i:

$$\mathrm{degree}_{i;r}(\cdot) := \sum_{j=1}^{n} \chi_{i,j;r}(\cdot)\,; \quad i = 1, \ldots, n\,. \tag{2.25}$$

The *distance* $\mathrm{dist}_{i,j}(\cdot)$ between i and j is the number of minimum links from i to j. The *eccentricity* of i is $\max_{1 \le j \le n}[\mathrm{dist}_{i,j}(\cdot)]$, $i = 1, \ldots, n$, and the *diameter* of the network is $\max_{1 \le i < j \le n}[\mathrm{dist}_{i,j}(\cdot)]$. The *average distance* $\sum_{j=1}^{n} \mathrm{dist}_{i,j}(\cdot)\,/\,n$ of bank i indicates where the bank i is allocated in the network: a bank with smaller average distance locates closer to the center of network.

There are other statistics such as *influential domain* (the number (or percentage to the total number of nodes) of maximal connected nodes in the network), and *betweenness centrality* of i which is defined by $\sum_{1 \le k,\ell \le n}[\mathfrak{p}_{k,\ell,i}\,/\,\sum_{j=1}^{n} \mathfrak{p}_{k,\ell,j}]$, where $\mathfrak{p}_{k,\ell,i}$ is the number of paths from k to ℓ that contain i in-between for $i, k, \ell = 1, \ldots, n$. These statistics are studied by MARKOSE [26] in Network simulations, by MÜLLER [27] for the Swiss interbank system, by SORAMÄKI ET.AL. [32] for the FRB interbank system, by SANTOS & CONT [30] for the Brazilian interbank system.

Figure 2.3 shows graphs of the network of $n = 100$ banks for different times, simulated based on (2.1) with constant lending preferences $p_{i,j}(\cdot)$ given by (2.11) for $1 \le i < j \le n$ and $\delta_i = 2\,/\,n$. Each line segments outside represents a size of money reserved by bank $X_i(\cdot)$ for $i = 1, \ldots, n$ and each line segment inside represents the link $\chi_{i,j;r}(\cdot)$ from i to j, defined by (2.23). The strength $|X_j(\cdot) - X_i(\cdot)| \cdot p_{i,j}(X(\cdot))$ of the link is considered in gray-scale: black line shows stronger link than the gray line. The links with strength weaker than some threshold r are not shown. For the sake of presentation the threshold depends on the range of the strength for each time

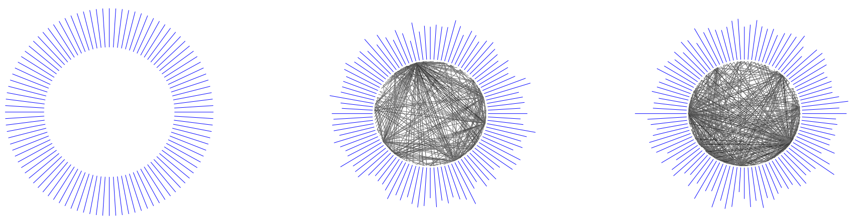

FIG. 2.3. *The graphs of the network for the initial state, after 200 steps and 400 steps, respectively from left to right.*

in Figure 2.3. As it is expected, the strong links are observed between relatively big banks and relatively small banks.

We shall study the dynamics of the random graphs $\{\chi_r\}_{\{r\ge 0\}}$ and the statistics listed above. Note that each statistic can be written as a function of the monetary reserve $X(t) = (X_1(t), \dots, X_n(t))$ for $0 \le t < \infty$. Let us denote such function by φ on $[0,\infty)^n$ and each statistic by $\varphi(X(t))$ for $0 \le t < \infty$. It follows from Corollary 2.1 that if the function $x := (x_1, \dots, x_n) \in [0,\infty)^n \to \varphi(x)$ depends only on the differences $(x_i - x_j)_{1\le i,j\le n}$, then the process $\varphi(X(t))$ is stochastically stable.

COROLLARY 2.2. *In addition to the model assumption and the condition (2.21) of Proposition 2.4, assume that the function $x := (x_1, \dots, x_n) \in [0,\infty)^n \to p_{i,j}(x)$ depends only on the difference $x_i - x_j$ for every $1 \le i, j \le n$, then the instantaneous monetary flow $(F_{i,j}(t))_{1\le i,j\le n}$ from j to i:*

$$F_{i,j}(t) := (X_i(t) - X_j(t)) \cdot p_{i,j}(X(t)); \quad 1 \le i, j \le n, \; 0 \le t < \infty \tag{2.26}$$

of the drift coefficient in (2.1) is stochastically stable, that is, there exists a unique invariant probability measure $\tilde{\mu}(\cdot)$ on the space of $(n\times n)$ matrices with nonnegative elements, such that the Law of Large Numbers holds

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f((F_{i,j}(t))_{1\le i,j\le n})dt = \int_{[0,\infty)^n} f((y_{i,j})_{1\le i,j\le n})\ \tilde{\mu}\Big(\prod_{i,j=1}^n dy_{i,j}\Big) \quad a.s.$$

for every bounded continuous function $f : [0,\infty)^n \otimes [0,\infty)^n \to \mathbb{R}$ and for $1 \le i \le n$.

EXAMPLE 2.2. *Let us consider $(p_{i,j})_{1\le i,j\le n}$ of (2.12). The lending preference $p_{ij}(x) = h(|x_i - x_j|)$ only depends on the difference $x_i - x_j$ for $1 \le i, j \le n$. If we choose the function $h(\cdot)$ as $h(s) = c_1(1 + s^{-1})/2$ for $s \ge 1$ and $h(s) = c_1 s$ for $s \le 1$, we can make the resulting lending preference satisfy the condition (2.21) with $c_3 = c_1/2$ and $c_4 = 1$. It follows from Corollary 2.2 that the instantaneous monetary flow $(F_{i,j}(\cdot))_{1\le i,j\le n}$ in (2.26) is stochastically stable, and hence so are the derived statistics (degree, distance, eccentricity, diameter). For example, for some particular choices of $r \in \{3\times 10^{-5}, 4\times 10^{-5}\}$, we simulate the degree distribution with Monte Carlo simulation (the number of simulation is 10^5). The expected degree is compared with the ranking of banks in their size (from the larger bank to the smaller bank) in Figure 2.4.* □

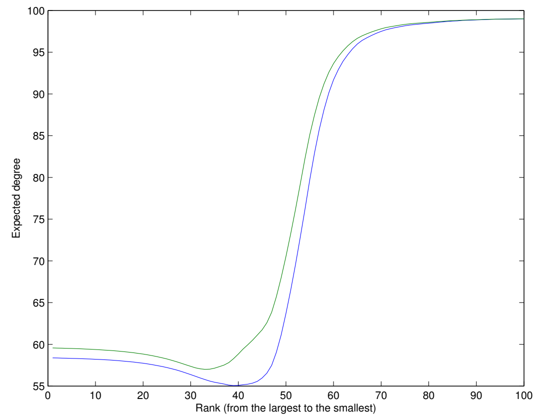

FIG. 2.4. *Expected degree* (2.25) *over the ranking of banks for different* $r = 3 \times 10^{-5}$ (*lower green curve*) $r = 4 \times 10^{-5}$ (*upper blue curve*) *from the Monte Carlo simulation.*

2.9. Exit system. As an extension of model (2.1) we consider subsystems where default banks are removed from the whole system to the *cemetery* state Δ. Let us denote the zero sets by $\mathcal{Z}_i := \{x \in [0, \infty)^n : x_i = 0\}$, $i = 1, \ldots, n$, and the initial index set $I_0 := \{i : 1 \le i \le n\}$ with size $|I_0| = n$. Let us define the index process $I_t := \{i : X_i(t) \neq 0 \text{ nor } X_i(t) \neq \Delta\}$ as an $\mathbb{F}$-adapted càdlàg process for $0 \le t < \infty$.

The default banks are removed from the system every time when their monetary reserves become zero, or in the zero set $\cup_{i=1}^n \mathcal{Z}_i$, that is, at the first default time

$$\sigma_1 := \inf\{t > \sigma_0 = 0 : X(t) \in \cup_{i=1}^n \mathcal{Z}_i\} \tag{2.27}$$

we remove all the default banks and keep the survivors index $I_{\sigma_1} := \{i : X_i(\sigma_1) \neq 0\}$. The monetary reserves $X_i(\cdot)$, $i \notin I_{\sigma_1}$ of the default banks are immediately removed and stay in Δ after this removal; for each survivor $i \in I_{\sigma_1}$ we restart the process with the following SDE:

$$X_i(t) = X_i(\sigma_1) + \int_{\sigma_1}^t \Big[\widehat{\delta}_i(u) + \sum_{j \in I_{\sigma_1}} (X_j(u) - X_i(u)) \cdot p_{i,j}(X(u))\Big] d\,u + 2 \int_{\sigma_1}^t \sqrt{X_i(u)}\, d\,W_i(u)\,;$$

for $i \in I_{\sigma_1}, \sigma_1 \le t < \sigma_2$, where $X(\cdot) := \{X_i(\cdot) : i \in I_{\sigma_1}\}$ until the next default time $\sigma_2 := \inf\{t > \sigma_1 : X(t) \in \cup_{i \in I_{\sigma_1}} \mathcal{Z}_i\}$. We evaluate the lending preference $p_{i,j}(\cdot) \equiv 0$ if the bank i or j is broke, i.e., $i \in I_0 \setminus I_{\sigma_1}$ or $j \in I_0 \setminus I_{\sigma_1}$. Here the individual growth rate $\widehat{\delta}_i(\cdot)$ may depend on the surviving banks in I_t. For example, we can choose $\widehat{\delta}_i(\cdot)$ as

$$\widehat{\delta}_i(t) := \big(n \sum_{k \in I_t} \delta_k\big)^{-1} \big(\sum_{j=1}^n \delta_j\big) \cdot \delta_i\,; \quad i \in I_t\,,\ t \ge 0\,, \tag{2.28}$$

so that the average growth rate is fixed : $\sum_{i \in I_t} \widehat{\delta}_i(t) \,/\, |I_t| \equiv \sum_{j=1}^n \delta_j \,/\, n$.

We continue this exit rule: at the stopping time σ_m we define the survivors index

$I_{\sigma_m} := \{i : X_i(\sigma_m) \neq 0\}$, and redefine the process

$$X_i(t) = X_i(\sigma_m) + \int_{\sigma_m}^{t} \Big[\widehat{\delta}_i(u) + \sum_{j \in I_{\sigma_m}} (X_j(u) - X_i(u)) \cdot p_{i,j}(X(u))\Big] du + 2 \int_{\sigma_m}^{t} \sqrt{X_i(u)} d W_i(u) \, ; \tag{2.29}$$

for $i \in I_{\sigma_m}, \sigma_m \le t < \sigma_{m+1}$, with $X(\cdot) := \{X_i(\cdot) : i \in I_{\sigma_m}\}$ until the next default time $\sigma_{m+1} := \inf\{t > \sigma_m : X(t) \in \cup_{i \in I_{\sigma_m}} \mathcal{Z}_i\}$, for $m = 1, 2, \ldots$, and build a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}, \mathbb{P})$ by pasting the probability measure locally at every stopping times $\{\sigma_1, \sigma_2, \ldots\}$ of defaults.

We shall study this exit system (2.29). Since the sample paths of $X(\cdot)$ are continuous, the interval $\sigma_m - \sigma_{m-1}$ between the stopping times σ_{m-1} and σ_m are positive almost surely for $m = 1, \ldots$. By the construction of the stopping times, $0 < \sigma_1 \le \inf\{t \ge 0 : |I_t| \le j\}$ almost surely for $j = 1, \ldots, n$. Along with the same line as Proposition 2.2 we obtain a similar result on multiple default times.

PROPOSITION 2.5. *Under the same conditions (2.7) as in Proposition 2.2 with* $\mathrm{x} \in (0, \infty)^n$ *the probability of multiple default*

$$\mathbb{P}_{\mathrm{x}}\big(X_{\ell_1}(t) = \cdots = X_{\ell_k}(t) = 0 \, , \ \textit{for some } t \in (0, \sigma_*)\big)$$

for the system (2.29) with (2.28) is strictly positive, where σ_* *is the first time that one of the banks indexed in the set* $\{\ell_1, \ldots, \ell_k\}$ *is broke, i.e.,* $\sigma_* := \inf\{t > 0 : \{\ell_1, \ldots, \ell_k\} \not\subset I_t\} > 0$.

Proof. We apply the comparisons with the Bessel process as in Proposition 2.2. In fact, it can be shown as in (2.9) that the dynamics of the process $\sum_{i=1}^{n-k} X_{\ell_i}(\cdot)$ has the drift strictly dominated by $\overline{\delta} < 2$ (see (2.9)). Hence by the comparison argument the process $\sum_{i=1}^{n-k} X_{\ell_i}(\cdot)$ is dominated by the squared Bessel process $\widetilde{\mathcal{X}}_k(\cdot)$ in (2.10) driven by the same Brownian motion with the dimension $\overline{\delta}$ and with the same initial condition. Since this squared Bessel process $\widetilde{\mathcal{X}}_k(\cdot)$ hits the origin during every positive time interval with positive probability, we conclude the desired result. □

The interpretation is that under the condition of (2.7), even if some weaker banks are removed from the system, there is still a positive probability of multiple defaults.

EXAMPLE 2.3 (Only one bank survives.). *If the lending preference* $(p_{i,j}(\cdot), 1 \le i, j \le n)$ *satisfies (2.7) with* $k = n - 1$ *and* $\delta_i \equiv 2 / n$ *for* $i = 1, \ldots, n$ *, that is,*

$$\max_{1 \le i, j \le n} \sup_{x \in [0, \infty)^n} |x_i - x_j| \cdot p_{i,j}(x) < \frac{2}{n(n-1)^2} \, ,$$

then under the model (2.29) with $\delta_* = 2$ *, the multiple defaults occur with positive probability, and moreover,* $\lim_{t \to \infty} |I_t| = 1$ *and* $\inf\{t > 0 : |I_t| = 1\} < \infty$ *a.s., i.e., only one bank can survive at the end.*

EXAMPLE 2.4 (All the banks are cooperating together.). *If the lending preference* $(p_{i,j}(\cdot), 1 \le i, j \le n)$ *satisfy*

$$\min_{1 \le i \le n} \inf_{x \in [0, \infty)^n} \sum_{j=1}^{n} (x_j - x_i) \cdot p_{i,j}(x) > \frac{2(n-1)}{n} \, ,$$

and $\delta_i = 2 / n$ *, for* $i = 1, \ldots, n$ *, then under the model (2.29) all the banks can survive by the comparison argument, i.e.,* $\mathbb{P}(|I_t| = n \textit{ for all } t) = 1$.

3. Conclusion. We analyze the simple model (2.1) and its exit system (2.29). Propositions 2.1-2.5 show that the growth rate and the lending preference are important to understand the systemic risk of inter-bank lending. Especially, the results spotlight how the lending preference leads illiquid monetary flow and multiple defaults. Our study of the Feller diffusion model (2.1) for inter-bank lending and borrowing dynamics has revealed the important quantities (2.15) and (2.16) which can be used by regulators to assess systemic risk. These indicators are model-independent but they can be rigorously analyzed under the model proposed in this paper. Some other quantities of interest such as the distribution of number of defaults are not given explicitly, but they can be obtained relatively easily under our model by simulations using Interacting Particle System methods.

This simple model presented in the paper may be extended to more general models with (i) the drift term that contains explicit interest rates, with (ii) the correlated random noise in the volatility term, and with (iii) controls by each bank. These extensions are on-going research topics.

4. Appendix. **[Proof of Proposition 2.4]**. In the case of the identical growth rates $\delta_i \equiv \delta$ and the symmetric lending preference $p_{i,j}(\cdot) = p_{j,i}(\cdot)$ for $i, j = 1, \ldots, n$ the deviation $Y_i(\cdot) = X_i(\cdot) - n^{-1}\mathcal{X}(\cdot)$ from the average $n^{-1}\mathcal{X}(\cdot) = n^{-1}\sum_{i=1}^n X_i(\cdot)$ satisfies

$$d\,Y_i(t) = \sum_{j=1}^n (X_j - X_i)\cdot p_{i,j}(X(t))d\,t + 2\sqrt{X_i(t)}d\,W_i(t) - \frac{2}{n}\sum_{j=1}^n \sqrt{X_j(t)}d\,W_j(t)$$

for $1 \le i \le n$, $0 \le t < \infty$, and hence by Itô's formula

$$\begin{aligned} d\sum_{i=1}^n (Y_i(t))^2 =& 2\Big[\sum_{i=1}^n Y_i(t)\sum_{j=1}^n (Y_j(t) - Y_i(t))\cdot p_{i,j}(X(t)) + 2\Big(1 - \frac{1}{n}\Big)\sum_{k=1}^n X_k(t)\Big]d\,t \\ & + 2\sum_{i=1}^n Y_i(t)\sqrt{X_i(t)}d\,W_i(t)\,; \quad 0 \le t < \infty\,. \end{aligned}$$

The process $\sum_{i=1}^n (Y_i(\cdot))^2$ is non-negative. Its diffusion coefficient becomes zero only when $X_1(\cdot) = X_2(\cdot) = \cdots = X_n(\cdot)$. Since we assume $\delta > 0$, the amount of time in which the process $\mathcal{X}(\cdot)$ in (2.2)-(2.3) spends at the origin is at most zero Lebesgue measure almost surely (recall Section 2.2). This implies that the amount of time in which the diffusion coefficient of $\sum_{i=1}^n (Y_i(\cdot))^2$ becomes zero is at most zero Lebesgue measure. In the following we show first that the squared sum $\sum_{i=1}^n (Y_i(\cdot))^2$ is positive recurrent.

For notational simplicity let us define the maximum $Z_1(\cdot) := \max_{1\le i\le n} X_i(\cdot)$, the minimum $Z_n(\cdot) := \min_{1\le i\le n} X_i(\cdot)$ and the range $\Xi(\cdot) := Z_1(\cdot) - Z_n(\cdot)$. In general, let us write $Z_k(\cdot)$ for the kth largest among $(X_1(\cdot), \ldots, X_n(\cdot))$, and let us also introduce the stochastic number $N_k(\cdot)$ of processes whose sizes are the same as $Z_k(\cdot)$ by $N_k(\cdot) := |\{i : X_i(\cdot) = Z_k(\cdot)\}$ for $k = 1, \ldots, n$.

It follows from Theorem 2.3 (page 1246) of Banner & Ghomrasni [3] that the rankings $Z_1(\cdot), \ldots, Z_n(\cdot)$ with $Z_1(\cdot) \ge \cdots \ge Z_n(\cdot)$ satisfy

$$d\,Z_k(t) = \sum_{i=1}^n (N_k(t))^{-1}\mathbf{1}_{\{Z_k(t) = X_i(t)\}}d\,X_i(t) + \sum_{j=k+1}^n (N_k(t))^{-1}d\,L_t(Z_k - Z_j)$$

$$-\sum_{j=1}^{k-1}(N_k(t))^{-1}d\,L_t(Z_j - Z_k)\,; \quad 0 \le t < \infty\,,\, k = 1, \dots, n\,,$$

where $L_t(U)$ is the local time accumulated at the origin by semimarttingale U until time $t \ge 0$. Taking the difference $\Xi(\cdot) = Z_1(\cdot) - Z_n(\cdot)$, we obtain the dynamics

$$d\,\Xi(t) = d\,(Z_1(t) - Z_n(t)) = d\,M(t) + \mathbf{G}(X(t))d\,t + d\,(\text{ local times })\,, \tag{4.1}$$

for $0 \le t < \infty$, where

$$d\,M(\cdot) := \sum_{i=1}^{n}\mathbf{1}_{\{Z_1(\cdot)=X_i(\cdot)\}}\sqrt{X_i(\cdot)}d\,W_i(\cdot) - \sum_{\ell=1}^{n}\mathbf{1}_{\{Z_n(\cdot)=X_\ell(\cdot)\}}\sqrt{X_\ell(\cdot)}d\,W_\ell(\cdot)\,,$$

$$\mathbf{G}(x) := \sum_{1\le i,j\le n}\mathbf{1}_{\{z_1=x_i\}}(x_j - x_i)p_{i,j}(x) - \sum_{1\le k,\ell\le n}\mathbf{1}_{\{z_n=x_\ell\}}(x_k - x_\ell)p_{\ell,k}(x)\,,$$

and $z_1 := \max_{1\le i\le n} x_i$, $z_n := \min_{1\le i\le n} x_i$ for $x = (x_1, \dots, x_n) \in [0,\infty)^n$. Here the local times work as minimum amount of push which keeps the range process $\Xi(\cdot)$ in the positive real line. Note that $(x_j - x_i)p_{i,j}(x) \le 0$ for $1 \le i, j \le n$ on the set $\{z_1 = x_i\}$, and $-(x_k - x_\ell)p_{\ell,k}(x) \le 0$ for $1 \le k, \ell \le n$ on the set $\{z_n = x_\ell\}$. Thus

$$\mathbf{G}(x) \le -2(z_1 - z_n)p_{i,\ell}(x)\mathbf{1}_{\{z_1=x_i, z_n=x_\ell\}} < 0\,; \quad x \in [0,\infty)^n\,.$$

because of the symmetry assumption $p_{i,j}(\cdot) = p_{j,i}(\cdot)$. Moreover, using the positive constants c_3, c_4 in the assumption (2.21), we estimate $\mathbf{G}(X(t))$ by

$$\mathbf{G}(X(t)) \le -c_4\Xi(t) \qquad \text{whenever } \Xi(t) = Z_1(t) - Z_n(t) \ge c_3\,. \tag{4.2}$$

Now let us define the time-changed process $\xi_t := \Xi(C_t)$ of $\Xi(\cdot)$ in (4.1) by the stochastic clock $C_t := \inf\{s : \langle\Xi\rangle(s) \ge t\}$ derived from its quadratic variation process $\langle\Xi\rangle(t) = \langle Z_1 - Z_n\rangle(t) = \int_0^t (4\Xi(s))d\,s$ for $t \ge 0$. It follows that

$$d\,\xi_t = d\,[\,M(C_t)\,] + [\,(4\xi_t)^{-1}\mathbf{G}(X(C_t))\,]d\,t + dL_t\,; \quad 0 \le t < \infty$$

where $L_\cdot$ consists of the corresponding time-changed local times obtained from (4.1). The local martingale $M(C_\cdot)$ is a Brownian motion (on a possibly extended probability space), because of the F. Knight Theorem. The drift part $(4\xi_\cdot)^{-1}\mathbf{G}(X(C_\cdot))$ is dominated by $-c_4/4 < 0$, whenever $\xi_t \ge c_3$, because of the estimate (4.2). The time-changed local time $L_\cdot$ is almost surely carried by the random set $\{t : X_i(C_t) = X_j(C_t) \text{ for some } i,j\}$ of zero Lebesgue measure. Thus by the comparison Theorem [22] the process $\xi_\cdot$ is dominated by a Brownian motion with the negative drift $-c_3/4$, whenever $\xi_\cdot > c_4$ and $X_i(C_\cdot) \ne X_j(C_\cdot)$ for all $1 \le i,j \le n$. It implies that $\xi_\cdot$ is positive recurrent and hence so does the range process $\Xi(\cdot) = Z_1(\cdot) - Z_n(\cdot) = \max_{1\le i\le n} X_i(\cdot) - \min_{1\le j\le n} X_j(\cdot)$. Furthermore, because of an equality $\sum_{i=1}^n (Y_i(\cdot))^2 \le n(\Xi(\cdot))^2$, the process $\sum_{i=1}^n (Y_i(\cdot))^2$ is also positive recurrent.

Finally, by extending Theorem 4.1 & 5.1 on page 119-121 of K'hasminskii [24] (also see Proof of Theorem 1 of [21]) for this degenerate case (but the process $Y(\cdot)$ stays the region $\{y \in \mathbb{R}^n : y_1 = y_2 = \cdots = y_n = 0\}$ of degeneracy only in the the time amount of zero Lebesgue measure) we can conclude that $Y(\cdot)$ is stochastically stable in the sense of (2.22). □

REFERENCES

[1] D.A. Adams, R.M. Ziff, and L.M. Sander. Computation of nucleation at a nonequilibrium first-order phase transition using a rare-event algorithm. *J. Chem. Phys.*, 133:174107–1–9, 2010.

[2] T. Adrian and H.S. Shin. Liquidity and leverage. *J. Financ. Intermed.*, **19**:418–437, 2010.

[3] A.D. Banner and R. Ghomrasni. Local times of ranked continuous semimartingales. *Stoch. Proc. Appl.*, **118**:1244–1253, 2008.

[4] R. F. Bass and E. A. Perkins. Degenerate stochastic differential equations with Hölder continuous coefficients and super-Markov chains. *Trans. Amer. Math. Soc.*, 355(1):373–405 (electronic), 2003.

[5] R. Carmona, J.-P. Fouque, and D. Vestal. Interacting particle systems for the computation of rare credit portfolio losses. *Finance Stoch.*, 13(6):613–633, 2009.

[6] R. Carmona, J.-P. Fouque, and L.-H. Sun. Mean Field Games and Systemic Risk. *Communications in Mathematical Sciences*, to appear.

[7] R. Cont and J.P. Bouchaud. Herd behavior and aggregate fluctuations in financial markets. *Macroecon. Dyn.*, 4:170–196, 2000.

[8] J. T. Cox, A. Greven, and T. Shiga. Finite and infinite systems of interacting diffusions. *Probab. Theory Related Fields*, 103(2):165–197, 1995.

[9] S. Deng, K. Giesecke, and T.L. Lai. Sequential importance sampling and resampling for dynamic portfolio credit risk. *Oper. Res.*, 60:78–91, 2012.

[10] T. Eisert, C. Eufinger. Interbank network and bank bailouts: insurance mechanism for non-insured creditors? *Available at SSRN: http://ssrn.com/abstract=1976383*, 2013.

[11] K. D. Elworthy, Xue-Mei Li, and M. Yor. The importance of strictly local martingales; applications to radial Ornstein-Uhlenbeck processes. *Probab. Theory Related Fields*, **115**(3):325–355, 1999.

[12] P. Erdös and A. Rényi. On random graphs. *Publ. Math. Debrecen*, 6:290–297, 1960.

[13] J.-P. Fouque and J. Langsam. Handbook on Systemic Risk. *Cambridge University Press*, 2013.

[14] J.-P. Fouque and L.-H. Sun. Systemic Risk Illustrated. *Handbook on Systemic Risk*, Cambridge University Press, Eds J.-P. Fouque and J. Langsam, 2013.

[15] H. Föllmer. Random economics with many interacting agents. *J. Math. Econ.*, 1:51–62, 1974.

[16] J. Garnier, G. Papanicolaou and T-W. Yang. Large deviations for a mean field model of systemic risk. *arXiv:1204.3536*, 2012.

[17] K. Giesecke and B. Kim. Systemic risk: What defaults are telling us. *Management Scienc*, **57**:1387–1405, 2011.

[18] E.N. Gilbert. Random graphs. *Annals of Mathematical Statistics*, 30:1141–1144, 1959.

[19] A. Göing-Jaeschke and M. Yor. A survey and some generalizations of Bessel processes. *Bernoulli*, **9**(2):313–349, 2003.

[20] T. Ichiba and I. Karatzas. On collisions of Brownian particles. *Ann. Appl. Probab.*, 20(3):951–977, 2010.

[21] T. Ichiba, V. Papathanakos, A. Banner, I. Karatzas, and R. Fernholz. Hybrid Atlas models. *Ann. Appl. Probab.*, 21(2):609–644, 2011.

[22] N. Ikeda and S. Watanabe. A comparison theorem for solutions of stochastic differential equations and its applications. *Osaka Journal of Mathematics*, 14:619–633, 1977.

[23] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*, volume **113** of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[24] R.Z. Khas'miniskii. *Stochastic stability of differential equations*. Sijthoff and Noerdhoff, Amsterdam, 1980.

[25] J. Lorenz, S. Battiston, and F. Schweitzer. Systemic risk in a unifying framework for cascading processes on networks. *arXiv*, 0907.5325v1:1–43, 2009.

[26] S.M. Markose, A. Alentorn, and A. Krause. Dynamic learning, herding and guru effects in networks. *University of Essex, Economics Department Working Paper*, 582, 2004.

[27] J. Muller. Interbank credit lines as a channel of contagion. *J. Financ. Res.*, 29(1):37–60, 2006.

[28] E. Nier, Jing Yang, T. Yorulmazer, and A. Alentorn. Network models and financial stability. *Bank of England Working Paper*, 346:1–28, 2008.

[29] L.C.G. Rogers, L.A.M. Veraart. Failure and rescue in an interbank network. *Manage. Sci.* online 2012.

[30] E.B. Santos and R. Cont. The Brazilian interbank network structure and systemic risk. *Banco Central do Brasil Working Paper.*, 219:1–54, 2010.

[31] T. Shiga and A. Shimizu. Infinite-dimensional stochastic differential equations and their applications. *J. Math. Kyoto Univ.*, 20(3):395–416, 1980.

[32] K. Soramaki, M.L. Bech, J. Arnold, R.J. Glass, and W.E. Beyeler. The topology of interbank payment flows. *Federal Reserve Bank of New York Staff Reports*, 243:1–13, 2006.