Uncertain Volatility Models with Stochastic Bounds*

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Abstract. In this paper, we study a class of uncertain volatility models with stochastic bounds. Like in the regular uncertain volatility models, we know only that volatility stays between two bounds, but instead of using two deterministic bounds, the uncertain volatility fluctuates between two stochastic bounds generated by its inherent stochastic volatility process. This brings better accuracy and is consistent with the observed volatility path such as for the VIX as a proxy for instance. We apply a regular perturbation analysis upon the worst-case scenario price, and derive the first order approximation in the regime of slowly varying stochastic bounds. The original problem which involves solving a fully nonlinear PDE in dimension two for the worst-case scenario price, is reduced to solving a nonlinear PDE in dimension one and a linear PDE with source, which gives a tremendous computational advantage. Numerical experiments show that this approximation procedure performs very well, even in the regime of moderately slow varying stochastic bounds.

Key words. uncertain volatility, stochastic bounds, nonlinear Black–Scholes–Barenblatt PDE, approximation

AMS subject classifications. 60H10, 91G80, 35Q93

1. Introduction.

In the standard Black–Scholes model of option pricing (Black and Scholes [1973]), volatility is assumed to be known and constant over time. Since then, it has been widely recognized and well-documented that this assumption is not realistic. Extensions of the Black–Scholes model to model ambiguity have been proposed, such as the stochastic volatility approach (Heston [1993], Hull and White [1987]), the jump diffusion model (Andersen and Andreasen [2000], Merton [1976]), and the uncertain volatility model (Avellaneda et al. [1995], Lyons [1995]). Among these extensions, the uncertain volatility model has received intensive attention in Mathematical Finance for risk management purpose.

In the uncertain volatility models (UVMs), volatility is not known precisely and is assumed to lie between constant upper and lower bounds $\underline{\sigma}$ and $\overline{\sigma}$. These bounds could be inferred from extreme values of the implied volatilities of the liquid options, or from high-low peaks in historical stock- or option-implied volatilities. Under the risk-neutral measure, the price process of the risky asset satisfies the following stochastic differential equation (SDE):

$$dX_t = rX_t dt + \alpha_t X_t dW_t, \tag{1}$$

where r is the constant risk-free rate, (W_t) is a Brownian motion and the volatility process (α_t) belongs to a family $\mathcal{A}$ of progressively measurable and $[\underline{\sigma}, \overline{\sigma}]$-valued processes.

When pricing a European derivative written on the risky asset with maturity T and nonnegative payoff $h(X_T)$, the seller of the contract is interested in the worst-case scenario. By assuming the worst case, sellers are guaranteed coverage against adverse market behavior, if the realized volatility belongs to the candidate set. Soner et al. [2013] showed that the seller

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of the derivative can superreplicate it with initial wealth $\operatorname{ess\,sup}_{\alpha\in\mathcal{A}} \mathbb{E}_t[h(X_T)]$, whatever the true volatility process is. The worst-case scenario price at time $t < T$ is given by

$$P(t, X_t) := \exp(-r(T-t)) \operatorname{ess\,sup}_{\alpha\in\mathcal{A}} \mathbb{E}_t[h(X_T)], \tag{2}$$

where $\mathbb{E}_t[\cdot]$ is the conditional expectation given $\mathcal{F}_t$ with respect to the risk neutral measure.

Following the arguments in stochastic control theory, $P(t, x)$ is the viscosity solution to the following Hamilton-Jacobi-Bellman (HJB) equation, which is the generalized Black–Scholes–Barenblatt (BSB) nonlinear equation in Financial Mathematics,

$$\begin{aligned} \partial_t P + r(x\partial_x P - P) + \sup_{\alpha\in[\underline{\sigma},\overline{\sigma}]} \left[\frac{1}{2}x^2\alpha^2\partial^2_{xx}P\right] &= 0, \\ P(T, x) &= h(x). \end{aligned} \tag{3}$$

It is well known that the worst-case scenario price is equal to its Black–Scholes price with constant volatility $\overline{\sigma}$ (resp. $\underline{\sigma}$) for convex (resp. concave) payoff function (see Pham [2009] for instance). For general terminal payoff functions, an asymptotic analysis of the worst-case scenario option prices as the volatility interval degenerates to a single point is derived in Fouque and Ren [2014].

However, for longer time-horizons, it is no longer consistent with observed volatility to assume that the bounds are constant (for instance by looking at the VIX over years, which is a popular measure of the implied volatility of SP500 index options). Therefore, instead of modeling α_t fluctuating between two deterministic bounds i.e. $\underline{\sigma} \le \alpha_t \le \overline{\sigma}$, it is reasonable to consider the case that the uncertain volatility moves between two stochastic bounds i.e. $\underline{\sigma}_t \le \alpha_t \le \overline{\sigma}_t$. Nutz [2013] introduced the notion of random G-expectation, which successfully extended the G-expectation (see Peng [2007]) by allowing the range of the volatility uncertainty to be stochastic. Later Neufeld and Nutz [2013] established the duality formula for the superreplication price, in a setting of volatility uncertainty including random G-expectation. Nutz and van Handel [2013] consolidated the foundation of this new area, by providing a general construction of time-consistent sublinear expectations on the space of continuous paths, which yields the existence of the conditional G-expectation of a Borel-measurable random variable and an optional sampling theorem. Neufeld and Nutz [2017] further provided the PDE characterization of the superreplication price in a jump diffusion setting, in which the link between the worst-case scenario price under stochastic bounds and its associated BSB equation is established for the first time.

In this paper, we study a class of models where the bounds are stochastic and slowly moving. The "center" of the bounds follows a stochastic process $F(Z_t)$, where F is a positive increasing and differentiable function on the domain of a regular diffusion of the form

$$dZ_t = \delta\mu(Z_t)dt + \sqrt{\delta}\beta(Z_t)dW^Z_t. \tag{4}$$

Here, W^Z is a Brownian motion possibly correlated to the Brownian motion W driving the stock price, with $d\langle W, W^Z\rangle_t = \rho dt$ for $|\rho| \le 1$. The parameter $\delta > 0$ represents the reciprocal of the time-scale of the process Z and will be small in the asymptotics that we consider in the paper. The volatility bound itself is given by

$$\underline{\sigma}_t := d\,F(Z_t) \le \alpha_t \le \overline{\sigma}_t := u\,F(Z_t), \quad \text{for} \quad 0 \le t \le T, \tag{5}$$

with u and d two constants such that $0 < d < 1 < u$. In the following, we will use the popular CIR process for Z, that is $\mu(z) = \kappa(\theta - z)$ and $\beta(z) = \sqrt{z}$ in (4), under the Feller condition $2\kappa\theta \geq 1$ to ensure that Z_t stays positive:

$$dZ_t = \delta\kappa(\theta - Z_t)dt + \sqrt{\delta}\sqrt{Z_t}dW_t^Z, \quad Z_0 = z > 0. \tag{6}$$

Our asymptotic analysis will reveal that, to the order $\sqrt{\delta}$, only the vol-vol value $\beta(z)$, the volatility level $F(z)$ and its slope $F'(z)$ are involved, but not the drift function μ. In the spirit of the Heston model we will use $F(z) = \sqrt{z}$ on $(0, \infty)$, and we will also give the corresponding formulas for our approximation in terms of a general function F. We denote $\alpha_t := q_t\sqrt{Z_t}$ so that the uncertainty in the volatility can be absorbed in the uncertain adapted slope as follows

$$d \leq q_t \leq u, \quad \text{for} \quad 0 \leq t \leq T.$$

One realization of the bounds is shown in Figure 1 with $\delta = .05$.

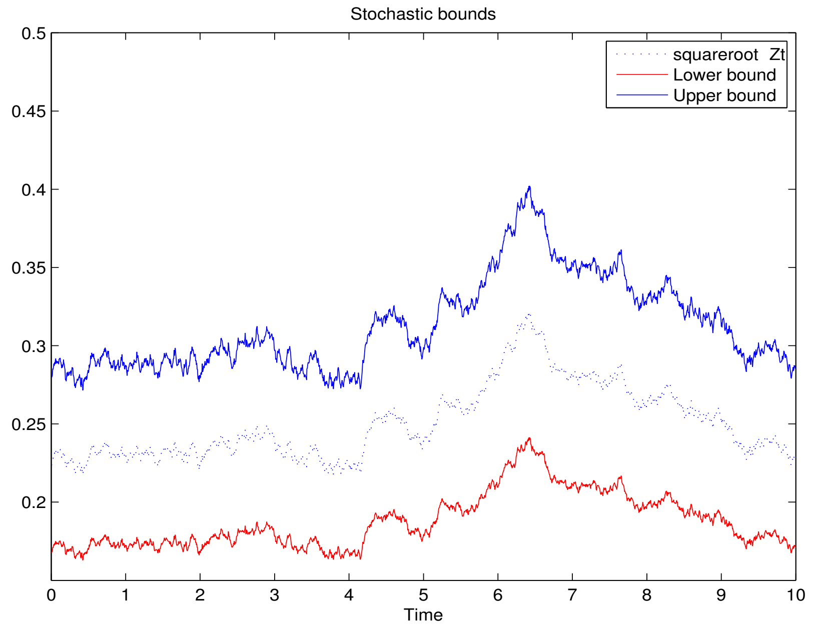

Figure 1. *Simulated stochastic bounds* $[0.75\sqrt{Z_t}, 1.25\sqrt{Z_t}]$ *where* Z_t *is the (slow) mean-reverting CIR process* (6).

In order to study the asymptotic behavior, we emphasize the importance of δ and reparameterize the SDE of the risky asset price process as

$$dX_t^\delta = rX_t^\delta dt + q_t\sqrt{Z_t}X_t^\delta dW_t. \tag{7}$$

When $\delta = 0$, note that the CIR process Z_t is frozen at z, and then the risky asset price process follows the dynamic

$$dX_t^0 = rX_t^0 dt + q_t\sqrt{z}X_t^0 dW_t, \tag{8}$$

both X_t^δ and X_t^0 starting at the same point x.

We denote the smallest riskless selling price (worst-case scenario) at time $t < T$ as

$$P^\delta(t,x,z) := \exp(-r(T-t))\,\mathrm{ess\,sup}_{q.\in[d,u]}\,\mathbb{E}_{(t,x,z)}[h(X_T^\delta)], \tag{9}$$

where $\mathbb{E}_{(t,x,z)}[\cdot]$ is the conditional expectation given $\mathcal{F}_t$ with $X_t^\delta = x$ and $Z_t = z$. When $\delta = 0$, we represent the smallest riskless selling price as

$$P_0(t,x,z) = \exp(-r(T-t))\,\mathrm{ess\,sup}_{q.\in[d,u]}\,\mathbb{E}_{(t,x,z)}[h(X_T^0)], \tag{10}$$

where the subscripts in $\mathbb{E}_{(t,x,z)}[\cdot]$ also means that $X_t^0 = x$ and $Z_t = z$ given the same filtration $\mathcal{F}_t$. Notice that $P_0(t,X_t,z)$ corresponds to $P(t,X_t)$ in (2) with constant volatility bounds given by $d\sqrt{z}$ and $u\sqrt{z}$.

Before displaying our result, it is worth mentioning some related new literatures. The result of Nutz [2015] can be used to derive a robust superhedging duality and the existence of an optimal superhedging strategy for general contingent claims. Neufeld and Nutz [2016] studied a robust portfolio optimization problem under model uncertainty for an investor with logarithmic or power utility, where uncertainty is specified by a set of possible Lévy triplets. Muhle-Karbe and Nutz [2016] analyzed the formation of derivative prices in equilibrium between risk neutral agents with heterogeneous beliefs, in the spirit of uncertain volatility with stochastic bounds.

The rest of the paper proceeds as follows. In Section 2, we first explore the convergence of the worst-case scenario price P^δ and its second derivative $\partial_{xx}^2 P^\delta$ as δ goes to 0. We then write down the pricing nonlinear parabolic PDE (11) which characterizes the option price $P^\delta(t,x,z)$ as a function of the present time t, the value x of the underlying asset, and the levels z of the volatility driving process. At last, we introduce the main result that the first order approximation to P^δ is $P_0 + \sqrt{\delta}P_1$ with accuracy in the order of $\mathcal{O}(\delta)$, where we define P_0 and P_1 in (13) and (18) respectively. The proof of the main result is presented in Section 3. In Section 4, a numerical illustration is presented. We conclude in Section 5. Some technical proofs are given in the Appendices.

2. Main Result.

In this section, we first prove the Lipschitz continuity of the worst-case scenario price P^δ with respect to the parameter δ. Then, we derive the main BSB equation that the worst-case scenario price should follow and further identify the first order approximation when δ is small enough. We reduce the original problem of solving the fully nonlinear PDE (11) in dimension two to solving the nonlinear PDE (13) in dimension one and a linear PDE (18) with source. The accuracy of this approximation is given in Theorem 2.14, the main theorem of this paper.

2.1. Convergence of P^δ.

It is established in Appendix A that X_t^δ and Z_t have finite moments for δ sufficiently small, which leads to the following result:

Proposition 2.1. *Let X^δ satisfies the SDE (7) and X^0 satisfies the SDE (8), then, uniformly in $(q.)$,*

$$\mathbb{E}_{(t,x,z)}(X_T^\delta - X_T^0)^2 \le C_0\delta$$

where C_0 is a positive constant independent of δ.

Proof. See Appendix B. ■

In order to carry out our asymptotic analysis, we need to impose some regularity of the payoff function h. Note that our numerical example in Section 4, a "butterfly" profile, does not satisfy these assumptions but we mention there a possible regularization step.

Assumption 2.2. *We assume that the terminal function h is four times differentiable, with a bounded first derivative and polynomial growth of the fourth derivative:*

$$\begin{cases} |h'(x)| \leq K_1, \\ |h^{(4)}(x)| \leq K_4(1+|x|^l), \end{cases}$$

for constants K_1 and K_4, and an integer l.

Remark 2.3. *The polynomial growth condition on $h^{(4)}$ implies polynomial growth of h'' and h''', and the bounded first derivative assumption implies that h is Lipschitz.*

Remark 2.4. *Note that for convex or concave payoff functions, such as for vanilla European Calls or Puts, if $h(\cdot)$ is convex (resp. concave), for the reason that supremum and expectation preserves convexity (resp. concavity), one can see that the worst-case scenario price*

$$P^\delta(t,x,z) = \exp(-r(T-t)) \operatorname{ess\,sup}_{q.\in[d,u]} \mathbb{E}_{(t,x,z)}[h(X_T)],$$

is convex (resp. concave) with $\partial^2_{xx} P^\delta > 0$ (resp. < 0), and thus $q^{*,\delta} = u$ (resp. $= d$). In these two cases, we are back to perturbations around Black–Scholes prices which have been treated in Fouque et al. [2011]. In this paper, we work with general terminal payoff functions, neither convex nor concave, therefore the signs of the second derivatives of the option prices cannot be easily determined. In order to proceed we impose regularity conditions on the payoff functions (Assumption 2.2) as in Fouque and Ren [2014].

Theorem 2.5. *Under Assumption 2.2, $P^\delta(\cdot,\cdot,\cdot)$, as a family of functions of (t,x,z) indexed by δ, converges to $P_0(\cdot,\cdot,\cdot)$ with rate $\sqrt{\delta}$, uniformly in $(t,x,z) \in [0,T]\times\mathbb{R}\times\mathbb{R}^+$.*

Proof. For P^δ given by (9) and P_0 given by (10), using the Lipschitz continuous of $h(\cdot)$ and by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |P^\delta - P_0| =& \exp(-r(T-t)) \left| \operatorname{ess\,sup}_{q.\in[d,u]} \mathbb{E}_{(t,x,z)}[h(X_T^\delta)] - \operatorname{ess\,sup}_{q.\in[d,u]} \mathbb{E}_{(t,x,z)}[h(X_T^0)] \right| \\ \leq& \exp(-r(T-t)) \operatorname{ess\,sup}_{q.\in[d,u]} \left| \mathbb{E}_{(t,x,z)}[h(X_T^\delta)] - \mathbb{E}_{(t,x,z)}[h(X_T^0)] \right| \\ \leq& \exp(-r(T-t)) \operatorname{ess\,sup}_{q.\in[d,u]} \mathbb{E}_{(t,x,z)} \left| h(X_T^\delta) - h(X_T^0) \right| \\ \leq& K_0 \exp(-r(T-t)) \operatorname{ess\,sup}_{q.\in[d,u]} \mathbb{E}_{(t,x,z)} \left| X_T^\delta - X_T^0 \right| \\ \leq& K_0 \exp(-r(T-t)) \operatorname{ess\,sup}_{q.\in[d,u]} \left[\mathbb{E}_{(t,x,z)} (X_T^\delta - X_T^0)^2 \right]^{1/2}. \end{aligned}$$

Therefore, by Proposition 2.1, we have

$$|P^\delta - P_0| \leq C_1\sqrt{\delta},$$

where C_1 is a positive constant independent of δ, as desired. ■

2.2. Pricing Nonlinear PDEs. We now derive P_0 and P_1, the leading order term and the first correction for the approximation of the worst-case scenario price P^δ, which is the solution to the HJB equation associated to the corresponding control problem given by the generalized BSB nonlinear equation:

$$\begin{aligned} \partial_t P^\delta + r(x\partial_x P^\delta - P^\delta) + \sup_{q\in[d,u]} \{\frac{1}{2}q^2 z x^2 \partial^2_{xx} P^\delta + \sqrt{\delta}(q\rho z x \partial^2_{xz} P^\delta)\} \\ +\delta(\frac{1}{2} z \partial^2_{zz} P^\delta + \kappa(\theta - z)\partial_z P^\delta) = 0, \end{aligned} \tag{11}$$

with terminal condition $P^\delta(T,x,z) = h(x)$. For simplicity and without loss of generality, $r = 0$ is assumed for the rest of paper.

In this section, we use the regular perturbation approach to formally expand the value function $P^\delta(t,x,z)$ as follows:

$$P^\delta = P_0 + \sqrt{\delta}P_1 + \delta P_2 + \cdots . \tag{12}$$

Inserting this expansion into the main BSB equation (11), by Theorem 2.5, the leading order term P_0 is the solution to

$$\begin{aligned} \partial_t P_0 + \sup_{q\in[d,u]} \{\frac{1}{2}q^2 z x^2 \partial^2_{xx} P_0\} = 0, \\ P_0(T,x,z) = h(x). \end{aligned} \tag{13}$$

In this case, z is just a positive parameter, and we have existence and uniqueness of a smooth solution to (13) (we refer to Pham [2009]). Note that in the general model given by (4) and (5), the equation for P_0 would be

$$\partial_t P_0 + \sup_{q\in[d,u]} \{\frac{1}{2}q^2 \sigma^2 x^2 \partial^2_{xx} P_0\} = 0, \quad \sigma := F(z).$$

2.2.1. Convergence of $\partial^2_{xx} P^\delta$. In what follows, we will assume regularity of the solution P^δ of the nonlinear PDE (11).

Assumption 2.6. *Throughout the paper, we make the following assumptions on P^δ:*
(i) $P^\delta(\cdot,\cdot,\cdot)$ belongs to $C_p^{1,2,2}$ (p for polynomial growth), for δ fixed.
(ii) $\partial_x P^\delta$ and $\partial^2_{xx} P^\delta$ are uniformly bounded in δ.

In Appendix C, we discuss these assumptions in a simpler setting, in particular, for a function $F(z)$ in (5) which is bounded and bounded away from zero, and a smooth terminal condition. After a change of variable $x = \log y$, and using the fact that $q \in [d,u]$ with $0 < d < u < \infty$, equation (11) is uniformly parabolic and classical results from Fleming and Soner [2006] and Krylov [1987] can be applied to deduce (i). (ii) follows from the derivation of estimates for first and second order derivatives in Sections IV.8 and IV.9 of Fleming and Soner [2006].

Remark 2.7. *In the present paper, we are concerned with a practical approximation of the superreplication problem viewed as a perturbation around the classical case of fixed volatility bounds. Our starting point is a superreplication price given as the classical solution of a non-linear PDE. Regarding the link between the worst-case scenario option price with its associated BSB equation, as well as regularity conditions and uniform boundedness of derivatives, we refer to Neufeld and Nutz [2013] and Lemma 3.2 in Muhle-Karbe and Nutz [2016] in a different context.*

Then, under Assumption 2.6, we have the following Proposition:

Proposition 2.8. *Under Assumptions 2.2 and 2.6, the family $\partial^2_{xx}P^\delta(\cdot,\cdot,\cdot)$ of functions of (t,x,z) indexed by δ, converges to $\partial^2_{xx}P_0(\cdot,\cdot,\cdot)$ as δ tends to 0 with rate $\sqrt{\delta}$, uniformly on compact sets in (x,z) and $t\in[0,T]$.*

Proof. Under Assumptions 2.2 and 2.6, and by Theorem 2.5, the Proposition can be obtained by following the arguments in Theorem 5.2.5 of Giga et al. [2010]. ∎

Denote the zero sets of $\partial^2_{xx}P_0$ as

$$S^0_{t,z} := \{x = x(t,z) \in \mathbb{R}^+ | \partial^2_{xx}P_0(t,x,z) = 0\}.$$

Define the set where $\partial^2_{xx}P^\delta$ and $\partial^2_{xx}P_0$ take different signs as

$$\begin{aligned} A^\delta_{t,z} :=&\{x = x(t,z) | \partial^2_{xx}P^\delta(t,x,z) > 0, \partial^2_{xx}P_0(t,x,z) < 0\} \\ &\cup \{x = x(t,z) | \partial^2_{xx}P^\delta(t,x,z) < 0, \partial^2_{xx}P_0(t,x,z) > 0\}. \end{aligned} \tag{14}$$

Assumption 2.9. *We make the following assumptions:*
(i) There is a finite number of zero points of $\partial^2_{xx}P_0(t,x,z)$, for any $t\in[0,T]$ and $z>0$, that is, $S^0_{t,z} = \{x_1 < x_2 < \cdots < x_{m(t,z)}\}$, where we assume that the number $m(t,z)$ is uniformly bounded in $t \le T$ and $z \in \mathbb{R}$.
(ii) There exists a constant C such that the set $A^\delta_{t,z}$ defined in (14) is included in $\cup_{i=1}^{m(t,z)} I^\delta_i$, where

$$I^\delta_i := [x_i - C\sqrt{\delta}, x_i + C\sqrt{\delta}], \quad \text{for } x_i \in S^0_{t,z} \text{ and } 1 \le i \le m(t,z).$$

Furthermore, we assume that for every $M>0$ there exists $B>0$ such that $|x_i| \le B$ for any $x_i \in S^0_{t,z}$, $1\le i\le m(t,z)$, $z \le M$.

Remark 2.10. *Here we explain the rationale for Assumption 2.9 (ii).*

Suppose P_0 has a third derivative with respect to x, which does not vanish on the set $S^0_{t,z}$. By Proposition 2.8, $\partial^2_{xx}P^\delta$ converges to $\partial^2_{xx}P_0$ with rate $\sqrt{\delta}$, therefore we conclude that there exists a constant C such that on the set $(\cup_{i=1}^{m(t,z)} I^\delta_i)^c$, $\partial^2_{xx}P^\delta(t,x,z)$ and $\partial^2_{xx}P_0(t,x,z)$ have the same sign, and Assumption 2.9 (ii) would follow. This is illustrated in Figure 6 by an example with two zero points for $\partial^2_{xx}P_0(t,x,z)$.

Otherwise, I^δ_i would have a larger radius of order $\mathcal{O}(\delta^\alpha)$ for $\alpha \in (0,\frac{1}{2})$, and then the accuracy in the main Theorem 2.14 would be $\mathcal{O}(\delta^{\alpha+1/2})$, but in any case of order $o(\sqrt{\delta})$.

In the sequel, in order to simplify the expressions, we denote

$$P_0 := P_0(t,x,z) \quad \text{and} \quad P^\delta := P^\delta(t,x,z),$$

and similar notations apply to the corresponding derivatives.

2.2.2. Optimizers. The optimal control in the nonlinear PDE (13) for P_0, denoted as

$$q^{*,0}(t,x,z) := \arg\max_{q\in[d,u]}\{\frac{1}{2}q^2 zx^2\partial^2_{xx}P_0\},$$

is given by

$$q^{*,0}(t,x,z) = \begin{cases} u, \partial^2_{xx}P_0 \geq 0 \\ d, \partial^2_{xx}P_0 < 0 \end{cases}. \tag{15}$$

The optimizer to the main BSB equation (11) is given in the following lemma:

Lemma 2.11. *Under Assumption 2.9, for δ sufficiently small and for $x \notin S^0_{t,z}$, the optimal control in the nonlinear PDE (11) for P^δ, denoted as*

$$q^{*,\delta}(t,x,z) := \arg\max_{q\in[d,u]}\{\frac{1}{2}q^2 zx^2\partial^2_{xx}P^\delta + \sqrt{\delta}(q\rho zx\partial^2_{xz}P^\delta)\},$$

is given by

$$q^{*,\delta}(t,x,z) = \begin{cases} u, \partial^2_{xx}P^\delta \geq 0 \\ d, \partial^2_{xx}P^\delta < 0 \end{cases}. \tag{16}$$

Proof. To find the optimizer $q^{*,\delta}$ to

$$\sup_{q\in[d,u]}\{\frac{1}{2}q^2 zx^2\partial^2_{xx}P^\delta + \sqrt{\delta}(q\rho zx\partial^2_{xz}P^\delta)\},$$

we firstly relax the restriction $q \in [d,u]$ to $q \in \mathbb{R}$.

Denote

$$f(q) := \frac{1}{2}q^2 zx^2\partial^2_{xx}P^\delta + \sqrt{\delta}(q\rho zx\partial^2_{xz}P^\delta).$$

By the result of Proposition 2.8 that $\partial^2_{xx}P^\delta$ uniformly converge to $\partial^2_{xx}P_0$ as δ goes to 0, for $x \notin S^0_{t,z}$, the optimizer of $f(q)$ is given by

$$\hat{q}^{*,\delta} = -\frac{\rho\sqrt{\delta}\partial^2_{xz}P^\delta}{x\partial^2_{xx}P^\delta}.$$

Since X_t and Z_t are strictly positive, the sign of the coefficient of q^2 in $f(q)$ is determined by the sign of $\partial^2_{xx}P^\delta$. We have the following cases represented in Figure 2, from which we can see that for δ sufficiently small such that $|\hat{q}^{*,\delta}| \leq d$, the optimizer is given by

$$q^{*,\delta} = u\mathbb{1}_{\{\partial^2_{xx}P^\delta \geq 0\}} + d\mathbb{1}_{\{\partial^2_{xx}P^\delta < 0\}}.$$

Plugging the optimizer $q^{*,\delta}$ given by Lemma 2.11, the BSB equation (11) can be rewritten as

$$\partial_t P^\delta + \frac{1}{2}(q^{*,\delta})^2 zx^2\partial^2_{xx}P^\delta + \sqrt{\delta}(q^{*,\delta}\rho zx\partial^2_{xz}P^\delta) + \delta(\frac{1}{2}z\partial^2_{zz}P^\delta + \kappa(\theta - z)\partial_z P^\delta) = 0, \tag{17}$$

with terminal condition $P^\delta(T,x,z) = h(x)$.

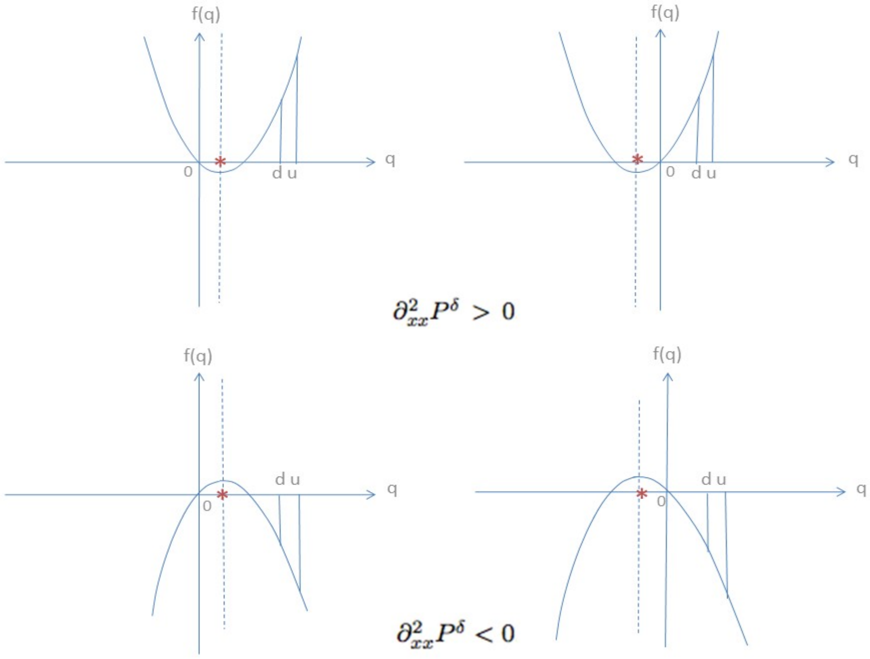

Figure 2. *Illustration of the derivation of $q^{*,\delta}$: if $\partial^2_{xx}P^\delta > 0$, whether $\hat{q}^{*,\delta}$ is positive or negative, with the requirement $q \in [d,u]$, $q^{*,\delta} = u$; otherwise $q^{*,\delta} = d$.*

2.2.3. Heuristic Expansion and Accuracy of the Approximation. We insert the expansion (12) into the main BSB equation (17) and collect terms in successive powers of $\sqrt{\delta}$. Under Assumption 2.9 that $q^{*,\delta} \to q^{*,0}$ as $\delta \to 0$, without loss of accuracy, the first order correction term P_1 is chosen as the solution to the linear equation:

$$\begin{aligned} \partial_t P_1 + \frac{1}{2}(q^{*,0})^2 z x^2 \partial^2_{xx} P_1 + q^{*,0}\rho z x \partial^2_{xz} P_0 = 0, \\ P_1(T,x,z) = 0, \end{aligned} \tag{18}$$

where $q^{*,0}$ is given by (15).

Since (18) is linear, the existence and uniqueness result of a smooth solution P_1 can be achieved by firstly change the variable $x \to \ln x$, and then use the classical result of Friedman [1975] for the parabolic equation (18) with diffusion coefficient bounded below by $d^2 z > 0$.

Note that in the general model given by (4) and (5), using the chain rule, the equation for P_1 would be

$$\partial_t P_1 + \frac{1}{2}(q^{*,0})^2 \sigma^2 x^2 \partial^2_{xx} P_1 + q^{*,0}\rho\sigma\sigma'\beta x \partial^2_{x\sigma} P_0 = 0, \quad \sigma = F(z), \sigma' := F'(z), \beta := \beta(z).$$

We shall show in the following that under additional regularity conditions imposed on the derivatives of P_0 and P_1, the approximation error $|P^\delta - P_0 - \sqrt{\delta}P_1|$ is of order $\mathcal{O}(\delta)$.

Assumption 2.12. *The following derivatives of P_0 and P_1 are of polynomial growth:*

$$\left\{\begin{array}{l} |\partial^2_{xx}P_0(t,x,z)| \leq a_{20}(1+x^{b_{20}}+z^{c_{20}}) \\ |\partial^2_{xz}P_0(t,x,z)| \leq a_{11}(1+x^{b_{11}}+z^{c_{11}}) \\ |\partial_{z}P_0(t,x,z)| \leq a_{01}(1+x^{b_{01}}+z^{c_{01}}) \\ |\partial^2_{xx}P_1(t,x,z)| \leq \bar{a}_{20}(1+x^{\bar{b}_{20}}+z^{\bar{c}_{20}}) \\ |\partial_{z}P_1(t,x,z)| \leq \bar{a}_{01}(1+x^{\bar{b}_{01}}+z^{\bar{c}_{01}}) \\ |\partial^2_{zz}P_1(t,x,z)| \leq \bar{a}_{02}(1+x^{\bar{b}_{02}}+z^{\bar{c}_{02}}) \end{array}\right. \tag{19}$$

where $a_i, b_i, c_i, \bar{a}_i, \bar{b}_i, \bar{c}_i$ are positive integers for $i \in (20, 11, 01, 02)$.

Remark 2.13. *As explained at the beginning of Section 2.2.1, regularity of P_0 and subsequently of P_1 given by (18), can be obtained from the assumed regularity of the payoff h (Assumption 2.2). The proof being outside the scope of this paper, we list these properties as assumptions and we introduce the notation for the constants needed later.*

Theorem 2.14 (Main Theorem). *Under Assumptions 2.2, 2.9 and 2.12, the residual function $E^\delta(t,x,z)$ defined by*

$$E^\delta(t,x,z) := P^\delta(t,x,z) - P_0(t,x,z) - \sqrt{\delta}P_1(t,x,z) \tag{20}$$

is of order $\mathcal{O}(\delta)$. In other words, $\forall (t,x,z) \in [0,T]\times\mathbb{R}^+\times\mathbb{R}^+$, there exists a positive constant C, such that $|E^\delta(t,x,z)| \leq C\delta$, where C may depend on (t,x,z) but not on δ.

3. Proof of the Main Theorem 2.14.

Define the following operator

$$\begin{aligned} \mathcal{L}^\delta(q) : &= \partial_t + \frac{1}{2}q^2 z x^2 \partial^2_{xx} + \sqrt{\delta} q\rho z x \partial^2_{xz} + \delta(\frac{1}{2}z\partial^2_{zz} + \kappa(\theta - z)\partial_z) \\ &= \mathcal{L}_0(q) + \sqrt{\delta}\mathcal{L}_1(q) + \delta\mathcal{L}_2, \end{aligned} \tag{21}$$

where $\mathcal{L}_0(q)$ contains the time derivative and is the Black–Scholes operator $\mathcal{L}_{BS}(q\sqrt{z})$, $\mathcal{L}_1(q)$ contains the mixed derivative due to the covariation between X and Z, and $\delta\mathcal{L}_2$ is the infinitesimal generator of the process Z, also denoted by $\delta\mathcal{L}_{CIR}$.

The main equation (17) can be rewritten as

$$\begin{aligned} \mathcal{L}^\delta(q^{*,\delta})P^\delta &= 0, \\ P^\delta(t,x,z) &= h(x). \end{aligned} \tag{22}$$

Equation (13) becomes

$$\begin{aligned} \mathcal{L}_0(q^{*,0})P_0 &= 0, \\ P_0(T,x,z) &= h(x). \end{aligned} \tag{23}$$

Equation (18) becomes

$$\begin{aligned} \mathcal{L}_0(q^{*,0})P_1 + \mathcal{L}_1(q^{*,0})P_0 &= 0, \\ P_1(T,x,z) &= 0. \end{aligned} \tag{24}$$

Applying the operator $\mathcal{L}^\delta(q^{*,\delta})$ to the error term, it follows that

$$\begin{aligned}
\mathcal{L}^\delta(q^{*,\delta})E^\delta =& \mathcal{L}^\delta(q^{*,\delta})(P^\delta - P_0 - \sqrt{\delta}P_1)\\
=& \underbrace{\mathcal{L}^\delta(q^{*,\delta})P^\delta}_{=0, \text{ eq. } (22).} - \mathcal{L}^\delta(q^{*,\delta})(P_0 + \sqrt{\delta}P_1)\\
=& -\left(\mathcal{L}_0(q^{*,\delta}) + \sqrt{\delta}\mathcal{L}_1(q^{*,\delta}) + \delta\mathcal{L}_{CIR}\right)(P_0 + \sqrt{\delta}P_1)\\
=& -\mathcal{L}_0(q^{*,\delta})P_0 - \sqrt{\delta}\left[\mathcal{L}_1(q^{*,\delta})P_0 + \mathcal{L}_0(q^{*,\delta})P_1\right] - \delta\left[\mathcal{L}_1(q^{*,\delta})P_1 + \mathcal{L}_{CIR}P_0\right]\\
& - \delta^{\frac{3}{2}}\left[\mathcal{L}_{CIR}P_1\right]\\
=& -\underbrace{\mathcal{L}_0(q^{*,0})P_0}_{=0, \text{ eq. } (23).} - (\mathcal{L}_0(q^{*,\delta}) - \mathcal{L}_0(q^{*,0}))P_0 - \sqrt{\delta}\Bigg[\underbrace{\mathcal{L}_1(q^{*,0})P_0 + \mathcal{L}_0(q^{*,0})P_1}_{=0, \text{ eq. } (24).}\\
& + (\mathcal{L}_1(q^{*,\delta}) - \mathcal{L}_1(q^{*,0}))P_0 + (\mathcal{L}_0(q^{*,\delta}) - \mathcal{L}_0(q^{*,0}))P_1\Bigg]\\
& - \delta\left[\mathcal{L}_1(q^{*,\delta})P_1 + \mathcal{L}_{CIR}P_0\right] - \delta^{\frac{3}{2}}(\mathcal{L}_{CIR}P_1)\\
=& -(\mathcal{L}_0(q^{*,\delta}) - \mathcal{L}_0(q^{*,0}))P_0 - \sqrt{\delta}\Bigg[(\mathcal{L}_1(q^{*,\delta}) - \mathcal{L}_1(q^{*,0}))P_0\\
& + (\mathcal{L}_0(q^{*,\delta}) - \mathcal{L}_0(q^{*,0}))P_1\Bigg] - \delta\left[\mathcal{L}_1(q^{*,\delta})P_1 + \mathcal{L}_{CIR}P_0\right] - \delta^{\frac{3}{2}}(\mathcal{L}_{CIR}P_1)\\
=& -\frac{1}{2}[(q^{*,\delta})^2 - (q^{*,0})^2]zx^2\partial^2_{xx}P_0\\
& - \sqrt{\delta}\left[\rho(q^{*,\delta} - q^{*,0})zx\partial^2_{xz}P_0 + \frac{1}{2}\left((q^{*,\delta})^2 - (q^{*,0})^2\right)zx^2\partial^2_{xx}P_1\right]\\
& - \delta\left[\rho(q^{*,\delta})zx\partial^2_{xz}P_1 + \frac{1}{2}z\partial^2_{zz}P_0 + \kappa(\theta - z)\partial_z P_0\right]\\
& - \delta^{\frac{3}{2}}\left[\frac{1}{2}z\partial^2_{zz}P_1 + \kappa(\theta - z)\partial_z P_1\right],
\end{aligned}$$

where $q^{*,0}$ and $q^{*,\delta}$ are given in (15) and (16) respectively.

The terminal condition of E^δ is given by

$$E^\delta(T,x,z) = P^\delta(T,x,z) - P_0(T,x,z) - \sqrt{\delta}P_1(T,x,z) = 0.$$

3.1. Feynman–Kac representation of the error term. For δ sufficiently small, the optimal choice $q^{*,\delta}$ to the main BSB equation (11) is given explicitly in Lemma 2.11. Correspondingly, the asset price in the worst-case scenario is a stochastic process which satisfies the SDE (1) with $(q_t) = (q^{*,\delta})$ and $r = 0$, i.e.,

$$dX^{*,\delta}_t = q^{*,\delta}\sqrt{Z_t}X^{*,\delta}_t dW_t. \tag{25}$$

Given the existence and uniqueness result of $X_t^{*,\delta}$ proved in Appendix D, we have the following probabilistic representation of $E^\delta(t,x,z)$ by Feynman–Kac formula:

$$E^\delta(t,x,z) = I_0 + \delta^{\frac{1}{2}} I_1 + \delta I_2 + \delta^{\frac{3}{2}} I_3,$$

where

$$I_0 := \mathbb{E}_{(t,x,z)}\left[\int_t^T \frac{1}{2}\left((q^{*,\delta})^2 - (q^{*,0})^2\right) Z_s (X_s^{*,\delta})^2 \partial_{xx}^2 P_0(s, X_s^{*,\delta}, Z_s) ds\right],$$

$$I_1 := \mathbb{E}_{(t,x,z)}\left[\int_t^T \left((q^{*,\delta} - q^{*,0})\rho Z_s X_s^{*,\delta} \partial_{xz}^2 P_0(s, X_s^{*,\delta}, Z_s) + \frac{1}{2}\left((q^{*,\delta})^2 - (q^{*,0})^2\right) Z_s (X_s^{*,\delta})^2 \partial_{xx}^2 P_1(s, X_s^{*,\delta}, Z_s)\right) ds\right],$$

$$I_2 := \mathbb{E}_{(t,x,z)}\left[\int_t^T \left(q^{*,\delta}\rho Z_s X_s^{*,\delta} \partial_{xz}^2 P_1(s, X_s^{*,\delta}, Z_s) + \frac{1}{2} Z_s \partial_{zz}^2 P_0(s, X_s^{*,\delta}, Z_s) + \kappa(\theta - Z_s)\partial_z P_0(s, X_s^{*,\delta}, Z_s)\right) ds\right],$$

$$I_3 := \mathbb{E}_{(t,x,z)}\left[\int_t^T \left(\frac{1}{2} Z_s \partial_{zz}^2 P_1(s, X_s^{*,\delta}, Z_s) + \kappa(\theta - Z_s)\partial_z P_1(s, X_s^{*,\delta}, Z_s)\right) ds\right].$$

Note that for $q^{*,0}$ given in (15) and $q^{*,\delta}$ given in (16) , we have

$$q^{*,\delta} - q^{*,0} = (u - d)(\mathbb{1}_{\{\partial_{xx}^2 P^\delta \geq 0\}} - \mathbb{1}_{\{\partial_{xx}^2 P_0 \geq 0\}}), \tag{26}$$

and

$$(q^{*,\delta})^2 - (q^{*,0})^2 = (u^2 - d^2)(\mathbb{1}_{\{\partial_{xx}^2 P^\delta \geq 0\}} - \mathbb{1}_{\{\partial_{xx}^2 P_0 \geq 0\}}). \tag{27}$$

Also note that $\{q^{*,\delta} \neq q^{*,0}\} = A_{t,z}^\delta$ defined in (14).

In order to show that E^δ is of order $\mathcal{O}(\delta)$, it suffices to show that I_0 is of order $\mathcal{O}(\delta)$, I_1 is of order $\mathcal{O}(\sqrt{\delta})$, and I_2 and I_3 are uniformly bounded in δ. Clearly, I_0 is the main term that directly determines the order of the error term E^δ.

3.2. Control of the term I_0. In this section, we are going to handle the dependence in δ of the process $X^{*,\delta}$ by a time-change argument.

Theorem 3.1. *Under Assumptions 2.2, 2.6 and 2.9, there exists a positive constant M_0, such that*

$$|I_0| \leq M_0 \delta$$

where M_0 may depend on (t,x,z) but not on δ. That is, I_0 is of order $\mathcal{O}(\delta)$.

Proof. Since $0 < d \le q^{*,\delta}, q^{*,0} \le u$, we have

$$\begin{aligned}
I_0 =&\mathbb{E}_{(t,x,z)}\left[\int_t^T \frac{1}{2}\left((q^{*,\delta})^2-(q^{*,0})^2\right) Z_s (X_s^{*,\delta})^2\partial_{xx}^2 P_0(s,X_s^{*,\delta},Z_s)ds\right]\\
\le&\frac{u^2}{2d^2}\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{X_s^{*,\delta}\in A^\delta_{s,Z_s}\}}(q^{*,\delta})^2 Z_s(X_s^{*,\delta})^2|\partial_{xx}^2 P_0(s,X_s^{*,\delta},Z_s)|ds\right]\\
=&\frac{u^2}{2d^2}\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{X_s^{*,\delta}\in A^\delta_{s,Z_s}\}}(q^{*,\delta})^2 Z_s(X_s^{*,\delta})^2|\partial_{xx}^2 P_0(s,X_s^{*,\delta},Z_s)|\mathbb{1}_{\{\sup_{t\le s'\le T} Z_{s'}\le M\}}ds\right]\\
&+\frac{u^2}{2d^2}\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{X_s^{*,\delta}\in A^\delta_{s,Z_s}\}}(q^{*,\delta})^2 Z_s(X_s^{*,\delta})^2|\partial_{xx}^2 P_0(s,X_s^{*,\delta},Z_s)|\mathbb{1}_{\{\sup_{t\le s'\le T} Z_{s'}> M\}}ds\right]\\
:=&\mathcal{M}_1+\mathcal{M}_2.
\end{aligned}$$

In the following, we are going to show that both terms $\mathcal{M}_1$ and $\mathcal{M}_2$ are of order $\mathcal{O}(\delta)$.

Step 1. Control of term $\mathcal{M}_1$

Recall that, under Assumption 2.9, the set $A^\delta_{t,z}$ defined in (14) is included in $\cup_{i=1}^{m(t,z)} I_i^\delta$, which is included in a compact set for $z \le M$. From Proposition 2.8, there exists a constant C_0 such that

$$|\partial_{xx}^2 P_0(s,X_s^{*,\delta},Z_s)| \le C_0\sqrt{\delta},\ \text{for } t\le s\le T,\ X_s^{*,\delta}\in A^\delta_{s,Z_s} \text{ and } \sup_{t\le s\le T} Z_s \le M,$$

which yields

$$\begin{aligned}
\mathcal{M}_1 \le&\frac{u^2}{2d^2}\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{X_s^{*,\delta}\in A^\delta_{s,Z_s}\}}(q^{*,\delta})^2 Z_s(X_s^{*,\delta})^2 C_0\sqrt{\delta}\mathbb{1}_{\{\sup_{t\le s'\le T} Z_{s'}\le M\}}ds\right]\\
\le&\frac{u^2}{2d^2}C_0\sqrt{\delta}\,\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{X_s^{*,\delta}\in A^\delta_{s,Z_s}\}}(q^{*,\delta})^2 Z_s(X_s^{*,\delta})^2 \mathbb{1}_{\{\sup_{t\le s'\le T} Z_{s'}\le M\}}ds\right].
\end{aligned} \tag{28}$$

In order to show that $\mathcal{M}_1$ is of order $\mathcal{O}(\delta)$, it suffices to show that there exists a constant C_1 such that

$$\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{\sup_{t\le s'\le T} Z_{s'}\le M\}}\mathbb{1}_{\{X_s^{*,\delta}\in A^\delta_{s,Z_s}\}}\sigma^2(X_s^{*,\delta})ds\right]\le C_1\sqrt{\delta}, \tag{29}$$

where $\sigma(X_s^{*,\delta}) := q^{*,\delta}\sqrt{Z_s}X_s^{*,\delta}$ and $dX_s^{*,\delta} = \sigma(X_s^{*,\delta})dW_s$ by (25). Define the stopping time

$$\tau(v) := \inf\{s > t; \langle X^{*,\delta}\rangle_s > v\},$$

where $\langle X^{*,\delta}\rangle_s = \int_t^s \sigma^2(X_u^{*,\delta})du$. Then according to Theorem 4.6 (time-change for martingales) in Karatzas and Shreve [2012], we know that $B_v := X^{*,\delta}_{\tau(v)}$ is a standard one-dimensional Brownian motion. In particular, the filtration $\mathcal{F}_v^B := \mathcal{F}_{\tau(v)}$ satisfies the usual condition and we have $\mathbb{Q}$-a.s. $X_t^{*,\delta} = B_{\langle X^{*,\delta}\rangle_t}$.

From the definition of $\tau(v)$ given above, we have

$$\int_t^{\tau(v)} \sigma^2(X_s^{*,\delta})ds = v,$$

which tells us that the inverse function of $\tau(\cdot)$ is

$$\tau^{-1}(T) = \int_t^T \sigma^2(X_s^{*,\delta})ds. \tag{30}$$

Next use the substitution $s = \tau(v)$ and for any $i \in [1, m(v,z)]$, we have

$$\begin{aligned}
&\int_t^T \mathbb{1}_{\{\sup_{t\le s'\le T} Z_{s'}\le M\}}\mathbb{1}_{\{|X_s^{*,\delta}-x_i|<C\sqrt{\delta}\}}\sigma^2(X_s^{*,\delta})ds \\
=&\int_0^{\tau^{-1}(T)} \mathbb{1}_{\{\sup_{t\le s'\le T} Z_{s'}\le M\}}\mathbb{1}_{\{|X_{\tau(v)}^{*,\delta}-x_i|<C\sqrt{\delta}\}}\sigma^2(X_{\tau(v)}^{*,\delta})\frac{1}{\sigma^2(X_{\tau(v)}^{*,\delta})}dv \\
=&\int_0^{\tau^{-1}(T)} \mathbb{1}_{\{\sup_{t\le s'\le T} Z_{s'}\le M\}}\mathbb{1}_{\{|X_{\tau(v)}^{*,\delta}-x_i|<C\sqrt{\delta}\}}dv \\
=&\int_0^{\tau^{-1}(T)} \mathbb{1}_{\{\sup_{t\le s'\le T} Z_{s'}\le M\}}\mathbb{1}_{\{|B_v+x-x_i|<C\sqrt{\delta}\}}dv.
\end{aligned} \tag{31}$$

Note that on the set $\{|B_v + x - x_i| < C\sqrt{\delta}\} \cap \{\sup_{t\le s'\le T} Z_{s'} \le M\}$, we have $(X_s^{*,\delta})^2 \le (x_i + C\sqrt{\delta})^2 \le D$, where D is a positive constant, and then by (30) we have

$$\tau^{-1}(T) = \int_t^T (q^{*,\delta}\sqrt{Z_s}X_s^{*,\delta})^2 ds \le Du^2TM. \tag{32}$$

From (31) and (32), we have

$$\begin{aligned}
&\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{\sup_{t\le s'\le T} Z_{s'}\le M\}}\mathbb{1}_{\{|X_s^{*,\delta}-x_i|<C\sqrt{\delta}\}}\sigma^2(X_s^{*,\delta})ds\right] \\
\le&\int_0^{Du^2TM} \mathbb{Q}^B\{|B_v+x-x_i|<C\sqrt{\delta}\}dv \\
\le&\int_0^{Du^2TM} \frac{2C\sqrt{\delta}}{\sqrt{2\pi v}}dv \\
\le&\sqrt{\delta}\left(\frac{4C}{\sqrt{2\pi}}\sqrt{Du^2TM}\right).
\end{aligned}$$

By finite union over the x_i's we deduce (29) and $\mathcal{M}_1 = \mathcal{O}(\sqrt{\delta})$ follows.

Step 2. Control of term $\mathcal{M}_2$

By the polynomial growth condition imposed in Assumption 2.12, one has

(33)

$$\begin{aligned}\mathcal{M}_2 =& \frac{u^2}{2d^2}\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{X_s^{*,\delta}\in A^\delta_{s,Z_s}\}}(q^{*,\delta})^2 Z_s (X_s^{*,\delta})^2 |\partial^2_{xx}P_0(s,X_s^{*,\delta},Z_s)|\mathbb{1}_{\{\sup_{t\le s'\le T} Z_{s'}>M\}}ds\right]\\ \le& \frac{u^2}{2d^2}\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{X_s^{*,\delta}\in A^\delta_{s,Z_s}\}}(q^{*,\delta})^2 Z_s (X_s^{*,\delta})^2 |a_{20}(1+(X_s^{*,\delta})^{b_{20}}+Z_s^{c_{20}})|\mathbb{1}_{\{\sup_{t\le s'\le T} Z_{s'}>M\}}ds\right].\end{aligned}$$

In order to show $\mathcal{M}_2 = \mathcal{O}(\delta)$, it suffices to show that, for any power $m, n \in \mathbb{N}$,

$$\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{\sup_{t\le s'\le T} Z_{s'}>M\}}(X_s^{*,\delta})^m Z_s^n ds\right] = \mathcal{O}(\delta). \tag{34}$$

By Cauchy-Schwarz inequality and the result established in Appendix A that X_t^δ and Z_t have finite moments for δ sufficiently small, we know that

$$\begin{aligned}&\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{\sup_{t\le s'\le T} Z_{s'}>M\}}(X_s^{*,\delta})^m Z_s^n ds\right]\\ \le& \int_t^T \mathbb{E}^{1/2}_{(t,x,z)}\left((X_s^{*,\delta})^{2m}Z_s^{2n}\right)\cdot \mathbb{Q}^{1/2}\left(\sup_{t\le s'\le T} Z_{s'} > M\right)ds\\ \le& C\mathbb{Q}^{1/2}\left(\sup_{t\le s'\le T} Z_{s'} > M\right),\end{aligned} \tag{35}$$

where C may depend on (t,x,z) and (m,n) but not on δ and we allow C to vary from line to line in the sequel. Integrating the SDE of the process Z over $[t,s]$ for $s \in [t,T]$, yields

$$Z_s = z + \int_t^s \delta\kappa(\theta - Z_v)dv + \Gamma_s,$$

with $\Gamma_s = \int_t^s \sqrt{\delta}\sqrt{Z_v}dW_v^Z$. Since $Z_t \ge 0$ and $0 \le \delta \le 1$, we have

$$\sup_{t\le s\le T} Z_s \le (z+\kappa\theta T) + \sup_{t\le s\le T}\Gamma_s, \tag{36}$$

and then let $M = z + \kappa\theta T + 1$, we have

$$\mathbb{1}_{\{\sup_{t\le s\le T} Z_s>M\}} \le \mathbb{1}_{\{\sup_{t\le s\le T}\Gamma_s>1\}}. \tag{37}$$

Therefore, from (36) and (37), by Chebyshev inequality, we obtain

$$\mathbb{Q}^{1/2}\left(\sup_{t\le s\le T} Z_s > M\right) \le \mathbb{Q}^{1/2}\left(\sup_{t\le s\le T}\Gamma_s > 1\right) \le \mathbb{E}^{1/2}\left(\sup_{t\le s\le T}\Gamma_s^4\right). \tag{38}$$

Note that Γ_s is a martingale and then Γ_s^4 is a nonnegative submartingale, thus by Doob's maximal inequality (Karatzas and Shreve [2012], page 14) and the result that the process Z

has finite moments uniformly in δ, we have

$$\begin{aligned}\mathbb{E}^{1/2}\left(\sup_{t\le s\le T}\Gamma_s^4\right) \le & C\mathbb{E}^{1/2}\left(\Gamma_T^4\right) \\ =& C\delta\mathbb{E}^{1/2}\left(\int_t^T \sqrt{Z_v}dW_v^Z\right)^4 \\ \le & C\delta\left(6T\mathbb{E}\int_t^T {Z_v}^2 dv\right)^{1/2} \\ \le & C\delta,\end{aligned} \tag{39}$$

where the second inequality established by the Martingale Moment Inequalities (Karatzas and Shreve [2012], page 163).

Now, we have (34) as desired, which completes the proof. ■

3.3. Control of the term I_1.

Theorem 3.2. *Under Assumptions 2.2, 2.6, 2.9 and 2.12, there exists a constant M_1, such that*

$$|I_1| \le M_1\sqrt{\delta}$$

where M_1 may depend on (t,x,z) but not on δ. That is, I_1 is of order $\mathcal{O}(\sqrt{\delta})$.

Proof. Under Assumption 2.12 and $0 < d \le q^{*,\delta}, q^{*,0} \le u$, we have

$$\begin{aligned}|I_1| =& \left|\mathbb{E}_{(t,x,z)}\left[\int_t^T \left((q^{*,\delta}-q^{*,0})\rho Z_s X_s^{*,\delta}\partial_{xz}^2 P_0(s,X_s^{*,\delta},Z_s)\right.\right.\right. \\ &\left.\left.\left. + \frac{1}{2}\left((q^{*,\delta})^2-(q^{*,0})^2\right) Z_s (X_s^{*,\delta})^2\partial_{xx}^2 P_1(s,X_s^{*,\delta},Z_s)\right)ds\right]\right| \\ \le& \mathbb{E}_{(t,x,z)}\left[\int_t^T \left(|q^{*,\delta}-q^{*,0}|Z_s X_s^{*,\delta}|\partial_{xz}^2 P_0(s,X_s^{*,\delta},Z_s)|\right.\right. \\ &\left.\left. + \frac{1}{2}|(q^{*,\delta})^2-(q^{*,0})^2|Z_s(X_s^{*,\delta})^2|\partial_{xx}^2 P_1(s,X_s^{*,\delta},Z_s)|\right)ds\right] \\ \le& \frac{u}{d^2}\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{X_s^{*,\delta}\in A_{s,Z_s}^\delta\}}(q^{*,\delta})^2 Z_s X_s^{*,\delta} a_{11}(1+(X_s^{*,\delta})^{b_{11}}+Z_s^{c_{11}})ds\right] \\ &+ \frac{u^2}{2d^2}\mathbb{E}_{(t,x,z)}\left[\int_t^T \mathbb{1}_{\{X_s^{*,\delta}\in A_{s,Z_s}^\delta\}}(q^{*,\delta})^2 Z_s (X_s^{*,\delta})^2 \bar{a}_{20}(1+(X_s^{*,\delta})^{\bar{b}_{20}}+Z_s^{\bar{c}_{20}})ds\right].\end{aligned}$$

Using the same techniques in proving Theorem 3.1, the result that $X_s^{*,\delta}$ and Z_s have finite moments for δ sufficiently small, and $X_s^{*,\delta} \le C(X_s^{*,\delta})^2$ on $\{X_s^{*,\delta}\in A_{s,Z_s}^\delta\}$, we can deduce that I_1 is of order $\mathcal{O}(\sqrt{\delta})$. ■

With the result of theorem 3.1 that I_0 is of order $\mathcal{O}(\delta)$, the result of theorem 3.2 that I_1 is of order $\mathcal{O}(\sqrt{\delta})$, and the result that I_2 and I_3 are uniformly bounded in δ where derivation of these bounds are given in the appendix E, we can see that

$$E^{\delta}(t,x,z) = I_0 + \delta^{\frac{1}{2}} I_1 + \delta I_2 + \delta^{\frac{3}{2}} I_3,$$

is of order $\mathcal{O}(\delta)$, which completes the proof of the main Theorem 2.14.

4. Numerical Illustration. In this section, we use the nontrivial example in Fouque and Ren [2014], and consider a symmetric European butterfly spread with the payoff function

$$h(x) = (x-90)^+ - 2(x-100)^+ + (x-110)^+. \tag{40}$$

Although this payoff function does not satisfy the conditions imposed in this paper, we could consider a regularization of it, that is to introduce a small parameter for the regularization and then remove this small parameter asymptotically without changing the accuracy estimate. This can be achieved by considering $P_0(T-\epsilon, x)$ as the regularized payoff (see Fouque et al. [2003] for details on this regularization procedure in the context of the Black–Scholes equation).

The original problem is to solve the fully nonlinear PDE (11) in dimension two for the worst-case scenario price, which is not analytically solvable in practice. In the following, we use the Crank–Nicolson version of the weighted finite difference method in Guyon and Henry-Labordère [2013], which corresponds to the case of solving P_0 in one dimension. To extend the original algorithm to our two dimensional case, we apply discretization grids on time and two state variables. Denote $u^n_{i,j} := P_0(t_n, x_i, z_j)$, $v^n_{i,j} := P_1(t_n, x_i, z_j)$ and $w^n_{i,j} := P^{\delta}(t_n, x_i, z_j)$, where $n = 0, 1, \cdots, N$ stands for the index of time, $i = 0, 1, \cdots, I$ stands for the index of the asset price process, and $j = 0, 1, \cdots, J$ stands for the index of the volatility process. In the following, we build a uniform grid of size 100×100 and use 20 time steps.

We use the classical discrete approximations to the continuous derivatives:

$$\partial_x(w^n_{i,j}) = \frac{w^n_{i+1,j} - w^n_{i-1,j}}{2\Delta x} \qquad \partial^2_{zz}(w^n_{i,j}) = \frac{w^n_{i,j+1} + w^n_{i,j-1} - 2w^n_{i,j}}{\Delta z^2}$$

$$\partial^2_{xx}(w^n_{i,j}) = \frac{w^n_{i+1,j} + w^n_{i-1,j} - 2w^n_{i,j}}{\Delta x^2} \qquad \partial_z(w^n_{i,j}) = \frac{w^n_{i,j+1} - w^n_{i,j-1}}{2\Delta z}$$

$$\partial^2_{xz}(w^n_{i,j}) = \frac{w^n_{i+1,j+1} + w^n_{i-1,j-1} - w^n_{i-1,j+1} - w^n_{i+1,j-1}}{4\Delta x \Delta z} \qquad \partial_t(w^n_{i,j}) = \frac{u^{n+1}_{i,j} - u^n_{i,j}}{\Delta t}$$

To simplify our algorithms and facilitate the implementation by matrix operations, we denote the following operators without any parameters:

$$L_{xx} = zx^2\partial^2_{xx} \qquad L_{zz} = z\partial^2_{zz} \qquad L_{xz} = xz\partial^2_{xz}$$

$$L_x = x\partial_x \qquad L_{z1} = \partial_z \qquad L_{z2} = z\partial_z$$

4.1. Simulation of P_0 and P_1. Note that in the PDE (18) for P_1 , $q^{*,0}$ must be solved in the PDE (13) for P_0 . Therefore, we solve P_0 and P_1 together in each 100×100 space grids and iteratively back to the starting time.

Algorithm 1 Algorithm to solve P_0 and P_1

Set $u_{i,j}^N = h(x_I)$ and $v_{i,j}^N = 0$.

Solve $u_{i,j}^n$ (predictor)

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} + [\frac{1}{2}(q_{i,j}^{n+1})^2 L_{xx}](\frac{u_{i,j}^{n+1} + u_{i,j}^n}{2}) = 0$$

with

$$q_{i,j}^{n+1} = u\mathbb{1}_{\{u^2 L_{xx}(u_{i,j}^{n+1}) \geq d^2 L_{xx}(u_{i,j}^{n+1})\}} + d\mathbb{1}_{\{u^2 L_{xx}(u_{i,j}^{n+1}) < d^2 L_{xx}(u_{i,j}^{n+1})\}}$$

Solve $u_{i,j}^n$ (corrector)

$$\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\Delta t} + [\frac{1}{2}(q_{i,j}^{n})^2 L_{xx}](\frac{u_{i,j}^{n+1} + u_{i,j}^n}{2}) = 0$$

with

$$q_{i,j}^{n} = u\mathbb{1}_{\{u^2 L_{xx}(\frac{u_{i,j}^{n+1}+u_{i,j}^n}{2}) \geq d^2 L_{xx}(\frac{u_{i,j}^{n+1}+u_{i,j}^n}{2})\}} + d\mathbb{1}_{\{u^2 L_{xx}(\frac{u_{i,j}^{n+1}+u_{i,j}^n}{2}) < d^2 L_{xx}(\frac{u_{i,j}^{n+1}+u_{i,j}^n}{2})\}}$$

Solve $v_{i,j}^n$

$$\frac{v_{i,j}^{n+1} - v_{i,j}^n}{\Delta t} + \frac{1}{2}(q_{i,j}^n)^2 L_{xx}(\frac{v_{i,j}^{n+1} + v_{i,j}^n}{2}) + \rho(q_{i,j}^n) L_{xz}(\frac{u_{i,j}^{n+1} + u_{i,j}^n}{2}) = 0$$

Throughout all the experiments, we set $X_0 = 100$, $Z_0 = 0.04$, $T = 0.25$, $r = 0$, $d = 0.75$, and $u = 1.25$. Therefore, the two deterministic bounds for P_0 are given by $\underline{\sigma} = d\sqrt{Z_0} = 0.15$ and $\overline{\sigma} = u\sqrt{Z_0} = 0.25$, which are standard Uncertain Volatility model bounds setup. From Figure 3, we can see that P_0 is above the Black–Scholes prices with constant volatility 0.15 and 0.25 all the time, which corresponds to the fact that we need extra cash to superreplicate the option when facing the model ambiguity. As expected, the Black–Scholes prices with constant volatility 0.25 (resp. 0.15) is a good approximation when P_0 is convex (resp. concave).

4.2. Simulation of P_δ. Considering the main BSB equation given by (11), if we relax the restriction $q \in [d, u]$ to $q \in \mathbb{R}$, the optimizer of

$$f(q) := \frac{1}{2} q^2 z x^2 \partial_{xx}^2 P^\delta + q\rho z x \sqrt{\delta} \partial_{xz}^2 P^\delta$$

is given by $\hat{q}^{*,\delta} = -\frac{\rho\sqrt{\delta}\partial_{xz}^2 P^\delta}{x\partial_{xx}^2 P^\delta}$, and the maximum value of $f(q)$ is given by $f(\hat{q}^{*,\delta}) = -\frac{\rho^2 \delta z (\partial_{xz}^2 P^\delta)^2}{2\partial_{xx}^2 P^\delta}$. Therefore,

$$\sup_{q\in[d,u]} f(q) = f(u) \vee f(d) \vee f(\hat{q}^{*,\delta}).$$

To simplify the algorithm, we denote

$$L_A = \frac{1}{2} u^2 L_{xx} + u\rho\sqrt{\delta} L_{xz}, \qquad L_B = \frac{1}{2} d^2 L_{xx} + d\rho\sqrt{\delta} L_{xz}, \qquad L_C = -\frac{\rho^2 \delta (L_{xz})^2}{2L_{xx}}.$$

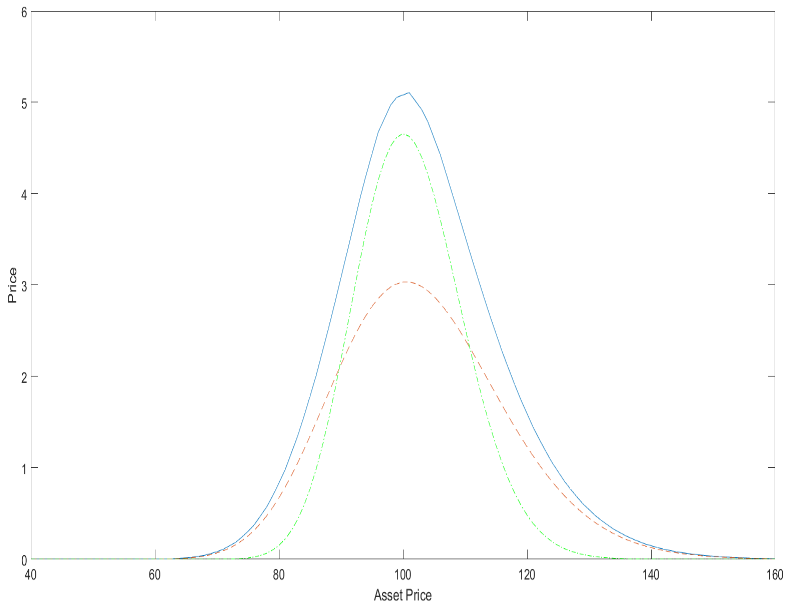

Figure 3. *The blue curve represents the usual uncertain volatility model price P_0 with two deterministic bounds 0.15 and 0.25, the red curve marked with "- -" represents the Black–Scholes prices with $\sigma = 0.25$, the green curve marked with '-.' represents the Black–Scholes prices with $\sigma = 0.15$.*

Algorithm 2 Algorithm to solve P^δ

1: Set $w_{i,j}^N = h(x_I)$.

2: Predictor:

$$\frac{w_{i,j}^{n+1} - w_{i,j}^n}{\Delta t} + [\frac{1}{2}(q_{i,j}^{n+1})^2 L_{xx} + (q_{i,j}^{n+1})\rho\sqrt{\delta}L_{xz} + \delta(\frac{1}{2}L_{zz} + \kappa\theta L_{z1} - \kappa L_{z2})](\frac{w_{i,j}^{n+1} + w_{i,j}^n}{2}) = 0$$

with

$$\begin{aligned} q_{i,j}^{n+1} =& u\mathbb{1}_{\{L_A(w_{i,j}^{n+1})=\max\{L_A,L_B,L_C\}(w_{i,j}^{n+1})\}} + d\mathbb{1}_{\{L_B(w_{i,j}^{n+1})=\max\{L_A,L_B,L_C\}(w_{i,j}^{n+1})\}} \\ &- \frac{\rho\sqrt{\delta}L_{xz}}{L_{xx}}(w_{i,j}^{n+1})\mathbb{1}_{\{L_C(w_{i,j}^{n+1})=\max\{L_A,L_B,L_C\}(w_{i,j}^{n+1})\}} \end{aligned}$$

3: Corrector:

$$\frac{w_{i,j}^{n+1} - w_{i,j}^n}{\Delta t} + [\frac{1}{2}(q_{i,j}^{n})^2 L_{xx} + (q_{i,j}^{n})\rho\sqrt{\delta}L_{xz} + \delta(\frac{1}{2}L_{zz} + \kappa\theta L_{z1} - \kappa L_{z2})](\frac{w_{i,j}^{n+1} + w_{i,j}^n}{2}) = 0$$

with

$$q_{i,j}^n = u\mathbb{1}_{\{L_A(\frac{w_{i,j}^{n+1}+w_{i,j}^n}{2})=\max\{L_A,L_B,L_C\}(\frac{w_{i,j}^{n+1}+w_{i,j}^n}{2})\}} + d\mathbb{1}_{\{L_B(\frac{w_{i,j}^{n+1}+w_{i,j}^n}{2})=\max\{L_A,L_B,L_C\}(\frac{w_{i,j}^{n+1}+w_{i,j}^n}{2})\}}$$
$$- \frac{\rho\sqrt{\delta}L_{xz}}{L_{xx}}(\frac{w_{i,j}^{n+1}+w_{i,j}^n}{2})\mathbb{1}_{\{L_C(\frac{w_{i,j}^{n+1}+w_{i,j}^n}{2})=\max\{L_A,L_B,L_C\}(\frac{w_{i,j}^{n+1}+w_{i,j}^n}{2})\}}$$

We set $\kappa = 15$ and $\theta = 0.04$, which satisfies the Feller condition required in this paper.

4.3. Error analysis. To visualize the approximation as δ vanishes, we plot P^δ, P_0 and $P_0 + \sqrt{\delta}P_1$ with ten equally spaced values of δ from 0.05 to 0, and consider a typical case of correlation $\rho = -0.9$ (see in't Hout and Foulon [2010]). In Figure 4, we see that the first order prices capture the main feature of the worst-case scenario prices for different values of δ. As can be seen, for δ very small, the approximation performs very well and it worth noting that, even for δ not very small such as 0.1, it still performs well.

To investigate the convergence of the error of our approximation as δ decrease, we compute the error of the approximation for each δ as following

$$\text{error}(\delta) = \sup_{x,z} |P^\delta(0,x,z) - P_0(0,x,z) - \sqrt{\delta}P_1(0,x,z)|.$$

As shown in Figure 5, the error decreases linearly as δ decreases (at least for δ small enough), as predicted by our Main Theorem 2.14.

Remark 4.1. *In Remark 2.10, for the case that P_0 has a third derivative with respect to x, which does not vanish on the set $S_{t,z}^0$, we have Assumption 2.9 (ii) as a direct result. In Figure 6, we can see that the slopes at the zero points of $\partial_{xx}^2 P^\delta$ and $\partial_{xx}^2 P_0$ are not 0, hence for this symmetric butterfly spread, Assumption 2.9 (ii) is satisfied.*

5. Conclusion. In this paper, we have proposed the uncertain volatility models with stochastic bounds driven by a CIR process. Our method is not limited to the CIR process and can be used with any other positive stochastic processes such as positive functions of an OU process. We further studied the asymptotic behavior of the worst-case scenario option prices in the regime of slowly varying stochastic bounds. This study not only helps understanding that uncertain volatility models with stochastic bounds are more flexible than uncertain volatility models with constant bounds for option pricing and risk management, but also provides an approximation procedure for worst-case scenario option prices when the bounds are slowly varying. From the numerical results, we see that the approximation procedure works really well even when the payoff function does not satisfy the requirements enforced in this paper, and even when δ is not so small such as $\delta = 0.1$.

Note that as risk evaluation in a financial management requires more accuracy and efficiency nowadays, our approximation procedure highly improves the estimation and still maintains the same efficiency level as the regular uncertain volatility models. Moreover, the worst-case scenario price P^δ (11) has to be recomputed for any change in its parameters κ, θ and δ. However, the PDEs (13) and (18) for P_0 and P_1 are independent of these parameters, so the approximation requires only to compute P_0 and P_1 once for all values of κ, θ and δ.

Appendix A. Moments of Z_t and X_t.

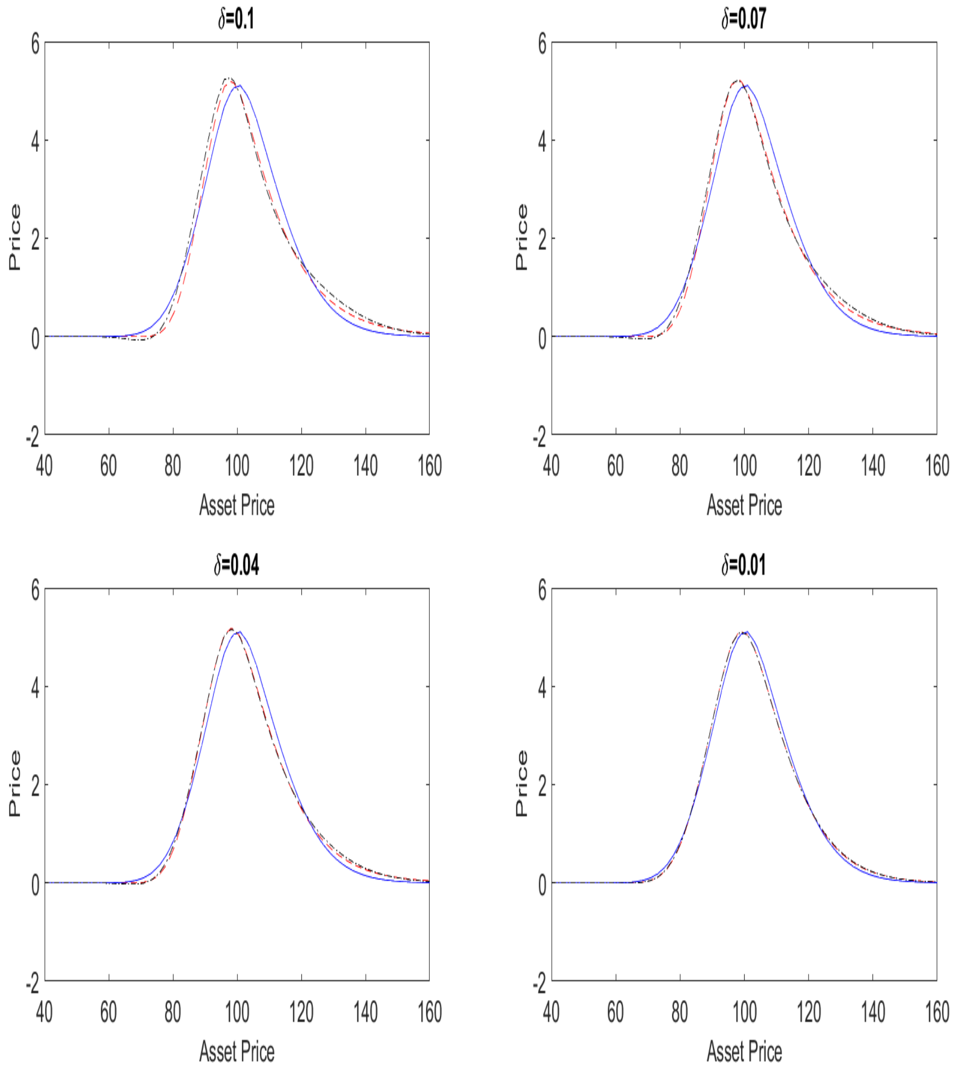

Figure 4. *The red curve marked with "- -" represents the worst-case scenario prices P^δ; the blue curve represents the leading term P_0; the black curve marked with '-.' represents the approximation $P_0 + \sqrt{\delta}P_1$.*

A.1. Moments of Z_t.

Proposition A.1. *The process Z_t given by (6) has finite moments of any order uniformly in $0 \le \delta \le 1$ for $t \le T$.*

The proof is given by Lemma 4.9 in Fouque et al. [2011]. Thus, for $k \in \mathbb{Z}$,

$$\mathbb{E}_{(0,z)}\left[\int_0^T |Z_s|^k ds\right] \le C_k(T,z), \quad Z_0 = z, \tag{41}$$

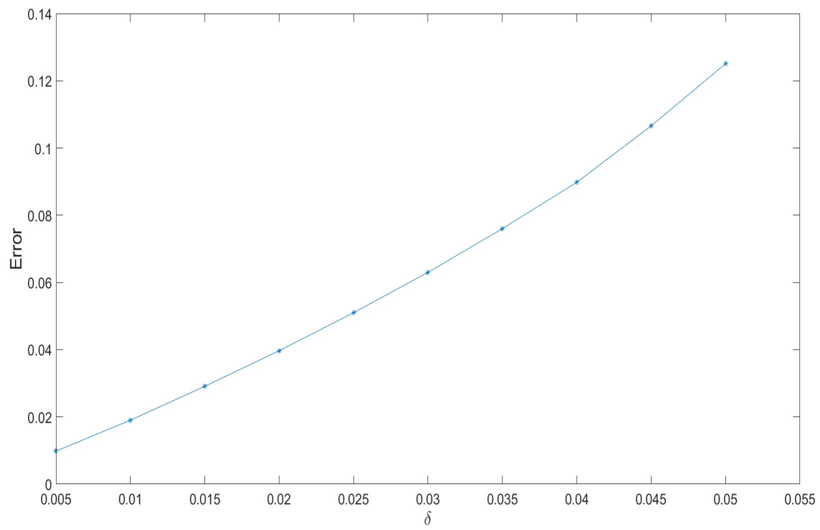

Figure 5. *Error for different values of δ*

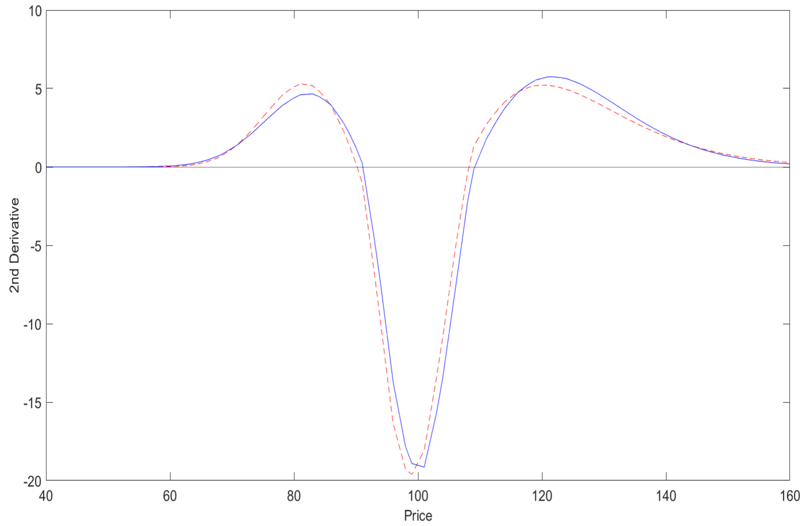

Figure 6. *The red curve marked with "- -" represents $\partial^2_{xx}P^\delta$; the blue curve represents $\partial^2_{xx}P_0$.*

where $C_k(T,z)$ may depend on (k,T,z) but not on δ.

A.2. Moments of X_t. In this subsection, we consider the process X_t evolving according to the SDE,

$$dX_t = rX_t dt + q_t\sqrt{Z_t}X_t dW_t, \quad X_0 = x, \tag{42}$$

where $q_t \in [d,u]$ and Z_t is the CIR process given by (6). In order to show that X_t has finite moments of any order, we will use a change of measure which will give rise to the following CIR process

$$d\widetilde{Z}_t = \left(\delta\kappa\theta - (\delta\kappa - nq_t\sqrt{\delta})\widetilde{Z}_t\right)dt + \sqrt{\delta}\sqrt{\widetilde{Z}_t}d\widetilde{W}_t^Z, \tag{43}$$

where the parameters κ, θ and δ are the same as the ones in the CIR process given by (6) and $n \in \mathbb{N}$.

Denote the moment generating function of the integrated $\widetilde{Z}_t$ process given $\widetilde{Z}_s|_{s=0} = z$ by

$$\widetilde{M}_z^\delta(\eta) := \mathbb{E}_{(0,z)}\left[\exp(\eta\int_0^t \widetilde{Z}_s ds)\right], \quad \text{for } \eta \in \mathbb{R}.$$

Then, we have the following preliminary result:

Proposition A.2. *For $\eta \in \mathbb{R}$, $\widetilde{M}_z^\delta(\eta)$ is bounded uniformly, for δ sufficiently small and for all $t \in [0,T]$, that is, there exists $\epsilon = \epsilon(n,u,d,\kappa,T,\eta) > 0$ and $\widetilde{N}(\kappa,\theta,T,z,\eta) < \infty$ such that $|\widetilde{M}_z^\delta(\eta)| \leq \widetilde{N}(\kappa,\theta,T,z,\eta) < \infty$, for all $\delta \leq \epsilon$.*

Proof. Note that under the Feller condition $2\kappa\theta \geq 1$, the $\widetilde{Z}_t$ process is strictly positive as it is the original CIR process given by (6) in the case $n = 0$, and for $n \geq 1$ the drift is positive for δ small enough. Therefore, $\widetilde{M}_z^\delta(\eta) \leq 1$ for $\eta \leq 0$, and we only need to focus on $\eta > 0$ in the following. Also, since $t = 0$ is a trivial case, we concentrate on $t \in (0,T]$ in the proof. By Corollary 3 of Albanese and Lawi [2005], we know that

$$\widetilde{M}_z^\delta(\eta) = \Psi(\eta,t)e^{-z\Xi(\eta,t)},$$

where

$$\Psi(\eta,t) = \left(\frac{\bar{b}e^{bt/2}}{\bar{b}\frac{e^{\bar{b}t/2}+e^{-\bar{b}t/2}}{2} + b\frac{e^{\bar{b}t/2}-e^{-\bar{b}t/2}}{2}}\right)^{2\kappa\theta},$$

$$\Xi(\eta,t) = -2\eta\left(\frac{\frac{e^{\bar{b}t/2}-e^{-\bar{b}t/2}}{2}}{\bar{b}\frac{e^{\bar{b}t/2}+e^{-\bar{b}t/2}}{2} + b\frac{e^{\bar{b}t/2}-e^{-\bar{b}t/2}}{2}}\right),$$

and

$$\bar{b} = \sqrt{b^2 - 2\eta\delta}, \quad b = \delta\kappa - nq_t\sqrt{\delta}.$$

Note that the sign of b depends on the value of $n \in \mathbb{N}$. That is, when $n \geq 1$, b is negative for δ sufficiently small, while when $n = 0$, b is always positive. We also need to discuss the sign of the term $b^2 - 2\eta\delta$, which determines whether $\bar{b}$ is a real number or a complex number.

Case $n \geq 1$ ($b < 0$).

- If $b^2 - 2\eta\delta \geq 0$, then $\bar{b} \geq 0$. Note that when $\delta < (nd/\kappa)^2$, we have

$$\bar{b}t = t\sqrt{(nq_t\sqrt{\delta} - \delta\kappa)^2 - 2\eta\delta} \leq |b|t \leq nq_t\sqrt{\delta}t \leq nu\sqrt{\delta}T,$$

and there exists $\epsilon_1 = \epsilon_1(n, u, d, \kappa, T)$ such that when $\delta < \epsilon_1$, we have $\bar{b}t \leq 1$ and $|bt + \mathcal{O}[(bt)^2]| < \frac{1}{2}$. Therefore, by the fact that $e^{bt/2} \leq 1$ and $\frac{e^{\bar{b}t/2} + e^{-\bar{b}t/2}}{2} \geq 1$, we have

$$\Psi(\eta, t) = \left(\frac{e^{bt/2}}{\frac{e^{\bar{b}t/2} + e^{-\bar{b}t/2}}{2} + b\frac{\bar{b}t + \mathcal{O}[(\bar{b}t)^2]}{\bar{b}}} \right)^{2\kappa\theta} = \left(\frac{e^{bt/2}}{\frac{e^{\bar{b}t/2} + e^{-\bar{b}t/2}}{2} + bt + \mathcal{O}[(bt)^2]} \right)^{2\kappa\theta}$$
$$\leq \left(\frac{1}{1 - \frac{1}{2}} \right)^{2\kappa\theta} = 2^{2\kappa\theta}$$

and

$$|\Xi(\eta, t)| = \left| -2\eta \left(\frac{\bar{b}t + \mathcal{O}[(\bar{b}t)^2]}{\bar{b}\frac{e^{\bar{b}t/2} + e^{-\bar{b}t/2}}{2} + b\{\bar{b}t + \mathcal{O}[(\bar{b}t)^2]\}} \right) \right| = \left| -2\eta \left(\frac{t + \mathcal{O}[\bar{b}t^2]}{\frac{e^{\bar{b}t/2} + e^{-\bar{b}t/2}}{2} + bt + \mathcal{O}[(bt)^2]} \right) \right|$$
$$\leq 2\eta \frac{t + t}{1 - \frac{1}{2}} = 8\eta t.$$

Therefore, for $\delta < \epsilon_1$, we have

$$\widetilde{M}_z^\delta(\eta) \leq 2^{2\kappa\theta} e^{8\eta T z}.$$

- If $b^2 - 2\eta\delta < 0$, then $\bar{b} = iv$, where $v = \sqrt{2\eta\delta - b^2}$. Note that $0 < vt \leq \sqrt{2\eta\delta}T$ and $\left|\frac{\sin(vt/2)}{vt/2}\right| \leq 1$. There exists $\epsilon_2 = \epsilon_2(n, u, T, \eta)$ such that when $\delta < \epsilon_2$, we have $\cos(vt/2) \geq \frac{3}{4}$ and $|bt| \leq 1$. Therefore,

$$\Psi(\eta, t) = \left(\frac{ive^{bt/2}}{iv\cos(vt/2) + ib\sin(vt/2)} \right)^{2\kappa\theta} = \left(\frac{e^{bt/2}}{\cos(vt/2) + \frac{bt}{2}\left(\frac{\sin(vt/2)}{vt/2}\right)} \right)^{2\kappa\theta},$$

$$\Xi(\eta, t) = -2\eta \left(\frac{\sin(vt/2)}{v\cos(vt/2) + b\sin(vt/2)} \right) = -2\eta \left(\frac{t}{2\cos(vt/2)\frac{vt/2}{\sin(vt/2)} + bt} \right)$$

and

$$\widetilde{M}_z^\delta(\eta) = \Psi(\eta, t)e^{-z\Xi(\eta, t)} \leq \left(\frac{1}{\frac{3}{4} - \frac{1}{2}} \right)^{2\kappa\theta} \exp\left(\frac{2\eta T z}{2 \times \frac{3}{4} - 1}\right) = 4^{2\kappa\theta} e^{4\eta T z}.$$

Case $n = 0$ ($b > 0$).

- If $b^2-2\eta\delta\geq 0$, then $\bar{b}\geq 0$. We have $\bar{b}t=t\sqrt{\delta^2\kappa^2-2\eta\delta}\leq\delta\kappa T$, and there exists $\epsilon_3=\epsilon_3(\kappa,T)$, such that when $\delta<\epsilon_3$, we have $\bar{b}t\leq 1$,

$$\Psi(\eta,t)\leq\left(\frac{\bar{b}e^{\delta\kappa t/2}}{\bar{b}\frac{e^{\bar{b}t/2}+e^{-\bar{b}t/2}}{2}}\right)^{2\kappa\theta}\leq\left(e^{\delta\kappa t/2}\right)^{2\kappa\theta}\leq e^{\kappa^2\theta T},$$

$$|\Xi(\eta,t)|\leq 2\eta\left(\frac{e^{\bar{b}t/2}-e^{-\bar{b}t/2}}{2\bar{b}}\right)\leq\eta(1+\mathcal{O}(\bar{b}t))T\leq 2\eta T$$

and

$$\widetilde{M}_z^{\delta}(\eta)\leq e^{\kappa^2\theta T}e^{2\eta Tz}.$$

- If $b^2-2\eta\delta<0$, then $\bar{b}=iv$, where $v=\sqrt{2\eta\delta-\delta^2\kappa^2}$. Note that $0<vt\leq\sqrt{2\eta\delta}T$ and there exists $\epsilon_4=\epsilon_4(n,u,T,\eta)$ such that when $\delta<\epsilon_4$, we have $\cos(vt/2)\geq\frac{3}{4}$ and $\sin(vt/2)\geq 0$. Therefore,

$$\Psi(\eta,t)=\left(\frac{ive^{\delta\kappa t/2}}{iv\cos(vt/2)+i\delta\kappa\sin(vt/2)}\right)^{2\kappa\theta}=\left(\frac{e^{\delta\kappa t/2}}{\cos(vt/2)+\frac{\delta\kappa t}{2}\left(\frac{\sin(vt/2)}{vt/2}\right)}\right)^{2\kappa\theta},$$

$$\Xi(\eta,t)=-2\eta\left(\frac{\sin(vt/2)}{v\cos(vt/2)+\delta\kappa\sin(vt/2)}\right)=-2\eta\left(\frac{t}{2\cos(vt/2)\frac{vt/2}{\sin(vt/2)}+\delta\kappa t}\right)$$

and

$$\widetilde{M}_z^{\delta}(\eta)=\Psi(\eta,t)e^{-z\Xi(\eta,t)}\leq\left(\frac{4e^{\kappa T/2}}{3}\right)^{2\kappa\theta}\exp(\frac{2\eta Tz}{2\times\frac{3}{4}})=\left(\frac{4e^{\kappa T/2}}{3}\right)^{2\kappa\theta}\exp(\frac{4\eta Tz}{3}).$$

In sum, there exists $\epsilon=\epsilon(n,u,d,\kappa,T,\eta)$ and $\widetilde{N}=\widetilde{N}(\kappa,\theta,T,z,\eta)$ which is independent of δ and t, such that when $\delta<\epsilon$, we have $\widetilde{M}_z^{\delta}(\eta)\leq\widetilde{N}$ as desired. ■

Proposition A.3. *The process X_t given by (42), has finite moments of any order, for $t\leq T$ and $\delta<\epsilon(n,u,d,\kappa,T,\eta)$ given in Proposition A.2, where n is a positive integer.*

Proof. For each $n\in\mathbb{N}$, we have

$$\begin{aligned}X_t^n=&x^n\exp\left(nrt-\frac{n}{2}\int_0^t(q_s\sqrt{Z_s})^2ds+n\int_0^t q_s\sqrt{Z_s}dW_s\right)\\=&x^k\exp\left(nrt+\frac{n^2-n}{2}\int_0^t(q_s\sqrt{Z_s})^2ds\right)\times\Lambda_t,\end{aligned}$$

where

$$\Lambda_t=\exp\left(-\frac{n^2}{2}\int_0^t(q_s\sqrt{Z_s})^2ds+n\int_0^t q_s\sqrt{Z_s}dW_s\right)$$

is a martingale, whose Novikov condition is satisfied thanks to Proposition A.2, i.e.

$$\mathbb{E}_{(0,x,z)}\left[\exp\left(\frac{1}{2}\int_0^t (nq\sqrt{Z_s})^2 ds\right)\right] \leq \mathbb{E}_{(0,x,z)}\left[\exp\left(\frac{n^2u^2}{2}\int_0^t Z_s ds\right)\right] < \infty.$$

By the corresponding change of measure and the inequality $q_s \leq u$, we get

$$\mathbb{E}_{(0,x,z)}[X_t^n] \leq x^n \exp(nrt)\widetilde{\mathbb{E}}_{(0,x,z)}\left[\exp\left(\frac{(n^2-n)u^2}{2}\int_0^t \widetilde{Z}_s ds\right)\right] \tag{44}$$

where, under the new measure $\widetilde{\mathbb{Q}}$, the process $\widetilde{Z}_t$ driven by a $\widetilde{\mathbb{Q}}$-Brownian motion $\widetilde{W}_t^Z$ evolves according to (43). Hence, by Proposition A.2, we have

$$\mathbb{E}_{(0,x,z)}[X_t^n] \leq x^n \exp(nrT)\widetilde{M}_z^\delta\left(\frac{(n^2-n)u^2}{2}\right) \leq x^n \exp(nrT)\widetilde{N},$$

where the upper bound $x^n \exp(nrT)\widetilde{N}$ is independent of δ and t. ∎

Therefore, for δ sufficiently small,

$$\mathbb{E}_{(0,x,z)}[\int_0^T |X_s|^n ds] \leq N_n, \tag{45}$$

where N_n does not on δ and $t \in [0,T]$.

Appendix B. Proof of Proposition 2.1. Integrating over $[t,T]$ the SDE (7) and the SDE (8), we have

$$X_T^\delta = x + \int_t^T rX_s^\delta ds + \int_t^T q_s\sqrt{Z_s}X_s^\delta dW_s \tag{46}$$

and

$$X_T^0 = x + \int_t^T rX_s^0 ds + \int_t^T q_s\sqrt{z}X_s^0 dW_s. \tag{47}$$

The difference of (46) and (47) is given by

$$\begin{aligned} X_T^\delta - X_T^0 &= \int_t^T r(X_s^\delta - X_s^0)ds + \int_t^T q_s(\sqrt{Z_s}X_s^\delta - \sqrt{z}X_s^0)dW_s \\ &= \int_t^T r(X_s^\delta - X_s^0)ds + \int_t^T q_s\sqrt{z}(X_s^\delta - X_s^0)dW_s + \int_t^T q_s(\sqrt{Z_s} - \sqrt{z})X_s^\delta dW_s. \end{aligned} \tag{48}$$

Let $Y_s = X_s^\delta - X_s^0$, then $Y_t = 0$ and

$$Y_T = \int_t^T rY_s ds + \int_t^T q_s\sqrt{z}Y_s dW_s + \int_t^T q_s(\sqrt{Z_s} - \sqrt{z})X_s^\delta dW_s. \tag{49}$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}_{(t,x,z)}[Y_T^2] \\
&\leq 3\mathbb{E}_{(t,x,z)}\left[\left(\int_t^T rY_s ds\right)^2 + \left(\int_t^T q_s\sqrt{z}Y_s dW_s\right)^2 + \left(\int_t^T q_s(\sqrt{Z_s}-\sqrt{z})X_s^\delta dW_s\right)^2\right] \\
&\leq \int_t^T \left(3Tr^2+3u^2z\right)\mathbb{E}_{(t,x,z)}[Y_s^2]ds + \underbrace{3u^2\int_t^T \mathbb{E}_{(t,x,z)}\left[(\sqrt{Z_s}-\sqrt{z})^2(X_s^\delta)^2\right]ds}_{R(\delta)}.
\end{aligned} \tag{50}
$$

Notice that only the upper bound of q is used, which gives the uniform convergence in q. Also note that using the result that X_t and Z_t have finite moments for δ sufficiently small, we can show that $|R(\delta)| \leq C\delta$, where $C = C(T, \theta, u, d, z)$ is independent of δ.

Denote $f(T) = \mathbb{E}_{(t,x,z)}(Y_T^2)$ and $\lambda = 3Tr^2+3u^2z > 0$, and by Gronwall inequality, equation (50) can be written as

$$
f(T) \leq \int_t^T \lambda f(s)ds + C\delta \leq \delta \int_t^T C\lambda e^{\lambda(T-s)}ds + C\delta
$$

Therefore,

$$
\mathbb{E}_{(t,x,z)}(X_T^\delta - X_T^0)^2 = \mathbb{E}_{(t,x,z)}Y_T^2 = f(T) \leq C'\delta,
$$

and the Proposition follows.

Appendix C. Regularity of P^δ. In this section, we prove that Assumption 2.6 is satisfied in the simpler setting of uniform parabolicity described in (5), where we assume that F is positive increasing, differentiable such that $0 < F_1 \leq F(z) \leq F_2 < \infty$ for two constants. Recall that the model is given by

$$
\begin{aligned}
dX_t^\delta &= q_t F(Z_t) X_t^\delta dW_t, \\
dZ_t &= \delta\mu(Z_t)dt + \sqrt{\delta}\beta(Z_t)dW_t^Z,
\end{aligned} \tag{51}
$$

where $d\langle W, W^Z\rangle_t = \rho dt$ for $|\rho| \leq 1$, and $0 < d \leq q_t \leq u < \infty$. Next, recall that the worst case scenario price is given by

$$
P^\delta(t,x,z) = \sup_{q_\cdot \in [d,u]} \mathbb{E}_{(t,x,z)}[h(X_T^\delta)], \tag{52}
$$

where h is a nonnegative function.

Proposition C.1. *For the problem given by (51) and (52), we further assume: β is bounded and bounded away from zero; μ, β and F are uniformly Lipschitz continuous, with continuous and bounded first and second order derivatives; $0 < F_1 \leq F(\cdot) \leq F_2 < \infty$, for F_1 and F_2 two positive constants; the terminal function h is in C_b^4 with bounded derivatives up to the fourth order such that $e^{ky}h^{(k)}(e^y)$ is bounded for $k = 1, \cdots, 4$. Then we have:*
(i) P^δ belongs to $C_b^{1,2,2}$, for $\delta > 0$ fixed.
(ii) $x\partial_x P^\delta$ and $x^2\partial_{xx}^2 P^\delta$ are uniformly bounded in $\delta \leq 1$.

Proof. (i) After the change of variable $Y = \ln(X^\delta)$, and denoting $\mathbf{X}_t = (Y_t, Z_t)$ and $\mathbf{x} = (y, z)$, (51) can be rewritten in the form of a two-dimensional SDE:

$$\begin{aligned} d\mathbf{X}_t &= f(\mathbf{X}_t, q_t)dt + \sigma(\mathbf{X}_t, q_t)d\mathbf{W}_t, \\ f(\mathbf{x}, q) &= \begin{pmatrix} -\frac{1}{2}q^2F^2(z) \\ \delta\mu(z) \end{pmatrix}, \\ \sigma(\mathbf{x}, q) &= \begin{pmatrix} \sqrt{1-\rho^2}qF(z) & \rho qF(z) \\ 0 & \sqrt{\delta}\beta(z) \end{pmatrix}, \end{aligned} \tag{53}$$

where $\mathbf{W} = (W^Y, W^Z)^T$, with $W^Y \perp\!\!\!\perp W^Z$. The BSB PDE becomes

$$\begin{aligned} \partial_t V^\delta(t, \mathbf{x}) + H(t, \mathbf{x}, DV^\delta(t, \mathbf{x}), D^2V^\delta(t, \mathbf{x})) = 0, \quad \forall (t, \mathbf{x}) \in [0, T) \times \mathbb{R}^2, \\ V^\delta(T, \cdot) = H(\cdot), \end{aligned} \tag{54}$$

where $V^\delta(t, \mathbf{x}) = P^\delta(t, e^y, z)$, $H(\mathbf{x}) = h(e^y)$, and for $(t, \mathbf{x}, p, M) \in [0, T] \times \mathbb{R}^2 \times \mathbb{R}^2 \times S_2$, we have

$$H(t, \mathbf{x}, p, M) = \sup_{q \in [d,u]} \left[f(\mathbf{x}, q) \cdot p + \frac{1}{2} tr(\sigma\sigma^T(\mathbf{x}, q)M) \right].$$

Under the assumptions in this Proposition, in particular, the control being in a compact set by $0 < d \le q_t \le u < \infty$, and uniform parabolicity, then, by Theorem 4.2 in Fleming and Soner [2006] (p.162) and Krylov [1987] (p.301), we know that (54) has a unique solution in $C_b^{1,2,2}$.

(ii) Applying Itô's formula, we get

$$\begin{aligned} J(t, \mathbf{x}, q) := \mathbb{E}_{(t,\mathbf{x})}[H(\mathbf{X}_T)] &= h(e^y) + \mathbb{E}_{(t,\mathbf{x})} \int_t^T \frac{1}{2} e^{2Y_s} h''(e^{Y_s}) q_s^2 F^2(Z_s) ds \\ &= h(e^y) + \mathbb{E}_{(t,\mathbf{x})} \int_t^T L(\mathbf{X}_s, q_s) ds, \end{aligned} \tag{55}$$

where we defined

$$L(\mathbf{x}, q) = \frac{1}{2} e^{2y} h''(e^y) q^2 F^2(z), \tag{56}$$

using the notations in Chapter IV of Fleming and Soner [2006]. The second form of J using the running cost L instead of the terminal value H will be needed to handle the time derivative $\partial_t V^\delta$.

Here, we explain the idea of the proof. First, naively, we could try to bound only $\partial_y V^\delta$ and $\partial^2_{yy} V^\delta$. For doing so, we introduce $\Delta_y J := \epsilon^{-1}[J(t, y+\epsilon, z, q) - J(t, y, z, q)]$, and, following the argument in Lemma 8.1 in Fleming and Soner [2006], we easily get a bound for $|\Delta_y J|$ uniform in $(\epsilon \le 1, \delta \le 1, q)$. Then, the inequality $|\Delta_y V^\delta| \le \sup_q |\Delta_y J|$ gives a bound for $|\Delta_y V^\delta|$ uniform in ϵ and δ, and, therefore, a bound for $|\partial_y V^\delta|$ uniform in δ. Next, we introduce

$\Delta_y^2 J := \epsilon^{-2}[J(t,y+\epsilon,z,q) + J(t,y-\epsilon,z,q) - 2J(t,y,z,q)]$, and, following the argument in Lemma 9.1 in Fleming and Soner [2006], we easily get a bound for $|\Delta_y^2 J|$ uniform in $(\epsilon \leq 1, \delta \leq 1, q)$. Unfortunately, this only provides a lower bound for $\Delta_y^2 V^\delta$ uniform in ϵ and δ, and, therefore, a lower bound for $\partial_{yy}^2 V^\delta$ uniform in δ. In order to obtain an upper bound for the second derivative, one needs a bound for the time derivative $\partial_t V^\delta$ and the full gradient $\partial_{\mathbf{x}} V^\delta$, and then use the HJB equation satisfied by V^δ. The running cost formulation (55) is essential to obtain a uniform bound for $|\Delta_t J|$ following the argument in Lemma 8.2 in Fleming and Soner [2006].

The full proof consists in following the lines of Fleming and Soner [2006] for the derivation of estimates for the first order derivatives in Section IV.8 and for second order derivatives in Section IV.9. The only thing to check is that the constants M_1, M_2, M_3, M_4 respectively in their Lemmas 8.1 about $\Delta_{\mathbf{x}} J$, Lemma 8.2 about $\Delta_t J$, Lemma 9.1 about $\Delta_{\mathbf{x}}^2 J$, and Theorem 9.2 about $D_{\mathbf{x}}^2 V$, can be chosen independent of $\delta \leq 1$. That is tedious but straightforward with the assumptions we made on the functions (F, μ, β), the terminal condition h, and the fact that $0 < d \leq q_s \leq u < \infty$.

More precisely, a careful reading of Sections IV.8 and IV.9 of Fleming and Soner [2006] (p.182–190 that we do not reproduce here) shows that the constants $M_i, i = 1, \cdots, 4$ depends only on T and the model parameter constants $C_i, i = 1, \cdots, 5$, k, l, and m satisfying:

$$
\begin{aligned}
&|f_t| + |f_{\mathbf{x}}| \leq C_1, \quad |\sigma_t| + |\sigma_{\mathbf{x}}| \leq C_1, \\
&|f(t, \mathbf{0}, q)| + |\sigma(t, \mathbf{x}, q)| \leq C_2, \\
&|L| \leq C_3(1 + |\mathbf{x}|^k), \\
&|L_t| + |L_{\mathbf{x}}| \leq C_4(1 + |\mathbf{x}|^l), \\
&|L_{\mathbf{xx}}| \leq C_5(1 + |\mathbf{x}|^m),
\end{aligned}
$$

where f, σ and L are defined in (53) and (56).

Under our model assumptions listed in Proposition C.1, one can obviously take $k = l = m = 0$, and $C_i, i = 1, \cdots, 5$ independent of $\delta \leq 1$. Note that the log-transform $y = \log x$ was crucial in order to get σ bounded, and the decay of the derivatives of h up to the fourth order ensured the existence of C_3, C_4, and C_5. Consequently, as a sub-product, $\partial_y V^\delta$ and $\partial_{yy}^2 V^\delta$ are uniformly bounded in $\delta \leq 1$ which, concludes the proof. ■

Appendix D. Existence and uniqueness of $(X_t^{*,\delta})$. For the existence and uniqueness of $X_t^{*,\delta}$, we consider the transformation $Y_t^{*,\delta} := \log X_t^{*,\delta}$, which is well defined for any $t < \tau^\epsilon$, where for any $\epsilon > 0$,

$$
\begin{aligned}
\tau^\epsilon :&= \inf\{t > 0 | X_t^{*,\delta} = \epsilon \quad \text{or} \quad X_t^{*,\delta} = 1/\epsilon\} \\
&= \inf\{t > 0 | Y_t^{*,\delta} = \log \epsilon \quad \text{or} \quad Y_t^{*,\delta} = -\log \epsilon\}.
\end{aligned}
$$

By Itô's formula, the process $Y_t^{*,\delta}$ satisfies the following SDE:

$$
dY_t^{*,\delta} = -\frac{1}{2}(q^{*,\delta})^2 Z_t dt + q^{*,\delta} \sqrt{Z_t} dW_t. \tag{57}
$$

Note that the diffusion coefficient satisfies $q^{*,\delta}\sqrt{Z_t} \geq d\sqrt{Z_t} > 0$, and is bounded away from 0, hence by Theorem 1 in section 2.6 of Krylov [2008] and the result 7.3.3 of Stroock and Varadhan [2007], the SDE (57) has a unique weak solution. Consequently, we have a unique solution to the SDE (25) until τ^ϵ for any $\epsilon > 0$. In order to show (25) has a unique solution, it suffices to prove that, for any $T > 0$,

$$\lim_{\epsilon\downarrow 0} \mathbb{Q}(\tau^\epsilon < T) = 0.$$

Note that the contribution of $Y_0^{*,\delta}(= \log x)$ is trivial on the term $\lim_{\epsilon\downarrow 0}\log(\frac{\epsilon}{x})$, for simplicity, we consider $Y_0^{*,\delta} = 0$ in the following. For any $t \in [0, T]$, one has

$$Y_t^{*,\delta} = \int_0^t -\frac{1}{2}(q^{*,\delta})^2 Z_s ds + \int_0^t q^{*,\delta}\sqrt{Z_s}dW_s.$$

Then

$$\begin{aligned}
&\mathbb{Q}(\sup_{t\in[0,T]}|Y_t^{*,\delta}| > |\log\epsilon|)\\
\leq&\mathbb{Q}\left(\sup_{t\in[0,T]}\left[\int_0^t \frac{1}{2}u^2 Z_s ds + \left|\int_0^t q^{*,\delta}\sqrt{Z_s}dW_s\right|\right] > |\log\epsilon|\right)\\
\leq&\mathbb{Q}\left(\frac{1}{2}u^2\int_0^T Z_s ds + \sup_{t\in[0,T]}\left|\int_0^t q^{*,\delta}\sqrt{Z_s}dW_s\right| > |\log\epsilon|\right)\\
\leq&\mathbb{Q}\left(\frac{1}{2}u^2\int_0^T Z_s ds > \frac{|\log\epsilon|}{2}\right) + \mathbb{Q}\left(\sup_{t\in[0,T]}\left|\int_0^t q^{*,\delta}\sqrt{Z_s}dW_s\right| > \frac{|\log\epsilon|}{2}\right)\\
:=&\mathcal{A} + \mathcal{B}.
\end{aligned}$$

By Markov inequality and by (41), we have

$$\mathcal{A} \leq \frac{u^2\mathbb{E}\int_0^T Z_s ds}{|\log\epsilon|} \leq \frac{u^2 T C_1(T,z)}{|\log\epsilon|}.$$

By Doob's martingale inequality and by (41), we have

$$\mathcal{B} \leq \frac{\mathbb{E}(\int_0^t q^{*,\delta}\sqrt{Z_s}dW_s)^2}{\left(\frac{\log\epsilon}{2}\right)^2} \leq \frac{\int_0^t \mathbb{E}\{(q^{*,\delta})^2 Z_s\}ds}{\left(\frac{\log\epsilon}{2}\right)^2} \leq \frac{u^2\int_0^T \mathbb{E}Z_s ds}{\left(\frac{\log\epsilon}{2}\right)^2} \leq \frac{u^2 T C_1(T,z)}{\left(\frac{\log\epsilon}{2}\right)^2}.$$

Therefore,

$$\lim_{\epsilon\downarrow 0}\mathcal{A} = \lim_{\epsilon\downarrow 0}\mathcal{B} = 0.$$

Finally, we can conclude that

$$\lim_{\epsilon\downarrow 0}\mathbb{Q}(\tau^\epsilon < T) = \lim_{\epsilon\downarrow 0}\mathbb{Q}(\sup_{t\in[0,T]}|Y_t^{*,\delta}| > |\log\epsilon|) = 0,$$

for any $T > 0$, as desired.

Appendix E. Proof of Uniform Boundedness of I_2 and I_3 on δ. With the help of Assumption 2.12, Cauchy–Schwarz inequality and the uniformly bounded moments of Z_t and X_t processes given in (41) and (45) respectively, we are going to prove that I_2 and I_3 are uniformly bounded in δ.

First recall that

$$\begin{aligned} I_2 =& \mathbb{E}_{(t,x,z)}\left[\int_t^T \left(\rho(q^{*,\delta}) Z_s X_s^{*,\delta} \partial^2_{xz} P_1(s, X_s^{*,\delta}, Z_s)\right.\right.\\ &\left.\left.+ \frac{1}{2} Z_s \partial^2_{zz} P_0(s, X_s^{*,\delta}, Z_s) + \kappa(\theta - Z_s)\partial_z P_0(s, X_s^{*,\delta}, Z_s)\right) ds\right] \\ \doteq& I_2^{(1)} + I_2^{(2)} + I_2^{(3)}. \end{aligned}$$

Then we have

$$\begin{aligned} I_2^{(1)} \leq& \mathbb{E}_{(t,x,z)}\left[\int_t^T \rho u Z_s X_s^{*,\delta} |\partial^2_{xz} P_1(s, X_s^{*,\delta}, Z_s)| ds\right] \\ \leq& \rho u \mathbb{E}^{1/2}_{(t,x,z)}\left[\int_t^T \left(Z_s X_s^{*,\delta}\right)^2 ds\right] \cdot \mathbb{E}^{1/2}_{(t,x,z)}\left[\int_t^T \left(\partial^2_{xz} P_1(s, X_s^{*,\delta}, Z_s)\right)^2 ds\right] \\ \leq& \rho u \mathbb{E}^{1/4}_{(t,x,z)}\left[\int_t^T (Z_s)^4 ds\right] \cdot \mathbb{E}^{1/4}_{(t,x,z)}\left[\int_t^T (X_s^{*,\delta})^4 ds\right] \\ &\cdot \bar{a}_{11}^2 \mathbb{E}^{1/2}_{(t,x,z)}\left[\int_t^T \left(1 + |X_s^{*,\delta}|^{\bar{b}_{11}} + |Z_s|^{\bar{c}_{11}}\right)^2 ds\right] \\ \leq& \rho u\, (C_4)^{1/4} \cdot (N_4)^{1/4} \cdot \bar{A}_{11}[C_{2\bar{b}_{11}} + N_{2\bar{c}_{11}}]^{1/2}, \end{aligned}$$

$$\begin{aligned} I^{(3)} \leq& \frac{1}{2} \mathbb{E}^{1/2}_{(t,x,z)}\left[\int_t^T (Z_s)^2 ds\right] \cdot \mathbb{E}^{1/2}_{(t,x,z)}\left[\int_t^T \left(\partial^2_{zz} P_0(s, X_s^{*,\delta}, Z_s)\right)^2 ds\right] \\ \leq& \frac{1}{2}\, (C_2)^{1/2} \cdot A_{02}[C_{2b_{02}} + N_{2c_{02}}]^{1/2} \end{aligned}$$

and

$$\begin{aligned} I^{(4)} \leq& \kappa \mathbb{E}^{1/2}_{(t,x,z)}\left[\int_t^T (\theta - Z_s)^2 ds\right] \cdot \mathbb{E}^{1/2}_{(t,x,z)}\left[\int_t^T \left(\partial_z P_0(s, X_s^{*,\delta}, Z_s)\right)^2 ds\right] \\ \leq& \kappa \mathbb{E}^{1/2}_{(t,x,z)}\left[\int_t^T \theta^2 + Z_s^2 ds\right] \cdot \mathbb{E}^{1/2}_{(t,x,z)}\left[\int_t^T \left(\partial_z P_0(s, X_s^{*,\delta}, Z_s)\right)^2 ds\right] \\ \leq& \frac{1}{2}\left(C_2 + \theta^2 T\right)^{1/2} \cdot A_{01}[C_{2b_{01}} + N_{2c_{01}}]^{1/2}, \end{aligned}$$

where $\bar{A}_{01}$, $\bar{A}_{11}$ and A_{02} are positive constants.

Next recall that

$$I_3 = \mathbb{E}_{(t,x,z)}\left[\int_t^T \frac{1}{2} Z_s \partial^2_{zz} P_1(s, X_s^{*,\delta}, Z_s) + \kappa(\theta - Z_s)\partial_z P_1(s, X_s^{*,\delta}, Z_s) ds\right] \doteq I_3^{(1)} + I_3^{(2)}.$$

Then we have

$$\begin{aligned} I_3^{(1)} \leq & \frac{1}{2}\mathbb{E}^{1/2}_{(t,x,z)}\left[\int_t^T (Z_s)^2 ds\right]\cdot \mathbb{E}^{1/2}_{(t,x,z)}\left[\int_t^T \left(\partial^2_{zz}P_1(s,X_s^{*,\delta},Z_s)\right)^2 ds\right] \\ \leq & (C_2)^{1/2}\cdot \bar{A}_{02}[C_{2\bar{b}_{02}}+N_{2\bar{c}_{02}}]^{1/2} \end{aligned}$$

and

$$\begin{aligned} I_3^{(2)} \leq & \, 2\kappa\mathbb{E}^{1/2}_{(t,x,z)}\left[\int_t^T \theta^2 + Z_s^2 ds\right]\cdot \mathbb{E}^{1/2}_{(t,x,z)}\left[\int_t^T \left(\partial_z P_1(s,X_s^{*,\delta},Z_s)\right)^2 ds\right] \\ \leq & \, 2\kappa[\theta^2 T + C_2]^{1/2}\cdot \bar{A}_{01}[C_{2\bar{b}_{01}}+N_{2\bar{c}_{01}}]^{1/2}, \end{aligned}$$

where $\bar{A}_{01}$, $\bar{A}_{02}$ are positive constants.

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