Pricing Asian Options with Stochastic Volatility

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Abstract

In this paper, we generalize the recently developed dimension reduction technique of Vecer for pricing arithmetic average Asian options. The assumption of constant volatility in Vecer's method will be relaxed to the case that volatility is randomly fluctuating and is driven by a mean-reverting (or ergodic) process. We then use the fast mean-reverting stochastic volatility asymptotic analysis introduced by Fouque, Papanicolaou and Sircar to derive an approximation to the option price which takes into account the skew of the implied volatility surface. This approximation is obtained by solving a pair of one-dimensional partial differential equations.

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1 Introduction

Asian options are known as path dependent options whose payoff depends on the average stock price and a fixed or floating strike price during a specific period of time before maturity. There has been an enormous amount of literature on the study of such options using numerical approaches for general Asian option problems. The two main streams are Monte Carlo approach with variance reduction techniques and Partial Differential Equation (PDE) approach. A brief review of recent results for both approaches can be found in [15, 20]. Within the PDE methods, the pricing equation is generally two-dimensional in space [14] under the Black-Scholes model with constant volatility. This is due to the introduction of a new state variable, which represents the running sum of the stock process. Methods using change of numeraire to reduce the dimension of the PDE for both floating and fixed strike Asian options have been proposed by Rogers and Shi [18] and Vecer [20]. Note that Vecer's reduction method is based on the use of options on traded account [20]. A similar formulation of the pricing PDE was independently derived by Hoogland and Neumann [11, 12]; see also [13]. The comparison of results of different methods for continuous-sampled Asian call options prices is shown in [19, 20, 17]. It comfirms that the method suggested by Vecer [20] has an efficient, accurate, and stable numerical performance. Moreover, Vecer's approach [20] is also applicable to discrete average Asian options and options on dividend paying stocks.
Regarding the lognomality assumption of the underlying stock price, a number of alternative types of stock dynamics have been suggested. Carr, Geman, Madan, and Yor [2] proposed the so-called CGMY jumps model for the stock price. Eberlein and Prause proposed a general hyperbolic model in [4]. Vecer and Xu [21] have used special semimartingale process models for pricing Asian options and derived pricing equations.
In addition, many empirical studies suggest that the volatility estimated from stock price returns exhibit the time-varying and random characteristics. Hence, stochastic volatility models are also considered. They can reproduce the skew effect of implied volatility, fat-tailed and asymmetric returns distributions (see for instance [5]).
In this paper, we generalize the Black-Scholes model into a particular class of diffusion models where the volatility is driven by a family of mean-reverting processes. Empirical studies (see for instance Alizadeh et al [1], and Chernov et al [3]) strongly support the presence of two well separated time scales

in volatility. Using simulated data, LeBaron in [16] shows that stochastic volatility models with three volatility factors (short, medium and long time scales) reproduce power laws in log returns and long memory behavior. In [7], Fouque-Papanicolaou-Sircar observed a fast time scale volatility factor in S&P 500 high frequency data. They use asymptotic analysis to approximate option pricing and hedging problems in finance as detailed in [5]. Recently Fouque et al [10] have used a combination of singular and regular perturbations analysis to incorporate both short and long time scales in their models showing that the long time scale factor is needed to handle options with longer maturities.

The results cited above motivate our paper to generalize Vecer's [20] dimension reduction technique on pricing arithmetic average Asian option (European style) by relaxing the assumption of constant volatility to the case that volatility is randomly fluctuating and is driven by an auxiliary mean-reverting process. We consider here only the case of a short time scale volatility factor and use a singular perturbation technique to approximate Asian option prices. A generalization along the lines of [10] would allow the presence of an additional long time scale volatility factor.

Based on the results from asymptotic analysis presented in the Appendix, the approximated price, or so-called corrected price, is derived. As a consequence, there is no need to estimate the current level of the unobserved volatility. All the parameters we need to compute the approximated price can be easily calibrated from the observed stock price and the implied volatility surface. Thus, this article describes a robust procedure to correct Asian option prices by taking the observed implied volatility skew into account. Numerical computation of the corrected Asian option price is certainly needed.

The organization of the paper is as follows. The introduction of the class of stochastic volatility models we consider is in Section 2. Two methods to derive the Asian option pricing PDEs are presented in Section 3. The asymptotics and the calibration of relevant parameters from the implied volatility skew are discussed in Section 4, which includes the probabilistic representation of the approximated Asian option price. A numerical illustration is presented in Section 5, the conclusions are in Section 6, and the details of the asymptotic analysis are given in the Appendix.

2 Mean Reverting Stochastic Volatility Models

In this section, the stochastic volatility models in which volatility $\sigma_t = f(Y_t)$ is driven by an ergodic process (Y_t) that approaches its unique invariant distribution at an exponential rate α is considered. The size of the exponential rate captures clustering effects. The function f is assumed to be sufficiently regular, positive, bounded, and bounded away from zero. In particular, we shall be interested in asymptotic approximation of price when α is large, which describes bursty volatility.

In the family of mean-reverting stochastic volatility model (S_t, Y_t), where S_t is the underlying price, we consider for instance that Y_t evolves as a one factor Ornstein-Uhlenbeck (OU) process, as a prototype of an ergodic diffusion. Under the physical probability measure P, our model can be written as

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma_t S_t dW_t, \\ \sigma_t &= f(Y_t), \\ dY_t &= \alpha(m - Y_t) + \beta(\rho dW_t + \sqrt{1-\rho^2}\, dZ_t). \end{aligned} \tag{1}$$

The stock price S_t evolves as a diffusion with a constant μ in the drift and the random process σ_t in the volatility. The driving volatility Y_t evolves with a mean m, a rate of mean reversion $\alpha > 0$, and a "volatility of the volatility" β. The processes W_t and Z_t are independent standard Brownian motions. The instant correlation $\rho \in (-1, 1)$ captures the leverage effect. Moreover, taking Y_t to be a diffusion process allows us to model the asymmetry of returns distributions by incorporating a negative correlation ρ.

The evidence of the appearance of a short time-scale, in the high-frequency intraday $S\&P$ 500 data, is discussed in [7]. This motivates the study of the asymptotic analysis in the option pricing problems. To perform an asymptotic analysis, we introduce a small parameter ε such that the rate of mean reversion defined by $\alpha = 1/\varepsilon$ becomes large. To capture the volatility clustering behaviors, we define $\nu^2 = \frac{\beta^2}{2\alpha}$ to be a fixed $O(1)$ constant. Since the rate of mean-reversion of the volatility process depends on ε, we denote this ε-dependence for S_t and Y_t by S_t^ε the stock price and Y_t^ε the volatility factor, respectively.

We assume that the market is pricing derivatives under a risk-neutral probability measure P^*. Using Girsanov theorem, our model under P^* can be written as

$$dS_t^\varepsilon = rS_t^\varepsilon dt + f(Y_t^\varepsilon)S_t^\varepsilon dW_t^*, \tag{2}$$

$$dY_t^\varepsilon = \left[\frac{1}{\varepsilon}(m - Y_t^\varepsilon) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(Y_t^\varepsilon)\right] dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}(\rho dW_t^* + \sqrt{1-\rho^2}\, dZ_t^*), \tag{3}$$

where W_t^* and Z_t^* are independent standard Brownian motions. The combined market price of risk is defined as

$$\Lambda(y) = \rho\frac{\mu - r}{f(y)} + \gamma(y)\sqrt{1-\rho^2},$$

which describes the relationship between the physical measure under which the stock price is observed, and the risk-neutral measure under which the market prices derivative securities are computed. We assume that the risk-free interest rate r is constant, and that the market price of volatility risk $\gamma(y)$ is bounded, and depends only on the volatility level y. At the leading order $1/\varepsilon$ in (3), that is omitting the Λ-term in the drift, Y_t^ε is an OU process which is fast mean-reverting with a normal invariant distribution $N(m, \nu^2)$. The volatility factor Y_t^ε fluctuates randomly around its mean level m and the long run magnitude ν of volatility fluctuations remains fixed for values of ε. Furthermore, due to the presence of the other Brownian motion Z^* in (2), there exists a γ-dependent family of equivalent risk-neutral measures. However, we assume that the market chooses one measure through the market price of volatility risk γ.

3 Asian Option Problems With Stochastic Volatility

Based on the Feynman-Kac formula, the pricing PDE can be derived from the conditional expectation under the risk-neutral probability measure P^*. The three-dimensional PDE's obtained in [5] are compared with the two-dimensional PDE's, which we derive by using Vecer's technique of dimension reduction [20].

3.1 Derivation of Three-Dimensional Pricing PDE with Stochastic Volatility

The usual way to deal with the Asian option problem is to introduce a new process

$$I_t^\varepsilon = \int_0^t S_s^\varepsilon ds, \tag{4}$$

which represents the running sum stock process. Here we assume the stochastic volatility model obey (2,3) in addition to the differential form of (4)

$$dI_t^\varepsilon = S_t^\varepsilon dt.$$

Under the risk-neutral probability measure P^* the process $(S_t^\varepsilon, Y_t^\varepsilon, I_t^\varepsilon)$ is a Markov process. The equation (4) remains unchanged under the change of measure. As shown in [5] the price of the Asian floating-strike call option with stochastic volatility at time $0 \le t \le T$ is given by

$$P^\varepsilon(t,s,y,I) = E^*\left\{ e^{-r(T-t)} \left(S_T^\varepsilon - \frac{I_T^\varepsilon}{T} \right)^+ \mid S_t^\varepsilon = s, Y_t^\varepsilon = y, I_t^\varepsilon = I \right\}.$$

From the Feynman-Kac formula, it is also obtained as the solution of the following PDE

$$\frac{\partial P^\varepsilon}{\partial t} + \frac{1}{2} f(y)^2 s^2 \frac{\partial^2 P^\varepsilon}{\partial s^2} + r\left(s\frac{\partial P^\varepsilon}{\partial s} - P^\varepsilon \right) + \frac{\rho\nu\sqrt{2}}{\sqrt{\varepsilon}} f(y) \frac{\partial^2 P^\varepsilon}{\partial s \partial y} + \frac{\nu^2}{\varepsilon}\frac{\partial^2 P^\varepsilon}{\partial y^2}$$
$$+ \left(\frac{1}{\varepsilon}(m-y) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(y) \right) \frac{\partial P^\varepsilon}{\partial y} + s\frac{\partial P^\varepsilon}{\partial I} = 0, \tag{5}$$

with the terminal condition

$$P^\varepsilon(T,s,y,I) = \left(s - \frac{I}{T} \right)^+ .$$

Notice that the Asian option price is obtained by solving this linear PDE (5) at any given current time t, stocks price S_t^ε, and driving volatility level Y_t^ε. However, the PDE (5) is three-dimensional in space and any numerical PDE scheme to solve it requires significant computation efforts. Our goal is to reduce the dimensionality of this PDE (5).

3.2 Derivation of Two-Dimensional Pricing PDE with Stochastic Volatility

The dimension reduction technique that Vecer [20] proposed in the case of constant volatility is to construct a wealth or portfolio process, which can replicate the stock price average by self-financing trading in the stock and bond. One feature of this technique is that the selected trading strategy function is only time dependent.

When the process (4) is considered in our model (2,3), it is important to distinguish whether the Asian option contract already started. When the current time t is exactly at the contract starting date 0, the Asian option is "fresh". When the current time t is between the contract starting date 0 and maturity date T, it is "seasoned." For ease of exposition, we confine our discussion to the case where the Asian option is fresh. The seasoned case and the Asian put-call parity will be considered in Section 3.3.

We generalize Vecer's method to the stochastic volatility model and find that the property of pre-determined time dependent trading strategy is preserved. Under the risk-neutral probability measure, the wealth or portfolio process, denoted by X_t^ε, will also be shown to replicate the stock price average by self-financing trading in the stock and bond.

The pre-determined time-dependent trading strategy function, corresponding to the strike being a continuously sampled arithmetic average stock price, is chosen to be

$$q(t) = \frac{1 - e^{-r(T-t)}}{rT}. \tag{6}$$

This is the number of units held at time t of the underlying stock and the constant T is the maturity time. Since the price of the bond at time t is e^{rt}, the quantity $e^{-rt}\left(X_t^\varepsilon - q(t)S_t^\varepsilon\right)$ is the number of units held in the savings account. We assume that this portfolio is to be self-financing so that the variation of the wealth process can be expressed in differential form as

$$\begin{aligned} dX_t^\varepsilon &= q(t)dS_t^\varepsilon + \frac{X_t^\varepsilon - q(t)S_t^\varepsilon}{e^{rt}} d(e^{rt}) \\ &= rX_t^\varepsilon dt + q(t)(dS_t^\varepsilon - rS_t^\varepsilon dt). \end{aligned} \tag{7}$$

Moreover, combining

$$d(e^{r(T-t)}X_t^\varepsilon) = e^{r(T-t)}q(t)(dS_t^\varepsilon - rS_t^\varepsilon dt) \tag{8}$$

with

$$d(e^{r(T-t)}q(t)S_t^\varepsilon) = e^{r(T-t)}q(t)(dS_t^\varepsilon - rS_t^\varepsilon dt) + e^{r(T-t)}S_t^\varepsilon dq(t), \tag{9}$$

and expressing equations (8) and (9) into integral forms, one obtains that the final wealth X_T^ε is equal to the arithmetic average of the underlying prices

$$\begin{aligned} X_T^\varepsilon &= e^{rT}X_0^\varepsilon + q(T)S_T^\varepsilon - e^{rT}q(0)S_0^\varepsilon - \int_0^T e^{r(T-t)}S_t^\varepsilon \left(-\frac{e^{-r(T-t)}}{T}\right) dt \\ &= \frac{1}{T}\int_0^T S_t^\varepsilon dt, \end{aligned}$$

if the initial wealth is chosen as $X_0^\varepsilon = q(0)S_0^\varepsilon$. Alternatively, if the initial wealth is chosen as $X_0 = q(0)S_0 - e^{-rT}K_2$, the final wealth X_T^ε becomes $\frac{1}{T}\int_0^T S_t^\varepsilon dt - K_2$. In the following we make that later choice so that the initial wealth X_0 is equal to

$$x = \frac{1-e^{-rT}}{rT}s - e^{-rT}K_2. \tag{10}$$

Hence the general payoff function for arithmetic average Asian options can be described as

$$h\left(\frac{1}{T}\int_0^T S_t^\varepsilon dt - K_1 S_T^\varepsilon - K_2\right) = h(X_T^\varepsilon - K_1 S_T^\varepsilon), \tag{11}$$

by choosing the initial wealth according to (10). When $K_1 = 0$, we have a fixed strike Asian option; when $K_2 = 0$, we have the floating strike Asian option. The payoff function h is assumed to be homogeneous, i.e.

$$h(xy) = xh(y), \tag{12}$$

for each nonnegative x. For example this is the case for

$$h(\cdot) = (\cdot)^{\pm}, \tag{13}$$

corresponding to a call or a put option, respectively.

The price $P^\varepsilon(0, s, y; T, K_1, K_2)$ of an arithmetic average Asian option with stochastic volatility, is given by

$$P^\varepsilon(0, s, y; T, K_1, K_2) = e^{-rT}E^*\{h(X_T^\varepsilon - K_1 S_T^\varepsilon) \mid S_0^\varepsilon = s, Y_0^\varepsilon = y\}, \tag{14}$$

where $(S_t^\varepsilon, Y_t^\varepsilon, X_t^\varepsilon)$ follow (2,3) and (7), respectively, under the pricing risk-neutral measure P^*. By change of numeraire

$$\Psi_t^\varepsilon = \frac{X_t^\varepsilon}{S_t^\varepsilon}, \tag{15}$$

and from Ito's formula, it can be derived that

$$d\Psi_t^\varepsilon \;=\; (q(t) - \Psi_t^\varepsilon) f(Y_t^\varepsilon) d\tilde{W}_t^{\,*}, \tag{16}$$

where

$$\tilde{W}_t^{\,*} = W_t^* - \int_0^t f(Y_s^\varepsilon) ds. \tag{17}$$

By Girsanov Theorem, under the probability measure $\tilde{P}^*$ defined by

$$\begin{aligned} \frac{d\tilde{P}^*}{dP^*} &= e^{-rT} \frac{S_T^\varepsilon}{S_0^\varepsilon} \\ &= \exp\left(\int_0^T f(Y_t^\varepsilon) dW_t^* - \frac{1}{2} \int_0^T f(Y_t^\varepsilon)^2 dt \right), \end{aligned}$$

the process $\tilde{W}_t^*$ given by (17) is a standard Brownian motion. Hence, the driving volatility process becomes

$$\begin{aligned} dY_t^\varepsilon &= \left[\frac{1}{\varepsilon}(m - Y_t^\varepsilon) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} \Lambda(Y_t^\varepsilon) + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} \rho f(Y_t^\varepsilon) \right] dt \\ &+ \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} (\rho d\tilde{W}_t^* + \sqrt{1-\rho^2}\, dZ_t^*). \end{aligned} \tag{18}$$

Using the homogeneous property (12) of the payoff function h, the Asian option price (14) is equal to

$$\begin{aligned} & sE^* \left\{ e^{-rT} \frac{S_T^\varepsilon}{S_0^\varepsilon} h\left(\frac{X_T^\varepsilon}{S_T^\varepsilon} - K_1 \right) \mid S_0^\varepsilon = s, Y_0^\varepsilon = y \right\} \\ & = s\tilde{E}^* \{ h(\Psi_T^\varepsilon - K_1) \mid \Psi_0^\varepsilon = \psi, Y_0^\varepsilon = y \}, \end{aligned}$$

where, by using (10), we have

$$\psi = \frac{x}{s} = \frac{1 - e^{-rT}}{rT} - e^{-rT} \frac{K_2}{s}.$$

We define the quantity of interest

$$u^\varepsilon(0,\psi,y;T,K_1,K_2) \equiv \tilde{E}^*\{h(\Psi_T^\varepsilon - K_1) \mid \Psi_0^\varepsilon = \psi, Y_0^\varepsilon = y\}, \tag{19}$$

such that the Asian option price (14) can be expressed as

$$P^\varepsilon(0,s,y;T,K_1,K_2) = su^\varepsilon(0,\psi,y;T,K_1,K_2). \tag{20}$$

Note that from (16) and (18) the joint process $(\Psi_t^\varepsilon, Y_t^\varepsilon)$ is Markovian. If we introduce

$$u^\varepsilon(t,\psi,y;T,K_1,K_2) \equiv \tilde{E}^*\{h(\Psi_T^\varepsilon - K_1) \mid \Psi_t^\varepsilon = \psi, Y_t^\varepsilon = y\},$$

by Feynman-Kac formula, $u^\varepsilon(t,\psi,y;T,K_1,K_2)$ is the solution of the two dimensional linear PDE

$$\begin{aligned}&\frac{\partial u^\varepsilon}{\partial t} + \frac{1}{2}(\psi - q(t))^2 f(y)^2 \frac{\partial^2 u^\varepsilon}{\partial \psi^2} + \frac{\rho\nu\sqrt{2}}{\sqrt{\varepsilon}}(q(t)-\psi)f(y)\frac{\partial^2 u^\varepsilon}{\partial y \partial \psi} \\ &+ \left(\frac{1}{\varepsilon}(m-y) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}(\Lambda(y) - \rho f(y))\right)\frac{\partial u^\varepsilon}{\partial y} + \frac{\nu^2}{\varepsilon}\frac{\partial^2 u^\varepsilon}{\partial y^2} = 0,\end{aligned} \tag{21}$$

with the terminal condition

$$u^\varepsilon(T,\psi,y;T,K_1,K_2) = h(\psi - K_1).$$

It is remarkable to observe that the PDE has one less spatial dimension than (5). However, the solution of this backward PDE can only be the price of the fresh Asian option when the current time $t=0$. The same PDE may not be a pricing equation for the seasoned Asian option, which is discussed in the next section.
Notice that our argument can be extended to the discretely sampled Asian options for cases considered in [20] and [21].

3.3 Seasoned Asian Option Prices and Asian Put-Call Parity

Suppose we are at the current time t, which is between the Asian option contract starting date 0 and the maturity date T. Denote by $\mathcal{F}_t$ the σ-algebra generated by the processes $(S_u^\varepsilon)_{u=0}^t$ and $(Y_u^\varepsilon)_{u=0}^t$. Conditioning on $\mathcal{F}_t$, the price of the Asian option is given at time t by

$$E^*\left\{e^{-r(T-t)}h\left(\frac{1}{T}\int_0^T S_u^\varepsilon du - K_1 S_T^\varepsilon - K_2\right) \mid \mathcal{F}_t\right\}.$$

Splitting the integral, the Asian option price is equal to

$$\frac{T-t}{T}E^*\left\{e^{-r(T-t)}h\left(\frac{\int_t^T S_u^\varepsilon du}{T-t}-\frac{T}{T-t}K_1S_T^\varepsilon-\frac{T}{T-t}K_2+\frac{\int_0^t S_u^\varepsilon du}{T-t}\right)\mid\mathcal{F}_t\right\}$$
$$=\frac{\tau}{T}E^*\left\{e^{-r\tau}h\left(\frac{1}{\tau}\int_t^T S_u^\varepsilon du-\hat{K}_1S_T^\varepsilon-\hat{K}_2\right)\mid\mathcal{F}_t\right\} \tag{22}$$

where, to simplify the notation, we define $\tau=T-t$ the time to maturity, and the updated strikes $\hat{K}_1=\frac{T}{T-t}K_1$ and $\hat{K}_2=\frac{T}{T-t}K_2+\frac{1}{T-t}\int_0^t S_u^\varepsilon du$. Repeating the argument given in section 3.2 but between times t and T, it follows that the seasoned Asain option price (22) is equal to

$$\frac{\tau s}{T}\tilde{E}^*\{h(\Psi_T^\varepsilon-\hat{K}_1)\mid\Psi_t^\varepsilon=\psi,Y_t^\varepsilon=y\}, \tag{23}$$

where

$$\psi=\frac{X_t^\varepsilon}{S_t^\varepsilon}=\frac{q(t)S_t^\varepsilon-e^{-r\tau}\hat{K}_2}{S_t^\varepsilon}=q(t)-\frac{e^{-r\tau}\hat{K}_2}{s}.$$

Furthermore, since the trading strategy $q(t)$ in (16) only depends on $T-t$ for fixed T and r, we can shift time frame of the contract between t and T to the time frame between 0 and t. This implies that the price of the seasoned Asian option (23) is equal to

$$\frac{\tau s}{T}\tilde{E}^*\{h(\Psi_\tau^\varepsilon-\hat{K}_1)\mid\Psi_0^\varepsilon=\psi,Y_0^\varepsilon=y\}. \tag{24}$$

Using the notation (19,20) introduced in section 3.2, the seasoned Asian option price can be represented by the multiplication of a fresh Asian option price with the updated strike prices $\hat{K}_1$ and $\hat{K}_2$ and a constant factor τ/T

$$\frac{\tau}{T}P^\varepsilon(0,s,y;\tau,\hat{K}_1,\hat{K}_2). \tag{25}$$

This result reveals that the seasoned Asian option price can be obtained by solving the pricing PDE (21) but with modified terminal condition and terminal time. Hoogland and Neumann [11] used the local scale invariant property of the contingent claim to derive a similar result for the case of constant volatility.

Remark 1: Using the dimension reduction technique, parameters and initial conditions in the pricing PDE (21) need to be changed with the fixed current time of the contract in order to get the seasoned Asian option price.

Compared to the pricing PDE (5), the parameters are always fixed whenever the current time is. Heuristically, what the dimension reduction technique in the stochastic volatility model does is breaking the three dimensional PDE into infinitely many two dimensional PDEs. Fortunately, for the purpose of pricing we are only interested in the price at discrete times.
We now consider the seasoned Asian put-call parity. Recall that the seasoned Asian call and put option prices are given from (22,25) by

$$
\begin{aligned}
&\frac{\tau}{T}P^{\varepsilon}_{call}(0,s,y;\tau,\hat{K}_1,\hat{K}_2) = \\
&\frac{\tau}{T}E^*\left\{e^{-r\tau}\left(\frac{1}{\tau}\int_0^{\tau} S^{\varepsilon}_t dt - \hat{K}_1 S^{\varepsilon}_{\tau} - \hat{K}_2\right)^+ \mid S^{\varepsilon}_0 = s, Y^{\varepsilon}_0 = y\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\tau}{T}P^{\varepsilon}_{put}(0,s,y;\tau,\hat{K}_1,\hat{K}_2) = \\
&\frac{\tau}{T}E^*\left\{e^{-r\tau}\left(\frac{1}{\tau}\int_0^{\tau} S^{\varepsilon}_t dt - \hat{K}_1 S^{\varepsilon}_{\tau} - \hat{K}_2\right)^- \mid S^{\varepsilon}_0 = s, Y^{\varepsilon}_0 = y\right\},
\end{aligned}
$$

respectively. A simple computation gives

$$
\begin{aligned}
&\frac{\tau}{T}P^{\varepsilon}_{call}(0,s,y;\tau,\hat{K}_1,\hat{K}_2) + \frac{\tau}{T}P^{\varepsilon}_{put}(0,s,y;\tau,\hat{K}_1,\hat{K}_2) \\
= \;& \frac{\tau}{T}E^*\left\{e^{-r\tau}\left(\frac{1}{\tau}\int_0^{\tau} S^{\varepsilon}_t dt - \hat{K}_1 S^{\varepsilon}_{\tau} - \hat{K}_2\right) \mid S^{\varepsilon}_0 = s, Y^{\varepsilon}_0 = y\right\} \\
= \;& \frac{s}{T}\frac{1-e^{-r\tau}}{r} + \frac{\tau}{T}\hat{K}_1 s - \frac{\tau}{T}e^{-r\tau}\hat{K}_2, \qquad (26)
\end{aligned}
$$

which is the seasoned Asian put-call parity.

4 Implied Volatilities and Calibration

When volatility is fast mean-reverting, on a time-scale smaller than typical maturities, one can apply asymptotic analysis or singular perturbation analysis on the pricing PDE (5) and (21) in order to obtain the approximated price. We find that the quantities of interest derived through the analysis are just functions of the parameters appearing in the approximated European option prices. Thus, in this section we describe a robust procedure to correct Black-Scholes Asian option prices to account for the observed implied volatility skew. The methodology is to observe both the underlying stock prices and the European option prices, which is encapsulated in the skew surface, such that the Asian option price under the stochastic volatility environment can be calculated.

4.1 Review of European Options Asymptotics

We give here a brief review of the main results in [5] from the asymptotic analysis of the European options problem under fast mean-reverting stochastic volatility model as appeared in (2,3). Let $P_e^\varepsilon(t,x,y)$ be the price of European option which solves

$$\begin{aligned}\frac{\partial P_e^\varepsilon}{\partial t} + \frac{1}{2}x^2 f(y)^2 \frac{\partial^2 P_e^\varepsilon}{\partial x^2} + r\left(x\frac{\partial P_e^\varepsilon}{\partial x} - P_e^\varepsilon\right) + \frac{\rho\nu\sqrt{2}}{\sqrt{\varepsilon}} x f(y)\frac{\partial^2 P_e^\varepsilon}{\partial x \partial y} \\ + \frac{\nu^2}{\varepsilon}\frac{\partial^2 P_e^\varepsilon}{\partial y^2} + \left(\frac{1}{\varepsilon}(m-y) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(y)\right)\frac{\partial P_e^\varepsilon}{\partial y} = 0, \qquad (27)\end{aligned}$$

with the terminal condition $P_e^\varepsilon(T,x,y) = h(x-K)$. The price P_e^ε of an European call or put option is shown in [9] to have the pointwise accuracy of the corrected Black-Scholes price

$$\left|P_e^\varepsilon(t,x,y) - \left(P_0^e(t,x) + \tilde{P}_1^e(t,x)\right)\right| = O(\varepsilon|\log\varepsilon|).$$

The first order term $P_0^e(t,x)$ solves the homogenized Black-Scholes PDE

$$\frac{\partial P_0^e}{\partial t} + \frac{1}{2}x^2\overline{\sigma}^2\frac{\partial^2 P_0^e}{\partial x^2} + r\left(x\frac{\partial P_0^e}{\partial x} - P_0^e\right) = 0,$$

with the terminal condition $P_0^e(T,x) = h(x-K)$ and a constant volatility $\overline{\sigma}$ defined in (30). The correction $\tilde{P}_1^e(t,x)$ satisfies

$$\frac{\partial \tilde{P}_1^e}{\partial t} + \frac{1}{2}x^2\overline{\sigma}^2\frac{\partial^2 \tilde{P}_1^e}{\partial x^2} + r\left(x\frac{\partial \tilde{P}_1^e}{\partial x} - \tilde{P}_1^e\right) = V_2 x^2\frac{\partial^2 P_0^e}{\partial x^2} + V_3 x^3\frac{\partial^3 P_0^e}{\partial x^3},$$

with zero terminal condition $\tilde{P}_1(T,x) = 0$. The parameters V_2 and V_3 are small quantities of order $\sqrt{\varepsilon}$ given by

$$V_2 = \frac{\nu\sqrt{\varepsilon}}{\sqrt{2}}(2\rho < f(y)\phi'(y) > - < \Lambda(y)\phi'(y) >), \qquad (28)$$

$$V_3 = \frac{\rho\nu\sqrt{\varepsilon}}{\sqrt{2}} < f(y)\phi'(y) >, \qquad (29)$$

where $< \cdot >$ denotes the averaging with respect to the invariant distribution $N(m, \nu^2)$ of the OU process Y_t introduced in (1):

$$< g >= \frac{1}{\sqrt{2\pi}\nu} \int g(y) e^{-(m-y)^2/2\nu^2} dy.$$

The effective constant volatility $\overline{\sigma}$ is defined as

$$\overline{\sigma}^2 =< f^2 >, \tag{30}$$

and the function $\phi(y)$ is a solution of the Poisson equation

$$\nu^2 \frac{\partial^2 \phi}{\partial y^2} + (m-y)\frac{\partial \phi}{\partial y} = f(y)^2 - < f^2 > .$$

Moreover, the parameters V_2 and V_3 can be calibrated from the term structure of the implied volatility surface [5]. The implied volatility I^ε of a European call option price with fast mean reverting stochastic volatility can be written as

$$I^\varepsilon = a\frac{log(K/x)}{T-t} + b + o(\sqrt{\varepsilon})$$

with

$$\begin{aligned} a &= -\frac{V_3}{\bar{\sigma}^3}, \\ b &= \bar{\sigma} + \frac{V_3}{\bar{\sigma}^3}(r + \frac{3}{2}\bar{\sigma}^2) - \frac{V_2}{\bar{\sigma}}. \end{aligned}$$

The parameters a and b are estimated as the slope and intercept of the line fit of the observed implied volatilities plotted as a function of LMMR ($logmoneyness - to - maturity - ratio$). From a and b calibrated on the observed implied volatility surface, the parameters V_2 and V_3 are given by

$$\begin{array}{l} V_2 = \bar{\sigma}((\bar{\sigma} - b) - a(r + \frac{3}{2}\bar{\sigma}^2)) \\ V_3 = -a\bar{\sigma}^3. \end{array} \tag{31}$$

Remark 2: Notice that the approximated price, $P_0^e + \tilde{P}_1^e$, does not depend on the current volatility level y, which is not directly observable. This will also be true in the Asian case.
Remark 3: As explained in [5], despite the many model parameters $\varepsilon, r, \nu, \rho, \gamma, m$, and $f(\cdot)$, the group of parameters that are needed to approximate the price of European options are:

- $\overline{\sigma}$: mean historical volatility of stock,
- V_2: a quantity related to the market price of volatility risk,
- V_3: a quantity related to the volatility skew.

4.2 Asian Option Asymptotics

Since the seasoned Asian option price can be deduced from the fresh Asian option prices as explained in section 3.3, it is sufficient to apply the asymptotic analysis on the fresh case. We look for a solution of (21) of the form

$$u^\varepsilon(t,\psi,y) = u_0(t,\psi,y) + \sqrt{\varepsilon}u_1(t,\psi,y) + \cdots .$$

Results from the Appendix show that the zero order term u_0 and the first order term $\tilde{u}_1 = \sqrt{\varepsilon}u_1$, both are volatility level y independent, solve individually the following two PDEs

$$\frac{\partial u_0}{\partial t} + \frac{1}{2}(q(t)-\psi)^2\overline{\sigma}^2\frac{\partial^2 u_0}{\partial \psi^2} = 0, \tag{32}$$

with the terminal condition

$$u_0(T,\psi) = h(\psi - K_1),$$

and

$$\frac{\partial \tilde{u}_1}{\partial t} + \frac{1}{2}(q(t)-\psi)^2\overline{\sigma}^2\frac{\partial^2 \tilde{u}_1}{\partial \psi^2} = \overline{V}_2(q(t)-\psi)^2\frac{\partial^2 u_0}{\partial \psi^2} + \overline{V}_3(q(t)-\psi)^3\frac{\partial^3 u_0}{\partial \psi^3}, \tag{33}$$

with

$$\tilde{u}_1(T,\psi) = 0.$$

The parameter $\overline{\sigma}$ given in (30) is the same as the mean historical volatility stock prices. The small parameters $\overline{V}_2$ and $\overline{V}_3$, defined in (47,48), are model dependent functions of $\varepsilon, r, \nu, \rho, \gamma, m$, and $f(\cdot)$. However, observe that these two parameters turn out to be linear functions of the other two small parameters V_2 and V_3, introduced in (28,29). From equations (47,48) in the Appendix, it is easy to see that

$$\begin{aligned} \overline{V_2} &= V_2 - 3V_3, \\ \overline{V_3} &= V_3. \end{aligned}$$

Using (31), we deduce that $(\overline{V_2}, \overline{V_3})$ needed for pricing the Asian option are obtained by the LMMR fit of the volatility skew:

$$\begin{aligned}\overline{V_2} &= \bar{\sigma}((\bar{\sigma} - b) - a(r - \frac{3}{2}\bar{\sigma}^2)) \\ \overline{V_3} &= -a\bar{\sigma}^3.\end{aligned}$$

This result provides a procedure to price the Asian options by using the European option prices. In order words, one can first estimate parameters V_2 and V_3 from the observed European option prices, or, by proxy, the implied volatility surface. Using $\bar{\sigma}$, $\bar{V_2}$, and $\bar{V_3}$, the approximated Asian option price consequently is comprised of $P_0(t, \psi)$ and $\tilde{P_1}(t, \psi)$, where $P_0(t, \psi) = S_0^\varepsilon u_0(t, \psi)$ and $\tilde{P_1}(t, \psi) = S_0^\varepsilon \tilde{u}_1(t, \psi)$. Each corresponds to solving one-dimensional linear PDE. At this level of accuracy $O(\sqrt{\varepsilon})$, it is convenient to use the Asian put-call parity (26) to compute the approximated put Asian price presuming we already compute the Asian call option price, and vice versa.

To demonstrate how the dimension reduction technique improves in the computational efforts for the case of fast mean reverting stochastic volatility, we recall results from [5], where the asymptotic analysis is applied to three-dimensional PDE (5). It is showed in [5] that the approximated Asian floating-strike call option $Q^\varepsilon(t, s, y, I)$ is the sum of two terms $Q_0(t, s, I)$ and $\tilde{Q}_1(t, s, I)$. The zero order price $Q_0(t, s, I)$ solves the two dimensional PDE

$$\frac{\partial Q_0}{\partial t} + \frac{1}{2}\overline{\sigma}^2 s^2 \frac{\partial^2 Q_0}{\partial s^2} + r\left(s\frac{\partial Q_0}{\partial s} - Q_0\right) + s\frac{\partial Q_0}{\partial I} = 0, \tag{34}$$

with the terminal condition

$$Q_0(T, s, I) = \left(s - \frac{I}{T}\right)^+.$$

The correction term $\tilde{Q}_1(t, x, I)$ solves the same two dimensional PDE but with a source term:

$$\frac{\partial \tilde{Q}_1}{\partial t} + \frac{1}{2}s^2\overline{\sigma}^2\frac{\partial^2 \tilde{Q}_1}{\partial s^2} + r\left(s\frac{\partial \tilde{Q}_1}{\partial s} - \tilde{Q}_1\right) + s\frac{\partial \tilde{Q}_1}{\partial I} = V_2 s^2\frac{\partial^2 Q_0}{\partial s^2} + V_3 s^3\frac{\partial^3 Q_0}{\partial s^3}, \tag{35}$$

and a zero terminal condition. Parameters V_2 and V_3 have the same definition as (28,29).

We have shown in one example the advantage of dimension reduction technique applied on the Asian option problem with fast mean reverting stochastic volatility. That is, rather than solving a pair of two-dimensional PDEs (34,35) to approximate the Asian floating strike call option, it is enough to solve a pair of one-dimensional PDEs (32,33). This fact can be easily generalized to other types of Asian option pricing problems, for example, the dividend paying stock case, the discrete-sampled average stock case, and so on.

4.3 Probabilistic Representation of The Approximated Asian Option Price

We have seen that u_0 in (32) and $\tilde{u}_1$ in (33) play the main roles in the approximated Asian option price. We know that such solutions can be represented as expectations of functionals of the "homogenized" process

$$d\overline{\Psi}_t = (q(t) - \overline{\Psi}_t)\overline{\sigma} d\overline{W}_t,$$

where $\overline{W}$ is a standard Brownian motion, and $\overline{\sigma}$ is the effective volatility given by (30).

The leading order term u_0 is obtained by writing the Feynman-Kac formula associated to (32)

$$u_0(t, \psi) = \overline{E}\left\{ h(\overline{\Psi}_T - K_1) \mid \overline{\Psi}_t = \frac{\overline{X}_t}{\overline{S}_t} = \frac{x}{s} = \psi \right\}. \tag{36}$$

To characterize the distribution of $\overline{\Psi}_T$, consider the substitution

$$\alpha_t = \overline{\Psi}_t - q(t)$$

such that the differential becomes an inhomogeneous linear stochastic differential equation:

$$d\alpha_t = q'(t)dt - \alpha_t \overline{\sigma} d\overline{W}_t.$$

Multiplying the equation by the suitable exponential martingale

$$F_t = e^{\overline{\sigma}^2 t/2 - \overline{\sigma}\overline{W}_t},$$

which plays the role of the "integrating factor," we introduce

$$d(F_t \alpha_t) = F_t q'(t)dt.$$

In the integral form, from t to T, we have

$$\begin{aligned} \alpha_T &= F_T^{-1} F_t^{-1} \alpha_t + \int_t^T q'(s) F_T^{-1} F_s ds \\ &= \alpha_t e^{-\overline{\sigma}^2 (T-t)/2 + \overline{\sigma}(\overline{W}_T - \overline{W}_t)} + \int_t^T q'(s) e^{-\overline{\sigma}^2 (T-s)/2 + \overline{\sigma}(\overline{W}_T - \overline{W}_s)} ds, \end{aligned} \quad (37)$$

where the initial value is $\alpha_t = \psi - q(t)$. Thus, the conditional expectation (36) is equal to

$$u_0(t, \psi) = \overline{E}\{h(\alpha_T + q(T) - K_1) | \alpha_t = \psi - q(t)\}. \quad (38)$$

Remark 4: Vecer and Xu [21] pointed out that European type options are special cases of Asian options by choosing $q(t) = 1$ and $K_1 = 0$. As a result, $\overline{\Psi}_T = \alpha_T + 1$ and the random variable α_T is simply a geometric Brownian motion.

The correction $\tilde{u}_1$ for the stochastic volatility is obtained by writing the Feynman-Kac formula with the source term

$$H(t, \phi) = \overline{V}_2 (q(t) - \psi)^2 \frac{\partial^2 u_0}{\partial \psi^2} + \overline{V}_3 (q(t) - \psi)^3 \frac{\partial^3 u_0}{\partial \psi^3}$$

and a zero terminal condition associated to (33). This gives

$$\tilde{u}_1(t, \psi) = \overline{E} \left\{ - \int_t^T H(s, \overline{\Psi}_s) ds \mid \overline{\Psi}_t = \psi \right\}. \quad (39)$$

Adding (36) and (39), we get the corrected formula

$$(u_0 + \tilde{u}_1)(t, \psi) = \overline{E} \left\{ h(\overline{\Psi}_T - K_1) - \int_t^T H(s, \overline{\Psi}_s) ds \mid \overline{\Psi}_t = \psi \right\}. \quad (40)$$

4.4 Accuracy of the Approximation

We have shown that the inherently path dependent problem of pricing Asian option with fast mean-reverting stochastic volatility can be transformed into an European type problem. Then we use the asymptotic analysis to formally approximate the volatility dependent Asian price $P^\varepsilon(t, s, y)$ by the non-volatility dependent corrected price $(P_0 + \tilde{P}_1)(t, s)$. In the case that the payoff function h in (11) is smooth, it follows straight forwardly from [5] that the accuracy of the approximation is

$$\left|P^{\varepsilon}(t,s,y)-\left(P_0+\tilde{P}_1\right)(t,s)\right|=O(\varepsilon),$$

in the pointwise convergent sense. The proof consists in introducing the next two corrections, writting the equation satisfied by the difference

$$P^{\varepsilon}-(P_0+\sqrt{\varepsilon}P_1+\varepsilon P_2+\varepsilon\sqrt{\varepsilon}P_3),$$

and using a probabilistic representation of this difference as in (40). To study the accuracy of the approximation for the case when the payoff function h corresponds to a call or a put (13), it requires a regularization of the payoff and properties of multiple derivatives of P_0 and $\tilde{P}_1$ with respect to the current underlying stock price s. This is done in the case of the European call options in [9]. However, these properties are difficult to obtain in the case of an Asian option due to the lack of an explicit form for the density function for α_T in (38). To stay within a reasonable length we here limit ourselves to the case of regularized payoffs.

5 Numerical Computation

We have seen in section 4 that the zero order price P_0 is of the form $P_0 = S_0^{\varepsilon} u_0(t,\psi)$, where u_0 solves equation (32) with an "effective" volatility $\overline{\sigma}$. A similar PDE has been solved numerically and shown to be stable for various volatilities and other parameters by Vecer [19]. For the price correction $\tilde{P}_1 = s\tilde{u}_1$, the factor $\tilde{u}_1$ solves equation (33) which is the equation (32) with a source term. Therefore, equation (33) can be solved by using a similar numerical scheme, which is already numerically stable. However, since the asymptotic approximation involves averaging effects for the fast mean-reverting process (Y_t), the corrected price will not be valid close to the expiration date of a contract.

To illustrate with examples, we consider an arithmetic average Asian option with the fixed strike price, i.e. $K_1 = 0$. Parameters are chosen so that the effective volatility $\overline{\sigma} = 0.5$, the risk-free interest rate $r = 0.06$, the strike price $K_2 = 2$, time to maturity $\tau = 1$, stock price $S \in [1, 2.5]$, and two sets of small parameters V_2 and V_3 are chosen. Numerical results for the homogenized price $P_0(0,S)$ are shown in Figure 1. Numerical results for $\tilde{P}_1(0,S)$, the price

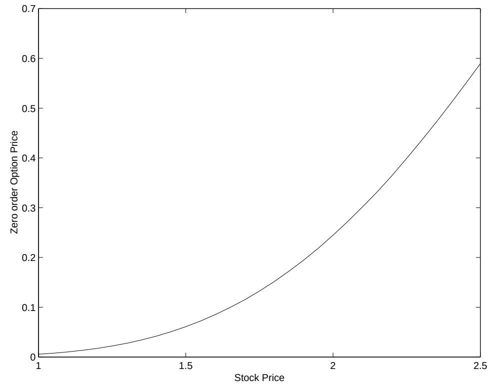

Figure 1: Finite difference numerical solution for the constant volatility price $P_0(0,S)$ of an arithmetic average Asian call option with parameters $\overline{\sigma} = 0.5, r = 0.06, K_1 = 0, K_2 = 2$, and time to maturity $T = 1$.

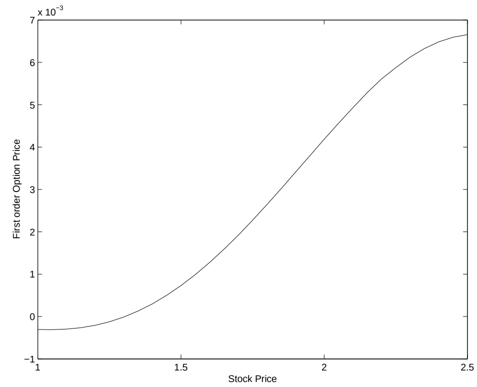

Figure 2: Finite difference numerical solution for the correction $\tilde{P}_1(0,S)$ for an arithmetic average Asian call option price with parameters $\overline{\sigma} = 0.5, r = 0.06, T = 1, V_2 = -0.01, V_3 = 0.004$. In practice the last two parameters would have been calibrated from the observed implied volatility skew.

correction with small parameters $V_2 = -0.01$ and $V_3 = 0.004$, are shown in Figure 2. The relative percentage of $\tilde{P}_1(0,S)$ to $P_0(0,S)$ is of order 10^{-2}.

Another set of results for $\tilde{P}_1(0,S)$ with small parameters $V_2 = -0.1$ and $V_3 = 0.05$ is shown in Figure 3. The relative percentage of $\tilde{P}_1(0,S)$ to $P_0(0,S)$ is of order 10^{-1}.

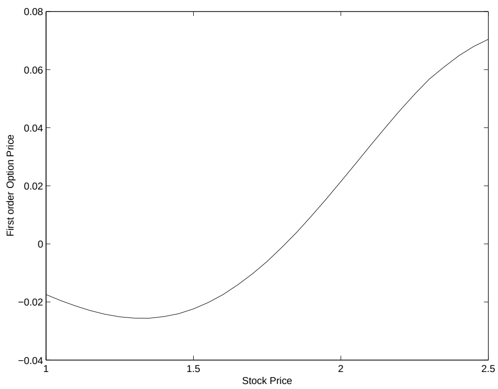

Figure 3: Finite difference numerical solution for the correction $\tilde{P}_1(0,S)$ for an arithmetic average Asian call option price with parameters $\overline{\sigma} = 0.5, r = 0.06, T = 1, V_2 = -0.1, V_3 = 0.05$. In practice the last two parameters would have been calibrated from the observed implied volatility skew.

6 Conclusion

We have shown that the dimension reduction technique introduced in [20] can be applied to stochastic volatility models for a class of arithmetic average Asian option problems. Results like shift property of the seasoned Asian option price and the Put-call Parity still hold. When the driving volatility is fast mean reverting, a singular perturbation asymptotic analysis can be applied such that the implied volatility skew can be taken into account. The mathematical justification for the accuracy of the approximation is obtained for regularized payoffs. The approximated price of Asian option is characterized by two one-dimensional PDEs (32,33). Compared to the usual two two-dimensional PDEs (34,35) derived in [5], our results reduce significantly the computational efforts. Furthermore, the main parameters $\bar{\sigma}$, $\overline{V}_2$, and $\overline{V}_3$

needed in the PDEs are estimated from the historical stock prices and the implied volatility surface. The procedure is robust and no specific model of stochastic volatility is actually needed.

7 Appendix

We look for an asymptotic solution of the form

$$u^\varepsilon(t,\psi,y) = u_0(t,\psi,y) + \sqrt{\varepsilon}u_1(t,\psi,y) + \cdots, \tag{41}$$

which solves the PDE (21). Differential operators on the left hand side of (21) can be decomposed as

$$\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1(t) + \mathcal{L}_2(t)$$

where the operators $\mathcal{L}_0, \mathcal{L}_1(t)$, and $\mathcal{L}_2(t)$ are defined by

$$\mathcal{L}_0 = \nu^2\frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y}, \tag{42}$$

$$\mathcal{L}_1(t) = \sqrt{2}\rho\nu(q(t)-\psi)f(y)\frac{\partial^2}{\partial y\partial\psi} - \sqrt{2}\nu(\Lambda(y)-\rho f(y))\frac{\partial}{\partial y}, \tag{43}$$

$$\mathcal{L}_2(t) = \frac{\partial}{\partial t} + \frac{1}{2}(q(t)-\psi)^2 f(y)^2\frac{\partial^2}{\partial\psi^2}. \tag{44}$$

Substituting (41,42,43,44) into equation (21), the expansion follows

$$\begin{aligned}&\frac{1}{\varepsilon}\mathcal{L}_0u_0 + \frac{1}{\sqrt{\varepsilon}}(\mathcal{L}_1(t)u_0 + \mathcal{L}_0u_1) + (\mathcal{L}_0u_2 + \mathcal{L}_1(t)u_1 + \mathcal{L}_2(t)u_2)\\ &+\sqrt{\varepsilon}(\mathcal{L}_2(t)u_1 + \mathcal{L}_1(t)u_1 + \mathcal{L}_0u_3) + \cdots = 0,\end{aligned} \tag{45}$$

with the terminal condition

$$u_0(T,\psi,y) + \sqrt{\varepsilon}u_1(T,\psi,y) + \cdots = (\psi - K_1)^+.$$

We obtain expression for u_0 and u_1 by successively equating the four leading order terms in (45) to zero. We let $< \cdot >$ denote the averaging with respect to the invariant distribution $N(m,\nu^2)$ of the OU process Y_t defined in (1), namely

$$< g >= \frac{1}{\nu\sqrt{2\pi}}\int g(y)e^{-(m-y)^2/2\nu^2}dy.$$

We will need to solve the Poisson equation associated with $\mathcal{L}_0$:

$$\mathcal{L}_0\chi + g = 0,$$

which requires the solvability condition

$$< g >= 0,$$

in order to admit solutions with reasonable growth at infinity. Equating terms of order $\frac{1}{\varepsilon}$, we have

$$\mathcal{L}_0 u_0 = 0.$$

By choosing u_0 independent of y, we can avoid solutions that exhibit unreasonable growth at infinity.
Equating next the order $\frac{1}{\sqrt{\varepsilon}}$ diverging term, we have

$$\mathcal{L}_0 u_1 + \mathcal{L}_1(t) u_0 = 0.$$

Since $\mathcal{L}_1(t)$ contains only terms with derivatives in y, it reduces to $\mathcal{L}_0 u_1 = 0$. Using the same argument for u_0, u_1 is chosen to be independent of y.
The order one term gives

$$\mathcal{L}_0 u_2 + \mathcal{L}_1(t) u_1 + \mathcal{L}_2(t) u_0 = 0$$

Since $\mathcal{L}_1(t) u_1 = 0$, this equation reduces to the Poisson equation in u_2:

$$\mathcal{L}_0 u_2 + \mathcal{L}_2(t) u_0 = 0.$$

Its solvability condition becomes

$$< \mathcal{L}_2(t) u_0 >=< \mathcal{L}_2(t) > u_0 = 0.$$

We denote by $\mathcal{L}_2(t;\bar{\sigma})$ the averaged differential operator $< \mathcal{L}_2(t) >$, for which $\overline{\sigma}$ is defined as $\sqrt{< f^2 >}$. Hence the leading order term u_0 solves

$$\mathcal{L}_2(t;\bar{\sigma}) u_0 = \frac{\partial u_0}{\partial t} + \frac{1}{2}(\psi - q(t))^2 \overline{\sigma}^2 \frac{\partial^2 u_0}{\partial \psi^2} = 0,$$

with the chosen terminal boundary condition

$$u_0(T, \psi, y) = (\psi - K_1)^+.$$

This is how we derive equation (32). Next, observe that the second order correction u_2 is given by

$$u_2 = -\mathcal{L}_0^{-1}(\mathcal{L}_2(t) - \mathcal{L}_2(t;\overline{\sigma}))u_0.$$

The order $\sqrt{\varepsilon}$ term gives the equation

$$\mathcal{L}_0 u_3 + \mathcal{L}_1(t)u_2 + \mathcal{L}_2(t)u_1 = 0.$$

This is again a Poisson equation in u_3, and its solvability condition reads

$$\begin{aligned} 0 &= < \mathcal{L}_2(t)u_1 + \mathcal{L}_1(t)u_2 > \\ &= \mathcal{L}_2(t;\overline{\sigma})u_1 - < \mathcal{L}_1(t)\mathcal{L}_0^{-1}(\mathcal{L}_2(t) - \mathcal{L}_2(t;\overline{\sigma})) > u_0. \end{aligned}$$

We thus derive

$$\begin{aligned} &\mathcal{L}_2(t;\overline{\sigma})u_1 \\ &= < \mathcal{L}_1(t)\mathcal{L}_0^{-1}\left(\frac{1}{2}(q(t)-\psi)^2(f(y)^2-\overline{\sigma}^2)\right) > \frac{\partial^2 u_0}{\partial \psi^2} \\ &= < \mathcal{L}_1(t)\phi(y) > \frac{1}{2}(q(t)-\psi)^2\frac{\partial^2 u_0}{\partial \psi^2}, \end{aligned} \quad (46)$$

with a zero terminal condition, and where the function ϕ solves the Poisson equation

$$\mathcal{L}_0\phi(y) = f(y)^2 - \overline{\sigma}^2.$$

Using (43) one can compute the differential operator in ψ,

$$\begin{aligned} &< \mathcal{L}_1(t)\phi(y) > \\ &= \sqrt{2}\rho\nu < f(y)\phi^{'}(y) > (q(t)-\psi)\frac{\partial}{\partial \psi} - \sqrt{2}\nu(< \Lambda(y)\phi'(y) > -\rho < f(y)\phi^{'}(y) >). \end{aligned}$$

Finally, we derive the PDE (33) for $\tilde{u}_1 = \sqrt{\varepsilon}\, u_1$:

$$\mathcal{L}_2(t;\overline{\sigma})\tilde{u}_1 = \overline{V}_2(q(t)-\psi)^2\frac{\partial^2 u_0}{\partial \psi^2} + \overline{V}_3(q(t)-\psi)^3\frac{\partial^3 u_0}{\partial \psi^3}$$

with a zero terminal condition, and where

$$\overline{V}_2 = \frac{\nu\sqrt{\varepsilon}}{\sqrt{2}}(-\rho < f(y)\phi'(y) > - < \Lambda(y)\phi'(y) >), \quad (47)$$

and

$$\overline{V}_3 = \frac{\rho\nu\sqrt{\varepsilon}}{\sqrt{2}} < f(y)\phi'(y) > . \quad (48)$$

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