Stochastic Volatility and Epsilon-Martingale Decomposition

Jean-Pierre Fouque, George Papanicolaou and Ronnie Sircar

Abstract. We address the problems of pricing and hedging derivative securities in an environment of uncertain and changing market volatility. We show that when volatility is stochastic but fast mean reverting Black-Scholes pricing theory can be corrected. The correction accounts for the effect of stochastic volatility and the associated market price of risk. For European derivatives it is given by explicit formulas which involve parsimonious parameters directly calibrated from the implied volatility surface. The method presented here is based on a martingale decomposition result which enables us to treat non-Markovian models as well.

1. Stochastic Volatility Models

We consider stochastic volatility models where the asset price $(X_t)_{t\geq 0}$ satisfies the stochastic differential equation

$$dX_t = \mu X_t dt + \sigma_t X_t dW_t,$$

and $(\sigma_t)_{t\geq 0}$ is called the volatility process. It must satisfy some regularity conditions for the model to be well-defined, but it does not have to be an Itô process: it can be a jump process, a Markov chain, etc. In order for it to be a volatility, it should be positive. Unlike the implied deterministic volatility models for which the volatility is a deterministic function $\sigma(t, X_t)$ of time and price, the volatility process is *not* perfectly correlated with the Brownian motion (W_t). Therefore, volatility is modeled to have an independent random component and since σ_t is not the price of a traded asset, the market is incomplete and there is no longer a unique equivalent martingale measure. We refer to [6], [5] or [3](Ch.2) for reviews of stochastic volatility models.

1.1. Mean-Reverting Stochastic Volatility Models

We consider first volatility processes which are Itô processes satisfying stochastic differential equations driven by a second Brownian motion. This is a convenient way to incorporate correlation with stock price changes.

One feature that most volatility models seem to like is *mean-reversion*. The term mean-reverting refers to the characteristic (typical) time it takes for a process to get back to the mean-level of its invariant distribution (the long-run distribution of the process). In other words we assume that σ_t is *ergodic* with additional mixing properties. From a financial modeling perspective, mean-reverting refers to a linear pull-back term in the drift of the volatility process itself, or in the drift of some (underlying) process of which volatility is a function. Let us denote $\sigma_t = f(Y_t)$ where f is some positive function. Then mean-reverting stochastic volatility means that the stochastic differential equation for (Y_t) looks like

$$dY_t = \alpha(m - Y_t)dt + \beta d\hat{Z}_t,$$

where $(\hat{Z}_t)_{t\geq 0}$ is a Brownian motion correlated with (W_t). Here α is called the *rate of mean-reversion* and m is the long-run mean-level of Y. The drift term pulls Y towards m and consequently we would expect that σ_t is pulled towards the mean value of $f(Y)$, with respect to the long-run distribution of Y.

Choosing $\beta > 0$ constant corresponds to the Ornstein-Uhlenbeck process which is a Gaussian process with the normal invariant distribution $\mathcal{N}(m, \beta^2/2\alpha)$. This choice, though not necessary, is particularly convenient to explain the concept of fast mean-reversion and to show through relatively explicit computations how to exploit this property in pricing and hedging problems. It is still very flexible since the function f is unspecified.

The second Brownian motion $(\hat{Z}_t)$ is correlated with the Brownian motion (W_t) driving the asset price equation. We denote by $\rho \in [-1, 1]$ the instantaneous correlation coefficient defined by

$$d\langle W, \hat{Z}\rangle_t = \rho dt.$$

It is also convenient to write

$$\hat{Z}_t = \rho W_t + \sqrt{1-\rho^2} Z_t,$$

where (Z_t) is a standard Brownian motion independent of (W_t). It is often found from financial data that $\rho < 0$, and there are economic arguments for a negative correlation or *leverage effect* between stock price and volatility shocks. From common experience and empirical studies, when volatility goes up, asset prices tend to go down. In general, the correlation may depend on time $\rho(t) \in [-1, 1]$, but we shall assume it a constant for notational simplicity and because, in most practical situations, it is taken to be such.

1.2. Pricing with Equivalent Martingale Measures

Because of the additional source of randomness in the volatility process, contingent claims cannot, in general, be replicated by self-financing portfolios made of stocks and riskfree bonds for which we assume a constant interest rate r for simplicity. There is no uniqueness of no-arbitrage prices. We take the point of view that the market is choosing one equivalent martingale measure to determine prices of derivatives. This translates into the introduction of a *market price of volatility risk*

γ_t, an adapted process such that

$$\frac{d\mathbb{P}^{\star(\gamma)}}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\int_0^T (\theta_s^2+\gamma_s^2)ds - \int_0^T \theta_s dW_s - \int_0^T \gamma_s dZ_s\right),$$

defines an equivalent probability $\mathbb{P}^{\star(\gamma)}$ with $\theta_t = (\mu - r)/f(Y_t)$. By Girsanov's theorem

$$W_t^\star = W_t + \int_0^t \frac{(\mu-r)}{f(Y_s)}ds \ , \ \ Z_t^\star = Z_t + \int_0^t \gamma_s ds,$$

are two independent Brownian motions under $\mathbb{P}^{\star(\gamma)}$. We assume that $\gamma_t = \gamma(Y_t)$ is a function of Y_t only and our model under the equivalent martingale measure $\mathbb{P}^{\star(\gamma)}$ becomes

$$\begin{aligned} dX_t &= rX_t dt + f(Y_t)X_t dW_t^\star, \\ dY_t &= \left[\alpha(m-Y_t) - \beta\Lambda(Y_t)\right]dt + \beta d\hat{Z}_t^\star, \end{aligned}$$

where $\hat{Z}_t^\star = \rho W_t^\star + \sqrt{1-\rho^2}\, Z_t^\star$ and $\Lambda(y) = \rho\frac{(\mu-r)}{f(y)} + \gamma(y)\sqrt{1-\rho^2}$ is a combined market price of risk.

Denoting by $\mathbb{E}^{\star(\gamma)}$ the expectation with respect to $\mathbb{P}^{\star(\gamma)}$, derivatives with time T payoff H are then priced by using the formula

$$V_t = \mathbb{E}^{\star(\gamma)}\{e^{-r(T-t)}H|\mathcal{F}_t\},$$

for all $t \leq T$, excluding arbitrage opportunities. The filtration $(\mathcal{F}_t)$ is generated by (W, Z) or $(W^\star, Z^\star)$. Even though, in these Markovian models, derivatives prices can be obtained as solutions of PDE's through the Feynman-Kac formula, we will not use this point of view. The martingale approach developed here has the advantage to generalize naturally to non-Markovian models.

1.3. Fast Mean-Reversion

We consider the "regime" where the rate of mean-reversion α is large while the variance of the invariant distribution of Y_t remains of order one ($\nu^2 = \beta^2/2\alpha$ in the OU case). In other words the volatility clock is running faster than typical maturities of order one unit of time (the year for instance). Empirical evidence for this regime in the S&P500 are given in [3](Ch.4) and in the forthcoming detailed analysis [4]. It is mathematically convenient to introduce the small parameter

$$\varepsilon = 1/\alpha\,,$$

and to rescale β accordingly to $\beta = \sqrt{2}\,\nu/\sqrt{\varepsilon}$ where ν does not vary with ε. With this notation our model under the pricing equivalent martingale measure becomes

$$\begin{aligned} dX_t^\varepsilon &= rX_t^\varepsilon dt + f(Y_t^\varepsilon)X_t^\varepsilon dW_t^\star, \\ dY_t^\varepsilon &= \left[\frac{1}{\varepsilon}(m-Y_t^\varepsilon) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(Y_t^\varepsilon)\right]dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}d\hat{Z}_t^\star, \end{aligned}$$

where we write $(X^\varepsilon, Y^\varepsilon)$ to indicate explicitly the dependence upon ε.

2. Asymptotic Pricing

We consider a European derivative given by its nonnegative payoff function $h(x)$ and its maturity time T. We assume that h is a smooth function in order to avoid technicalities in explaining the principle of our asymptotic method first introduced in [3](Ch.10). The nonsmooth case is more technical and not presented here. The price $P^\varepsilon(t)$ of this derivative at time $t < T$ is given by

$$P^\varepsilon(t) = I\!E^{\star(\gamma)}\{e^{-r(T-t)}h(X_T^\varepsilon)|\mathcal{F}_t\},$$

where the conditional expectation is with respect to the filtration $(\mathcal{F}_t)$ of the two Brownian motions. Instead of characterizing this price as a function of the current values (x, y) of $(X_t^\varepsilon, Y_t^\varepsilon)$ satisfying a two-dimensional PDE, we characterize $P^\varepsilon(t)$ by the fact that the process M^ε defined by

$$M_t^\varepsilon = e^{-rt}P^\varepsilon(t) = I\!E^{\star(\gamma)}\{e^{-rT}h(X_T^\varepsilon)|\mathcal{F}_t\},$$

is a martingale with a terminal value given by

$$M_T^\varepsilon = e^{-rT}h(X_T^\varepsilon).$$

Our goal is to show that $P^\varepsilon(t)$ can be approximated up to the order $\mathcal{O}(\varepsilon)$ by $Q^\varepsilon(t, X_t^\varepsilon)$ where $Q^\varepsilon(t, x)$ is a function, independent of y, to be determined.

2.1. The Epsilon-Martingale Decomposition Argument

For a function $Q^\varepsilon(t, x)$ to be determined which may depend on ε but not on y, we consider the process N^ε defined by

$$N_t^\varepsilon = e^{-rt}Q^\varepsilon(t, X_t^\varepsilon).$$

Requiring that the function Q^ε satisfies $Q^\varepsilon(T, x) = h(x)$ at the final time T, we have

$$M_T^\varepsilon = N_T^\varepsilon.$$

The method consists in finding $Q^\varepsilon(t, x)$ such that N^ε can be **decomposed** as

$$N_t^\varepsilon = \widetilde{M_t^\varepsilon} + R_t^\varepsilon,$$

where $\widetilde{M^\varepsilon}$ is a martingale and R_t^ε is of order ε. Observe that in this **decomposition** the terms of order $\sqrt{\varepsilon}$ are absorbed in the martingale part.

Supposing that this has been established, by taking a conditional expectation with respect to $\mathcal{F}_t$ on both sides of the equality

$$N_T^\varepsilon = \widetilde{M_T^\varepsilon} + R_T^\varepsilon,$$

and using the martingale property of $\widetilde{M^\varepsilon}$, one obtains

$$I\!E^\star\{N_T^\varepsilon|\mathcal{F}_t\} = \widetilde{M_t^\varepsilon} + I\!E^\star\{R_T^\varepsilon|\mathcal{F}_t\}.$$

From the terminal condition $M_T^\varepsilon = N_T^\varepsilon$ and the martingale property of M^ε we deduce that the left hand side is also equal to M_t^ε. From the decomposition of N_t^ε we have $\widetilde{M_t^\varepsilon} = N_t^\varepsilon - R_t^\varepsilon$ and therefore

$$M_t^\varepsilon = N_t^\varepsilon + \mathbb{E}^\star\{R_T^\varepsilon|\mathcal{F}_t\} - R_t^\varepsilon\,.$$

Multiplying by e^{rt} and using the definitions of M_t^ε and N_t^ε one deduces

$$P^\varepsilon(t) = Q^\varepsilon(t, X_t^\varepsilon) + \mathcal{O}(\varepsilon)\,,$$

which is the desired **approximation result**. Indeed it remains to determine Q^ε leading to the epsilon-decomposition of N^ε.

2.2. Decomposition Result

Assuming a priori sufficient smoothness of the function Q^ε we write

$$\begin{aligned} dN_t^\varepsilon &= d\left(e^{-rt}Q^\varepsilon(t, X_t^\varepsilon)\right) \\ &= e^{-rt}\left(\frac{\partial}{\partial t} + \frac{1}{2}f(Y_t^\varepsilon)^2(X_t^\varepsilon)^2\frac{\partial^2}{\partial x^2} + rX_t^\varepsilon\frac{\partial}{\partial x} - r\right)Q^\varepsilon(t, X_t^\varepsilon)dt \\ &+ e^{-rt}\left(\frac{\partial Q^\varepsilon}{\partial x}(t, X_t^\varepsilon)\right)f(Y_t^\varepsilon)X_t^\varepsilon dW_t^\star\,. \end{aligned}$$

The method consists in cancelling the bounded variation terms as much as possible. The first obvious step in that direction is to replace the volatility $f(Y_t^\varepsilon)$ by a constant volatility and to consider $Q^\varepsilon(t, x)$ as a perturbation of the corresponding Black-Scholes pricing function. A natural choice of a constant volatility is given by averaging f^2 with respect to the invariant distribution of Y^ε. Denoting this averaging by $\langle\cdot\rangle_\varepsilon$ we define

$$\bar{\sigma}_\varepsilon^2 = \langle f^2\rangle_\varepsilon\,.$$

In our example, Y^ε is a perturbed Ornstein-Uhlenbeck process and its invariant distribution admits the density

$$J_\varepsilon \exp\left\{-\frac{(y-m)^2}{2\nu^2} - \frac{\sqrt{2\varepsilon}}{\nu}\widetilde{\Lambda}(y)\right\}\,,$$

where $\widetilde{\Lambda}$ is an antiderivative of the market price of risk Λ and J_ε is the appropriate normalizing constant. Introducing the usual Black-Scholes operator

$$\mathcal{L}_{BS}(\sigma) = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right)\,,$$

we have

$$\begin{aligned} dN_t^\varepsilon &= e^{-rt}\left(\mathcal{L}_{BS}(\bar{\sigma}_\varepsilon) + \frac{1}{2}\left(f(Y_t^\varepsilon)^2 - \bar{\sigma}_\varepsilon^2\right)(X_t^\varepsilon)^2\frac{\partial^2}{\partial x^2}\right)Q^\varepsilon(t, X_t^\varepsilon)dt \\ &+ e^{-rt}\left(\frac{\partial Q^\varepsilon}{\partial x}(t, X_t^\varepsilon)\right)f(Y_t^\varepsilon)X_t^\varepsilon dW_t^\star\,. \end{aligned}$$

Setting $Q^\varepsilon = P_0^\varepsilon + \widetilde{Q_1^\varepsilon}$ where P_0^ε is the the solution of the Black-Scholes equation $\mathcal{L}_{BS}(\bar\sigma_\varepsilon)P_0^\varepsilon = 0$ with the terminal condition $P_0^\varepsilon(T,x) = h(x)$, we deduce

$$\begin{aligned} dN_t^\varepsilon &= e^{-rt}\left(\mathcal{L}_{BS}(\bar\sigma_\varepsilon)\widetilde{Q_1^\varepsilon}(t,X_t^\varepsilon) + \frac{1}{2}\left(f(Y_t^\varepsilon)^2 - \bar\sigma_\varepsilon^2\right)(X_t^\varepsilon)^2\,\frac{\partial^2 Q^\varepsilon}{\partial x^2}(t,X_t^\varepsilon)\right)dt \\ &+ e^{-rt}\left(\frac{\partial Q^\varepsilon}{\partial x}(t,X_t^\varepsilon)\right) f(Y_t^\varepsilon)X_t^\varepsilon dW_t^\star \,. \end{aligned}$$

The second term will be small of order $\mathcal{O}(\sqrt{\varepsilon})$ by a central limit type argument and $\widetilde{Q_1^\varepsilon}$ will be chosen to combine with it into a term of order $\mathcal{O}(\varepsilon)$ and a martingale term. For clarity we do that first by using the Markov property of Y^ε.

2.2.1. Markovian Case. We denote by $\varepsilon^{-1}\mathcal{L}_0$ (resp. $\varepsilon^{-1}\mathcal{L}_0^\varepsilon$) the infinitesimal generator of the unperturbed (resp. perturbed) Markov process Y^ε. In the particular case of an OU process one has:

$$\begin{aligned} \mathcal{L}_0 &= (m-y)\frac{\partial}{\partial y} + \nu^2\frac{\partial^2}{\partial y^2}\,, \\ \mathcal{L}_0^\varepsilon &= \mathcal{L}_0 - \nu\sqrt{2\varepsilon}\,\Lambda(y)\frac{\partial}{\partial y} \\ &= (m - y - \nu\sqrt{2\varepsilon}\,\Lambda(y))\frac{\partial}{\partial y} + \nu^2\frac{\partial^2}{\partial y^2}\,. \end{aligned}$$

We then consider a solution ϕ_ε of the Poisson equation

$$\mathcal{L}_0^\varepsilon\phi_\varepsilon(y) = f(y)^2 - \bar\sigma_\varepsilon^2\,,$$

which we assume to be well defined (up to an additive constant) and to have a bounded derivative as in the OU case with a bounded function f.

We then deduce from Itô's formula that

$$\left(f(Y_t^\varepsilon)^2 - \bar\sigma_\varepsilon^2\right)dt = \varepsilon\left\{d(\phi_\varepsilon(Y_t^\varepsilon)) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\,\phi_\varepsilon'(Y_t^\varepsilon)d\hat{Z}_t^\star\right\}\,,$$

leading to

$$\begin{aligned} dN_t^\varepsilon &= e^{-rt}\left(\mathcal{L}_{BS}(\bar\sigma_\varepsilon)\widetilde{Q_1^\varepsilon}(t,X_t^\varepsilon)dt + \frac{1}{2}\varepsilon(X_t^\varepsilon)^2\,\frac{\partial^2 Q^\varepsilon}{\partial x^2}(t,X_t^\varepsilon)d(\phi_\varepsilon(Y_t^\varepsilon))\right) \\ &+ e^{-rt}\left(\frac{\partial Q^\varepsilon}{\partial x}(t,X_t^\varepsilon)\right) f(Y_t^\varepsilon)X_t^\varepsilon dW_t^\star \\ &- \frac{\nu\sqrt{\varepsilon}}{\sqrt{2}}e^{-rt}\left((X_t^\varepsilon)^2\,\frac{\partial^2 Q^\varepsilon}{\partial x^2}(t,X_t^\varepsilon)\right)\phi_\varepsilon'(Y_t^\varepsilon)d\hat{Z}_t^\star\,. \end{aligned}$$

The second term is computed by using the integration by parts formula:

$$\begin{aligned} (X_t^\varepsilon)^2\frac{\partial^2 Q^\varepsilon}{\partial x^2}d\left(\phi_\varepsilon(Y_t^\varepsilon)\right) &= d\left((X_t^\varepsilon)^2\frac{\partial^2 Q^\varepsilon}{\partial x^2}\phi_\varepsilon(Y_t^\varepsilon)\right) - \phi_\varepsilon(Y_t^\varepsilon)d\left((X_t^\varepsilon)^2\frac{\partial^2 Q^\varepsilon}{\partial x^2}\right) \\ &- d\left\langle (X^\varepsilon)^2\frac{\partial^2 Q^\varepsilon}{\partial x^2}, \phi_\varepsilon(Y^\varepsilon)\right\rangle_t\,, \end{aligned}$$

where the covariation term is given by

$$d\left\langle (X^\varepsilon)^2 \frac{\partial^2 Q^\varepsilon}{\partial x^2}, \phi_\varepsilon(Y^\varepsilon) \right\rangle_t = \frac{\nu\rho\sqrt{2}}{\sqrt{\varepsilon}}\, \phi'_\varepsilon(Y_t^\varepsilon) \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 Q^\varepsilon}{\partial x^2}\right)(t, X_t^\varepsilon) f(Y_t^\varepsilon) X_t^\varepsilon dt\,.$$

Collecting the martingale terms and the bounded variation terms of order $\mathcal{O}(\varepsilon)$, we get

$$\begin{aligned} N_t^\varepsilon &= N_0^\varepsilon + \text{Martingale} \\ &+ \int_0^t e^{-rs}\left(\mathcal{L}_{BS}(\bar\sigma_\varepsilon)\widetilde{Q_1^\varepsilon}(s, X_s^\varepsilon) - \frac{\nu\rho\sqrt{\varepsilon}}{\sqrt{2}}\, \phi'_\varepsilon(Y_s^\varepsilon) f(Y_s^\varepsilon) X_s^\varepsilon \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_0^\varepsilon}{\partial x^2}\right)(t, X_s^\varepsilon)\right) ds \\ &+ \mathcal{O}(\varepsilon)\,. \end{aligned}$$

We choose now $\widetilde{Q_1^\varepsilon}$ such that the quantity to be integrated in time is centered with respect to the invariant distribution of Y^ε. Repeating the argument above, one can show that this integral can be replaced by the sum of a martingale and a term of order $\mathcal{O}(\varepsilon)$. The centering involves the third derivative of the Black-Scholes price P_0^ε. Introducing the small constant

$$V_3^\varepsilon = \frac{\rho\nu\sqrt{\varepsilon}}{\sqrt{2}} \langle f\phi'_\varepsilon \rangle_\varepsilon\ ,$$

and defining the source function

$$H^\varepsilon(t,x) = V_3^\varepsilon\, x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2 P_0^\varepsilon}{\partial x^2}\right) = V_3^\varepsilon \left(2x^2 \frac{\partial^2 P_0^\varepsilon}{\partial x^2} + x^3 \frac{\partial^3 P_0^\varepsilon}{\partial x^3}\right)\ ,$$

we choose $\widetilde{Q_1^\varepsilon}(t,x)$ to be the solution of the Black-Scholes equation

$$\mathcal{L}_{BS}(\bar\sigma_\varepsilon)\widetilde{Q_1^\varepsilon} = H^\varepsilon\ ,$$

with the zero terminal condition $\widetilde{Q_1^\varepsilon}(T,x) = 0$. The desired decomposition follows

$$N_t^\varepsilon = \widetilde{M_t^\varepsilon} + R_t^\varepsilon\ ,$$

where $\widetilde{M^\varepsilon}$ is a martingale and R^ε is small of order $\mathcal{O}(\varepsilon)$. Note that $Q^\varepsilon = P_0^\varepsilon + \widetilde{Q_1^\varepsilon}$ do not depend on y and satisfies $Q^\varepsilon(T,x) = h(x)$. The correction $\widetilde{Q_1^\varepsilon}$ is of order $\mathcal{O}(\sqrt{\varepsilon})$ and $P^\varepsilon(t) = Q^\varepsilon(t,x) + \mathcal{O}(\varepsilon)$. Note also that in the less interesting uncorrelated case $\rho = 0$, the correction $\widetilde{Q_1^\varepsilon}$ is simply zero and the approximated price is the Black-Scholes price P_0^ε with the modified volatility $\bar\sigma_\varepsilon$.

2.2.2. Non-Markovian Case. We briefly indicate how to obtain the decomposition of N^ε in a non-Markovian stochastic volatility environment. We assume that Y^ε has an invariant distribution and is strongly mixing with a fast exponential rate of mixing denoted by α. We again set $\varepsilon = 1/\alpha$ and we assume that the variance of the invariant distribution of the unperturbed process Y denoted by ν^2 is independent of ε. We also suppose that Y has a diffusion part driven by a

Brownian motion correlated to W through the constant ρ as in the Markovian case.

The decomposition result is then obtained as in the Markovian case except that we cannot define the "corrector" ϕ_ε by using the Poisson equation involving an infinitesimal generator. Instead $\phi_\varepsilon(Y_t^\varepsilon)$ is replaced by the random quantity $\widetilde{\phi_\varepsilon}(t)$ defined by the *conditional shift*

$$\widetilde{\phi_\varepsilon}(t) = -\frac{1}{\varepsilon} I\!E^\star \left\{ \int_t^T \left(f(Y_s^\varepsilon)^2 - \bar{\sigma}_\varepsilon^2 \right) ds | \mathcal{F}_t^\varepsilon \right\} .$$

It is very similar to the way the solutions of the Poisson equations are constructed in the Markovian case since the simple change of variable $u = s/\varepsilon$ shows that it is comparable to

$$\phi_\varepsilon(y) = -\int_0^{+\infty} I\!E^\star \left\{ \left(f(Y_t)^2 - \bar{\sigma}_\varepsilon^2 \right) | Y_0 = y \right\} dt \,,$$

used in the Markovian case. The method of conditional shift (or pseudo-generator) is treated in [7] for instance. The rest of the proof is very similar to the Markovian case once we observe that

$$\mathcal{M}_t^\varepsilon = \widetilde{\phi_\varepsilon}(t) - \frac{1}{\varepsilon} \int_0^t \left(f(Y_s^\varepsilon)^2 - \bar{\sigma}_\varepsilon^2 \right) ds \,,$$

is a martingale. Defining $\psi_\varepsilon(t)$ by

$$\frac{\sqrt{\varepsilon}}{2} \, d \left\langle W^\star, \mathcal{M}^\varepsilon \right\rangle_t = \psi_\varepsilon(t) dt \,,$$

so that it plays the role of $\frac{\nu\rho}{\sqrt{2}} \phi'(Y_t^\varepsilon)$ in the Markovian case, the small constant V_3^ε is given by the averaging

$$V_3^\varepsilon = \langle f \psi_\varepsilon \rangle_\varepsilon \,.$$

The source function $H^\varepsilon(t, x)$ and the correction $\widetilde{Q_1^\varepsilon}(t, x)$ are then defined exactly as in the Markovian case and the same conlusion follows: $\widetilde{Q_1^\varepsilon}$ is of order $\mathcal{O}(\sqrt{\varepsilon})$ and $P^\varepsilon(t) = P_0^\varepsilon(t, x) + \widetilde{Q_1^\varepsilon}(t, x) + \mathcal{O}(\varepsilon)$.

3. Practical Form of the Approximated Price

The computation of the approximated price $P_0^\varepsilon + \widetilde{Q_1^\varepsilon}$ requires the two parameters $\bar{\sigma}_\varepsilon$ and V_3^ε in order to compute first the Black-Scholes price P_0^ε and then deduce the correction $\widetilde{Q_1^\varepsilon}$. In fact it is easily shown, [3](Ch.5), that

$$\widetilde{Q_1^\varepsilon} = -(T-t) H^\varepsilon = -(T-t) V_3^\varepsilon \, x \frac{\partial}{\partial x} \left(x^2 \frac{\partial^2 P_0^\varepsilon}{\partial x^2} \right) .$$

The mean square volatility $\bar{\sigma}_\varepsilon^2$ would be easy to estimate except for the fact that we do not observe stock returns under the risk neutral probability $I\!P^{\star(\gamma)}$ but rather under the subjective probability $I\!P$ which does not involve the market price of volatility risk γ. For practical purpose we need to rewrite our approximation formula with a different set of parameters.

3.1. Perturbed Invariant Distribution

Denoting by $\langle\cdot\rangle$ the average with respect to the invariant distribution of the unperturbed Y process, we have the following expansion of the invariant distribution of the perturbed Y^ε process

$$\langle\cdot\rangle_\varepsilon = \langle\cdot\rangle - \frac{\sqrt{2\varepsilon}}{\nu}\left\langle\widetilde{\Lambda}(\cdot - \langle\cdot\rangle)\right\rangle + \mathcal{O}(\varepsilon)\,,$$

where $\widetilde{\Lambda}$ is an antiderivative of the combined market price of risk Λ. Defining $\bar{\sigma}^2 = \langle f^2\rangle$, it follows that

$$\begin{aligned}\mathcal{L}_{BS}(\bar{\sigma}_\varepsilon) &= \mathcal{L}_{BS}(\bar{\sigma}) + \frac{1}{2}\left(\langle f^2\rangle_\varepsilon - \langle f^2\rangle\right)x^2\frac{\partial^2}{\partial x^2} \\ &= \mathcal{L}_{BS}(\bar{\sigma}) - \frac{\sqrt{\varepsilon}}{\nu\sqrt{2}}\langle\widetilde{\Lambda}(f^2 - \langle f^2\rangle)\rangle x^2\frac{\partial^2}{\partial x^2} + \mathcal{O}(\varepsilon)\,.\end{aligned}$$

Introducing $P_0(t,x)$, the Black-Scholes solution with constant voaltility $\bar{\sigma}$, one can easily write

$$\widetilde{P_1} = -(T-t)\left(V_2x^2\frac{\partial^2 P_0}{\partial x^2} + V_3x^3\frac{\partial^3 P_0}{\partial x^3}\right)\,,$$

for two parameters V_2 and V_3, small of order $\mathcal{O}(\sqrt{\varepsilon})$, and such that the approximation $P^\varepsilon(t) = P_0(t,x) + \widetilde{P_1}(t,x) + \mathcal{O}(\varepsilon)$ holds. This is the form which arises naturally when performing the asymptotics on the pricing PDE in the Markovian case as described first in [1] and detailed in [3](Ch.5).

3.2. Calibration

Without going into details we see that, in order to use this formula, we need the three parameters $(\bar{\sigma}, V_2, V_3)$. The first one is easily estimated on historical returns data. The two others, the V's, can be obtained by using observed call option prices or equivalently by fitting the term structure of implied volatility. Explicit formulas are given in [2] and, in [3](Ch.6) with a stability analysis.

4. Hedging

To stay within a reasonable length we simply recall that, in presence of stochastic volatility, a perfect self-financing hedge of a derivative, with stocks and bonds only, is not possible in general. When volatility is fast mean-reverting as described in this paper, it is possible to correct a pure Black-Scholes hedge ratio in order to reduce the bias introduced in the cost of the hedging strategy. This hedging correction gives a *mean self-financing strategy* up to order $\mathcal{O}(\varepsilon)$. It is obtained by the method described in Section 2.2 but performed under the objective probability $I\!P$. We refer to [3](Ch.7) for details and formulas.

References

[1] J.P. Fouque, G. Papanicolaou and R. Sircar, *Asymptotics of a Two-Scale Stochastic Volatility Model,* Equations aux dérivées partielles et applications, Articles dédiés à Jacques-Louis Lions, Gauthier-Villars, Paris (1998) 517–526.

[2] J.P. Fouque, G. Papanicolaou and R. Sircar, *Mean-Reverting Stochastic Volatility,* International Journal of Theoretical and Applied Finance, **3**(1) (2000), 101–142.

[3] J.P. Fouque, G. Papanicolaou and R. Sircar, *Derivatives in Financial Markets with Stochastic Volatility,* Cambridge University Press, 2000.

[4] J.P. Fouque, G. Papanicolaou, R. Sircar and K. Solna, *Mean Reversion of S&P 500 Volatility,* preprint (2000).

[5] R. Frey, *Derivative Asset Analysis in Models with Level-Dependent and Stochastic Volatility,* CWI Quarterly, **10**(1) (1996), 1–34.

[6] E. Ghysels, A. Harvey and E. Renault, *Stochastic Volatility,* Statistical Methods in Finance. Handbook of Statistics, **14**(Ch.5) (1996) 119–191.

[7] A. Kushner, *Approximation and Weak Convergence Methods for random Processes,* MIT Press, 1984.

Jean-Pierre Fouque,
Department of Mathematics,
North Carolina State University,
Raleigh NC 27695-8205, USA
Partially supported by NSF/DMS-0071744
E-mail address: fouque@math.ncsu.edu

George Papanicolaou,
Department of Mathematics,
Stanford University,
Stanford CA 94305, USA
E-mail address: papanico@math.stanford.edu

Ronnie Sircar,
Department of Operations Research & Financial Engineering,
Princeton University, E-Quad
Princeton NJ 08544, USA
Partially supported by NSF/DMS-0090067
E-mail address: sircar@princeton.edu