MULTISCALE STOCHASTIC VOLATILITY ASYMPTOTICS

JEAN-PIERRE FOUQUE*, GEORGE PAPANICOLAOU†, RONNIE SIRCAR‡, AND KNUT SOLNA§

Abstract. In this paper we propose to use a combination of regular and singular perturbations to analyze parabolic PDEs that arise in the context of pricing options when the volatility is a stochastic process that varies on several characteristic time scales. The classical Black-Scholes formula gives the price of call options when the underlying is a geometric Brownian motion with a constant volatility. The underlying might be the price of a stock or an index say and a constant volatility corresponds to a fixed standard deviation for the random fluctuations in the returns of the underlying. Modern market phenomena makes it important to analyze the situation when this volatility is not fixed but rather is heterogeneous and varies with time. In previous work, see for instance [5], we considered the situation when the volatility is fast mean reverting. Using a singular perturbation expansion we derived an approximation for option prices. We also provided a calibration method using observed option prices as represented by the so-called term structure of implied volatility. Our analysis of market data, however, shows the need for introducing also a slowly varying factor in the model for the stochastic volatility. The combination of regular and singular perturbations approach that we set forth in this paper deals with this case. The resulting approximation is still independent of the particular details of the volatility model and gives more flexibility in the parametrization of the implied volatility surface. In particular, the introduction of the slow factor gives a much better fit for options with longer maturities. We use option data to illustrate our results and show how exotic option prices also can be approximated using our multiscale perturbation approach.

1. Introduction.

No-arbitrage prices of options written on a risky asset are mathematical expectations of present values of the payoffs of these contracts. These expectations are in fact computed with respect to one of the so-called risk-neutral probability measures, under which the discounted price of the underlying asset is a martingale. In a Markovian context these expectations, as functions of time, the current value of the underlying asset and the volatility level, are solutions of parabolic PDE's with final conditions at maturity times. These conditions are given by the contracts payoffs, and various boundary conditions are imposed depending on the nature of the contracts.

In [5] we considered a class of models where volatility is a mean-reverting diffusion with an intrinsic fast time-scale, i.e. a process which decorrelates rapidly and fluctuates on a fine time-scale. Using a singular perturbation technique on the pricing PDE, we were able to show that the option price is in fact a perturbation of the Black-Scholes price with an effective constant volatility. Moreover we derived a simple explicit expression for the first correction in the singular perturbation expansion. We have shown that this correction is universal in this class of models and that it involves two effective parameters which can easily be calibrated by using prices of liquid call options represented by the implied volatility surface.

In this paper we introduce a class of *multiscale* stochastic volatility models. More precisely we consider volatility processes which are driven by two diffusions, one fluctuating on a fast time-scale as in [5], and the other fluctuating on a slow time-scale

*Department of Mathematics, NC State University, Raleigh NC 27695-8205, *fouque@math.ncsu.edu*. Work partially supported by NSF grant DMS-0071744.
†Department of Mathematics, Stanford University, Stanford CA 94305, *papanico@math.stanford.edu*.
‡Department of Operations Research & Financial Engineering, Princeton University, E-Quad, Princeton, NJ 08544, *sircar@princeton.edu*. Work supported by NSF grant DMS-0090067.
§Department of Mathematics, University of California, Irvine CA 92697, *ksolna@math.uci.edu*.

or in other words a slowly varying diffusion process. We show that it is possible to combine a *singular perturbation expansion* with respect to the fast scale, with a *regular perturbation expansion* with respect to the slow scale. This again leads to a leading order term which is the Black-Scholes price with a constant effective volatility. The first correction is now made up of two parts which derive respectively from the fast and the slow factors, and involves four parameters which still can be easily calibrated from the implied volatility surface. We show with options data that the addition of the slow factor to the model greatly improves the fit to the longer maturities.

The paper is organized as follows: in Section 2 we introduce the class of stochastic volatility models that we consider and we discuss the concepts of fast and slow time scales. This is done under the physical measure which describes the actual evolution of the asset price. In this section we also rewrite the model under the risk-neutral pricing measure which now involves two market prices of volatility risk. In Section 2.4 we write down the pricing parabolic PDE which characterizes the option price $P(t, x, y, z)$ as a function of the present time t, the value x of the underlying asset, and the levels (y, z) of the two volatility driving processes. For a European option the final condition is of the form $P(T, x, y, z) = h(x)$. In Section 3 we carry out the asymptotic analysis in the regime of fast and slow time scales. We use a combination of singular and regular perturbations to derive the leading order term and the first corrections associated with the fast and slow factors. These corrections are nicely interpreted in terms of the *Greeks* (or sensitivities) of the leading order Black-Scholes price. The accuracy of this approximation is given in Theorem 3.6, the main result of this section. The proof is a generalization of the one presented in [8] where only the fast scale factor was considered. In Section 4 we recall the concept of implied volatility and we deduce its expansion in the regime of fast and slow volatility factors. This leads to a simple and accurate parametrization of the implied volatility surface. It involves four parameters which can be easily calibrated from the observed implied volatility surface. A main feature of our approach is that these calibrated parameters are explicitly related to the parameters needed in the price approximation formula, and that, in fact, only these four parameters and the effective constant volatility are needed rather than a fully specified stochastic volatility model. In Section 5 we illustrate the quality of the fit to the implied volatility surface by using options data. In particular we show that the introduction of the slow volatility factor is crucial for capturing the behavior of the surface for the longer maturities. In Section 6 we show how to use our perturbation approach to price exotic options which are contracts depending on the path of the underlying process.

2. Multiscale stochastic volatility models. In this section we introduce the class of two-scale stochastic volatility models which we consider and discuss the concept of a multiscale diffusion model. We also discuss the risk neutral or equivalent martingale measure that is used for pricing of options.

2.1. Background. Volatility models built on diffusions were introduced in the literature in the late 1980s by Hull & White [11], Wiggins [18] and Scott [16]. One popular class of models builds on the Feller process model introduced in this context by Heston [10] because call option prices can be solved for in closed form up to a Fourier inversion.

Typically a lot of emphasis is placed on fitting the models very closely to observed implied volatilities (see Section 4 for the definition), and not surprisingly, models with more degrees of freedom perform better in this regard. For example, the models studied in [2, 4] include jumps in stochastic volatility on top of a Heston-type model.

However, little attention is paid to the stability of the estimated parameters over time, and it is usual practice in the industry simply to re-calibrate each day.

The approach taken here, based on modelling volatility in terms of its characteristic scales rather than specific distributions, sacrifices some of the goodness of in-sample fit to current data for greater stability properties. It also allows for efficient computation of approximations to prices of exotic contracts, which otherwise have to be solved for by simulations or numerical solution of a high-dimensional PDE associated with the full stochastic volatility model.

2.2. Model under physical measure. We denote the price of the underlying by X_t and model it as the solution of the stochastic differential equation:

$$dX_t = \mu X_t dt + \sigma_t X_t dW_t^{(0)}, \tag{2.1}$$

where σ_t is the stochastic volatility which will be described below. Observe that when σ_t is constant then X_t is a geometric Brownian motion and corresponds to the classical model used in the Black-Scholes theory. We refer the reader to [15] for details concerning diffusion processes and the related stochastic calculus, and a brief review of this calculus and the Black-Scholes pricing theory can also be found in [5]. In the class of models that we consider the volatility process σ_t is driven by two diffusion processes Y_t and Z_t:

$$\sigma_t = f(Y_t, Z_t). \tag{2.2}$$

We assume that f is a smooth positive function that is bounded and bounded away from zero.

2.2.1. Fast scale volatility factor. The first factor driving the volatility σ_t is a fast mean reverting diffusion process. Here, we choose this diffusion to be the simple standard model diffusion corresponding to a Gaussian Ornstein-Uhlenbeck process. We denote by $1/\varepsilon$ the rate of mean reversion of this process, with $\varepsilon > 0$ being a small parameter which corresponds to the time scale of this process. It is an ergodic process and we assume that its invariant distribution is independent of ε. This distribution is Gaussian with mean denoted by m and variance denoted by ν^2. The stochastic differential equation that follows from these prescriptions is:

$$dY_t = \frac{1}{\varepsilon}(m - Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} dW_t^{(1)},$$

where $W_t^{(1)}$ is a standard Brownian motion, and its covariation with $W_t^{(0)}$ is given by:

$$d\langle W^{(0)}, W^{(1)}\rangle_t = \rho_1 dt.$$

We assume that the correlation coefficient ρ_1 is constant and that $|\rho_1| < 1$. This correlation gives the well documented leverage effect and we will see below that it plays a crucial role in our expansion for the option prices. Under its invariant distribution $\mathcal{N}(m, \nu^2)$, the autocorrelation of Y_t is given by

$$I\!E\left\{(Y_s - m)(Y_t - m)\right\} = \nu^2 e^{-\frac{|t-s|}{\varepsilon}}.$$

Therefore the process decorrelates exponentially fast on the time scale ε and thus we refer to Y_t as the fast volatility factor.

2.2.2. Slow scale volatility factor. The second factor Z_t driving the volatility σ_t is a slowily varying diffusion process. Here, we choose this diffusion to be the one resulting from the simple time change $t \to \delta t$ of a given diffusion process

$$d\tilde{Z}_t = c(\tilde{Z}_t)dt + g(\tilde{Z}_t)d\tilde{W}_t$$

where $\delta > 0$ is a small parameter. This means that $Z_t = \tilde{Z}_{\delta t}$, and that

$$dZ_t = \delta c(Z_t)dt + g(Z_t)d\tilde{W}_{\delta t}.$$

We assume that the coefficients $c(z)$ and $g(z)$ are smooth and at most linearly growing at infinity. In distribution, Z_t satisfies

$$dZ_t = \delta c(Z_t)dt + \sqrt{\delta} g(Z_t)dW_t^{(2)},$$

where $W_t^{(2)}$ is another standard Brownian motion. We allow a general correlation structure between the three standard Brownian motions $W^{(0)}, W^{(1)}$ and $W^{(2)}$ so that

$$\begin{pmatrix} W_t^{(0)} \\ W_t^{(1)} \\ W_t^{(2)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \rho_1 & \sqrt{1-\rho_1^2} & 0 \\ \rho_2 & \tilde{\rho}_{12} & \sqrt{1-\rho_2^2-\tilde{\rho}_{12}^2} \end{pmatrix} \mathbf{W}_t, \tag{2.3}$$

where $\mathbf{W}_t$ is a standard three-dimensional Brownian motion, and where the constant coefficients ρ_1, ρ_2 and $\tilde{\rho}_{12}$ satisfy $|\rho_1| < 1$ and $\rho_2^2 + \tilde{\rho}_{12}^2 < 1$. Observe that with this parametrization the covariation between $W_t^{(1)}$ and $W_t^{(2)}$ is given by $t\rho_{12}$ where $\rho_{12} := \rho_1\rho_2 + \tilde{\rho}_{12}\sqrt{1-\rho_1^2}$. However, only the two parameters ρ_1 and ρ_2 will play an explicit role in the correction derived from our asymptotic analysis. To summarize our class of stochastic volatility models we have

$$\begin{aligned} dX_t &= \mu X_t dt + f(Y_t, Z_t) X_t dW_t^{(0)} \\ dY_t &= \frac{1}{\varepsilon}(m - Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} dW_t^{(1)} \\ dZ_t &= \delta\ c(Z_t)dt + \sqrt{\delta}\ g(Z_t)dW_t^{(2)}. \end{aligned} \tag{2.4}$$

Note that the slow factor in the volatility model corresponds to a small perturbation situation and the resulting regular perturbation scenario has been considered in many different settings. The fast factor on the other hand leads to a singular perturbations situation and gives rise to a diffusion homogenization problem that is not so widely applied.

2.2.3. Empirical Evidence. Empirical evidence of a fast volatility factor (with a characteristic mean-reversion time of a few days) was found in the analysis of high-frequency S&P 500 data in [7]. Many empirical studies have looked at low-frequency (daily) data, with the data necessarily ranging over a period of years, and they have found a slow volatility factor. This does not contradict the empirical finding described above: analyzing data at lower frequencies over longer time periods primarily picks up a slower time-scale of fluctuation and cannot identify scales of length comparable to the sampling frequency.

Another recent empirical study [1], this time of exchange rate dynamics, finds "the evidence points strongly toward two-factor [volatility] models with one highly persistent factor and one quickly mean-reverting factor".

2.3. Model under risk-neutral measure. No arbitrage pricing theory (see [3], for example) states that option prices are expectations of discounted payoffs with respect to an equivalent martingale measure. A brief review of this in the present stochastic volatility context is presented in Chapter 2 of [5]. This measure is a probability measure which is equivalent to the physical measure modeled in the previous section, and under which the discounted value for the underlying is a martingale. In the context of a constant volatility, the market is complete and there is a unique equivalent martingale measure. We consider the case with a random, non-tradable volatility which gives rise to an incomplete market and a family of pricing measures that are parameterized by the market price of volatility risk. The market chooses one of these for pricing and we write next the stochastic differential equations that model this choice in terms of the following three-dimensional standard Brownian motion under the risk-neutral measure:

$$\mathbf{W}_t^\star = \mathbf{W}_t + \int_0^t \begin{pmatrix} (\mu - r)/f(Y_s, Z_s) \\ \gamma(Y_s, Z_s) \\ \xi(Y_s, Z_s) \end{pmatrix} ds,$$

where we assume that $\gamma(y,z)$ and $\xi(y,z)$ are smooth bounded functions of y and z only. We introduce the combined market prices of volatility risk Λ and Γ defined by

$$\begin{aligned} \Lambda(y,z) &= \frac{\rho_1(\mu - r)}{f(y,z)} + \gamma(y,z)\sqrt{1-\rho_1^2} \\ \Gamma(y,z) &= \frac{\rho_2(\mu - r)}{f(y,z)} + \gamma(y,z)\tilde{\rho}_{12} + \xi(y,z)\sqrt{1-\rho_2^2-\tilde{\rho}_{12}^2}, \end{aligned}$$

and we write the evolution under the risk-neutral measure as

$$\begin{aligned} dX_t &= rX_t dt + f(Y_t, Z_t) X_t dW_t^{(0)\star} \\ dY_t &= \left(\frac{1}{\varepsilon}(m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} \Lambda(Y_t, Z_t) \right) dt + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}} dW_t^{(1)\star} \\ dZ_t &= \left(\delta\ c(Z_t) - \sqrt{\delta}\ g(Z_t)\Gamma(Y_t, Z_t) \right) dt + \sqrt{\delta}\ g(Z_t) dW_t^{(2)\star}, \end{aligned} \tag{2.5}$$

where the correlated Brownian motions $W^{(i)\star}$ are defined as in (2.3) with $\mathbf{W}^\star$ replacing $\mathbf{W}$. Observe that the process (X, Y, Z) is Markovian. Denoting by $I\!E^\star\{\cdot\}$ the expectation with respect to the risk-neutral measure described above, the price of a European option with payoff function $h(x)$ is given by:

$$P^{\varepsilon,\delta}(t, X_t, Y_t, Z_t) = I\!E^\star \left\{ e^{-r(T-t)} h(X_T) \mid X_t, Y_t, Z_t \right\}, \tag{2.6}$$

where we explicitly show the dependence on the two small parameters ε and δ.

2.4. Pricing equation. By an application of the Feynman-Kac formula, we obtain a characterization of $P^{\varepsilon,\delta}(t,x,y,z)$ in (2.6) as the solution of the parabolic PDE with a final condition

$$\mathcal{L}^{\varepsilon,\delta} P^{\varepsilon,\delta} = 0 \tag{2.7}$$

$$P^{\varepsilon,\delta}(T, x, y, z) = h(x), \tag{2.8}$$

where the partial differential operator $\mathcal{L}^{\varepsilon,\delta}$ is given by

$$\mathcal{L}^{\varepsilon,\delta} = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3, \tag{2.9}$$

using the notation

$$\mathcal{L}_0 = (m-y)\frac{\partial}{\partial y} + \nu^2 \frac{\partial^2}{\partial y^2} \tag{2.10}$$

$$\mathcal{L}_1 = \nu\sqrt{2}\left(\rho_1 f(y,z) x \frac{\partial^2}{\partial x \partial y} - \Lambda(y,z)\frac{\partial}{\partial y}\right) \tag{2.11}$$

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2} f^2(y,z) x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right) \tag{2.12}$$

$$\mathcal{M}_1 = -g(z)\Gamma(y,z)\frac{\partial}{\partial z} + \rho_2 g(z) f(y,z) x \frac{\partial^2}{\partial x \partial z} \tag{2.13}$$

$$\mathcal{M}_2 = c(z)\frac{\partial}{\partial z} + \frac{g(z)^2}{2}\frac{\partial^2}{\partial z^2} \tag{2.14}$$

$$\mathcal{M}_3 = \nu\sqrt{2}\ \rho_{12} g(z) \frac{\partial^2}{\partial y \partial z}. \tag{2.15}$$

Note that $\mathcal{L}_2$ is the Black-Scholes operator, corresponding to a constant volatility level $f(y,z)$, which we denote $\mathcal{L}_{BS}(f(y,z))$. We shall also denote the Black-Scholes price by $C_{BS}(t,x;\sigma)$, that is the price of a European claim with payoff h at the volatility level σ. It is given as the solution of the following PDE problem

$$\mathcal{L}_{BS}(\sigma) C_{BS} = 0, \qquad C_{BS}(T,x;\sigma) = h(x). \tag{2.16}$$

We have now written the pricing equation as a *singular-regular perturbation* problem around a Black-Scholes operator. We carry out this double asymptotics in the next section.

3. Asymptotics. In the following subsections we give a formal derivation of the price approximation in the regime where ε and δ are small independent parameters. The main theorem stating the accuracy of the approximation is given at the end of this section along with its proof. In the formal derivation we choose to expand first with respect to δ and subsequently with respect to ε. This choice is more convenient for the proof than the reverse ordering which in fact gives the same result. In our notation, the term $P_{j,k}$ is associated with the term of order $\varepsilon^{j/2}\delta^{k/2}$. The leading order term is denoted simply P_0.

3.1. Long scale limit. In this section we consider an expansion of $P^{\varepsilon,\delta}$ in powers of $\sqrt{\delta}$:

$$P^{\varepsilon,\delta} = P_0^\varepsilon + \sqrt{\delta} P_1^\varepsilon + \delta P_2^\varepsilon + \cdots. \tag{3.1}$$

Recall that the volatility factor associated with δ small corresponds the **slow** factor Z_t. In the case of a single slow volatility factor such an expansion has been considered in [9], [13] and [17], for instance. See also [14] and [12] for related regular perturbation expansions, and [19] for approximations based on large strike-price limits.

DEFINITION 3.1. *The leading order term P_0^ε is defined as the unique solution to the problem*

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right) P_0^\varepsilon = 0 \tag{3.2}$$

$$P_0^\varepsilon(T,x,y) = h(x). \tag{3.3}$$

Definition 3.2. *The next term P_1^ε is defined as the unique solution to the problem*

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2\right) P_1^\varepsilon = -\left(\mathcal{M}_1 + \frac{1}{\sqrt{\varepsilon}}\mathcal{M}_3\right) P_0^\varepsilon \tag{3.4}$$

$$P_1^\varepsilon(T, x, y) = 0. \tag{3.5}$$

Here we will only consider the first correction P_1^ε. In the next section, we expand P_0^ε and P_1^ε in powers of $\sqrt{\varepsilon}$ to obtain an approximation for the price $P^{\varepsilon,\delta}$.

3.2. Expansion in the Fast-Scale. Consider first P_0^ε which we decompose as

$$P_0^\varepsilon = P_0 + \sqrt{\varepsilon} P_{1,0} + \varepsilon P_{2,0} + \varepsilon^{3/2} P_{3,0} + \cdots. \tag{3.6}$$

In this subsection we derive explicit expressions for P_0 and $P_{1,0}$. We insert the expansion (3.6) in the equation (3.2) and find that the equations associated with the first two leading terms are:

$$\mathcal{L}_0 P_0 = 0 \tag{3.7}$$

$$\mathcal{L}_0 P_{1,0} + \mathcal{L}_1 P_0 = 0. \tag{3.8}$$

These are two ordinary differential equations in y and the only solutions that have reasonable growth in y do not depend on y and we therefore take $P_0 = P_0(t, x, z)$ and $P_{1,0} = P_{1,0}(t, x, z)$. Note next that the order one terms give

$$\mathcal{L}_0 P_{2,0} + \mathcal{L}_2 P_0 = 0, \tag{3.9}$$

since $\mathcal{L}_1 P_{1,0} = 0$. This is a Poisson equation in $P_{2,0}$ with respect to the y-variable and there will be no solution unless $\mathcal{L}_2 P_0$ is in the orthogonal complement of the null space of $\mathcal{L}_0^*$ (Fredholm Alternative). This is equivalent to saying that $\mathcal{L}_2 P_0$ has mean zero with respect to the invariant measure of the OU process: $\langle \mathcal{L}_2 P_0 \rangle = 0$. Here the bracket notation means integration with respect to the invariant distribution of the OU-process with infinitesimal generator $\mathcal{L}_0$, that is, integration with respect to the Gaussian $\mathcal{N}(m, \nu^2)$ density. The leading order term P_0 does not depend on y and we define it as the solution of the problem described below.

Definition 3.3. *The problem that determines P_0 is*

$$\begin{aligned} \langle \mathcal{L}_2 \rangle P_0 &= 0 \\ P_0(T, x, z) &= h(x), \end{aligned} \tag{3.10}$$

where

$$\langle \mathcal{L}_2 \rangle = \frac{\partial}{\partial t} + \frac{1}{2}\langle f^2(\cdot, z)\rangle x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right), \tag{3.11}$$

the Black-Scholes operator with volatility

$$\langle f^2(\cdot, z)\rangle := \bar{\sigma}^2(z) \tag{3.12}$$

which depends on the slow factor z. Therefore, P_0 is the Black-Scholes price of the claim at the volatility level $\bar{\sigma}(z)$, that is

$$P_0(t, x, z) = C_{BS}(t, x; \bar{\sigma}(z)),$$

with C_{BS} being defined in (2.16). Next we derive an expression for $P_{1,0}$. From the Poisson equation (3.9) and the centering condition (3.10), we deduce that

$$P_{2,0} = -\mathcal{L}_0^{-1}\left(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle\right)P_0 \tag{3.13}$$

up to an additive function which does not depend on y and which will not play a role in the problem that defines $P_{1,0}$. The next order term in the ε expansion in (3.6) gives the following Poisson equation in $P_{3,0}$

$$\mathcal{L}_0 P_{3,0} + \mathcal{L}_1 P_{2,0} + \mathcal{L}_2 P_{1,0} = 0. \tag{3.14}$$

The centering condition for this equation

$$\langle \mathcal{L}_2 P_{1,0} + \mathcal{L}_1 P_{2,0}\rangle = 0$$

gives the following problem that determines $P_{1,0}$:

DEFINITION 3.4. *The function $P_{1,0}(t,x,z)$ satisfies the inhomogeneous problem*

$$\begin{aligned} \langle\mathcal{L}_2\rangle P_{1,0} &= \mathcal{A}P_0 \\ P_{1,0}(T,x,z) &= 0. \end{aligned} \tag{3.15}$$

where

$$\mathcal{A} := \left\langle \mathcal{L}_1\mathcal{L}_0^{-1}\left(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle\right)\right\rangle. \tag{3.16}$$

The function $P_{1,0}$ is in fact given by the expression

$$P_{1,0} = -(T-t)\mathcal{A}P_0. \tag{3.17}$$

We next compute the operator $\mathcal{A}$ explicitly. First, we introduce $\phi(y,z)$ that is a solution of the following Poisson equation with respect to the variable y:

$$\mathcal{L}_0\phi(y,z) = f^2(y,z) - \bar{\sigma}^2(z). \tag{3.18}$$

Note that ϕ is defined up to an additive function that depends only on the variable z and which will not affect $\mathcal{A}$. With this notation, we have

$$\mathcal{L}_0^{-1}(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle) = \frac{1}{2}\phi(y,z)x^2\frac{\partial^2}{\partial x^2} \tag{3.19}$$

and therefore

$$\mathcal{A} = \frac{\nu\rho_1}{\sqrt{2}}\left\langle f\frac{\partial\phi}{\partial y}\right\rangle x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2}{\partial x^2}\right) - \frac{\nu}{\sqrt{2}}\left\langle \Lambda\frac{\partial\phi}{\partial y}\right\rangle x^2\frac{\partial^2}{\partial x^2}. \tag{3.20}$$

Using the facts that the operator $\langle\mathcal{L}_2\rangle$ commutes with $x^k\partial^k/\partial x^k$ and that $\langle\mathcal{L}_2\rangle P_0 = 0$, it can be checked that the solution $P_{1,0}$ is indeed given by the expression (3.17).

We next carry out the expansion of P_1^ε, the second term in the δ expansion in (3.1), in the small parameter ε .

3.3. Expansion of P_1^ε. We write

$$P_1^\varepsilon = P_{0,1} + \sqrt{\varepsilon}P_{1,1} + \varepsilon P_{2,1} + \varepsilon^{3/2}P_{3,1} + \cdots, \tag{3.21}$$

and we derive below an explicit expression for $P_{0,1}$. Substituting the expansion (3.21) into equation (3.4) gives

$$\mathcal{L}_0 P_{0,1} = 0, \tag{3.22}$$

from the highest order terms. As before, this implies $P_{0,1}$ does not depend on y. The next order gives

$$\mathcal{L}_0 P_{1,1} = 0, \tag{3.23}$$

where we have used $\mathcal{M}_3 P_0 = 0$ because $\mathcal{M}_3$ takes derivatives in y and P_0 does not depend on y and that $\mathcal{L}_1 P_{0,1} = 0$ for the same reason. Therefore $P_{1,1}$ also does not depend on y.

Evaluating the terms of order one and using that $\mathcal{M}_3 P_{1,0} = \mathcal{L}_1 P_{1,1} = 0$, we find

$$\mathcal{L}_0 P_{2,1} + \mathcal{L}_2 P_{0,1} = -\mathcal{M}_1 P_0. \tag{3.24}$$

This is therefore a Poisson equation in y for $P_{2,1}$ and the associated solvability condition leads to:

Definition 3.5. *The term $P_{0,1}(t,x,z)$ is the unique solution to the problem*

$$\begin{aligned} \langle \mathcal{L}_2 \rangle P_{0,1} &= -\langle \mathcal{M}_1 \rangle P_0 \\ P_{0,1}(T,x,z) &= 0. \end{aligned} \tag{3.25}$$

This term $P_{0,1}$ is in fact given explicitly in terms of derivatives with respect to x and z of P_0:

$$P_{0,1} = \frac{T-t}{2}\langle \mathcal{M}_1 \rangle P_0. \tag{3.26}$$

The formula (3.26) for $P_{0,1}$ is obtained as follows. Observe first that the derivative of the Black-Scholes price $P_{BS}(t,x;\sigma)$ with respect to volatility σ, known as the *Vega*, can be expressed as

$$\frac{\partial P_{BS}}{\partial \sigma} = (T-t)\sigma x^2 \frac{\partial^2 P_{BS}}{\partial x^2}, \tag{3.27}$$

which implies

$$\frac{\partial P_0}{\partial z} = (T-t)\bar{\sigma}(z)\bar{\sigma}'(z)x^2 \frac{\partial^2 P_0}{\partial x^2}.$$

Introducing the operator M_1 defined by

$$\langle \mathcal{M}_1 \rangle = \left(-g\langle \Gamma \rangle + \rho_2 g \langle f \rangle x \frac{\partial}{\partial x} \right) \frac{\partial}{\partial z} := M_1 \frac{\partial}{\partial z}, \tag{3.28}$$

we then check that $P_{0,1}$ given in (3.26) solves the equation (3.25):

$$\begin{aligned}
\langle \mathcal{L}_2 \rangle P_{0,1} &= \langle \mathcal{L}_2 \rangle \left[\frac{T-t}{2} \left(M_1 \frac{\partial}{\partial z} \right) P_0 \right] \\
&= \langle \mathcal{L}_2 \rangle \left[\frac{(T-t)^2}{2} M_1 \left(\bar{\sigma}(z) \bar{\sigma}'(z) x^2 \frac{\partial^2 P_0}{\partial x^2} \right) \right] \\
&= -(T-t) M_1 \left(\bar{\sigma}(z) \bar{\sigma}'(z) x^2 \frac{\partial^2 P_0}{\partial x^2} \right) \\
&\quad + \frac{(T-t)^2}{2} M_1 \bar{\sigma}(z) \bar{\sigma}'(z) x^2 \frac{\partial^2}{\partial x^2} \langle \mathcal{L}_2 \rangle P_0 \\
&= -(T-t) M_1 \left(\bar{\sigma}(z) \bar{\sigma}'(z) x^2 \frac{\partial^2 P_0}{\partial x^2} \right) \\
&= -\langle \mathcal{M}_1 \rangle P_0,
\end{aligned}$$

where we have again used that the operator $\langle \mathcal{L}_2 \rangle$ commutes with $x^k \partial^k / \partial x^k$ and that $\langle \mathcal{L}_2 \rangle P_0 = 0$.

We next derive $P_{1,1}$ and $P_{2,1}$, which although not part of our approximation, will be needed in the proof of Theorem 3.6. We define

$$P_{2,1} = -\mathcal{L}_0^{-1} \left((\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) P_{0,1} + (\mathcal{M}_1 - \langle \mathcal{M}_1 \rangle) P_0 \right), \tag{3.29}$$

as a solution of the Poisson equation (3.24), up to an arbitrary function independent of y which will not play a role in the proof.

Comparing terms of order $\sqrt{\varepsilon}$ in (3.4), we have a Poisson equation for $P_{3,1}$:

$$\mathcal{L}_0 P_{3,1} + \mathcal{L}_1 P_{2,1} + \mathcal{L}_2 P_{1,1} = -\mathcal{M}_1 P_{1,0} - \mathcal{M}_3 P_{2,0}. \tag{3.30}$$

Its solvability condition is

$$\langle \mathcal{L}_2 \rangle P_{1,1} = \mathcal{A} P_{0,1} + \mathcal{B} P_0 - \langle \mathcal{M}_1 \rangle P_{1,0} - \langle \mathcal{M}_3 P_{2,0} \rangle, \tag{3.31}$$

where $\mathcal{A}$ is defined in (3.16), and $\mathcal{B}$ is defined similarly by

$$\mathcal{B} = \left\langle \mathcal{L}_1 \mathcal{L}_0^{-1} \left(\mathcal{M}_1 - \langle \mathcal{M}_1 \rangle \right) \right\rangle.$$

In the next section we summarize our expansion of the price and discuss its accuracy.

3.4. Price approximation and its accuracy. From the expansions of $P^{\varepsilon,\delta}$, P_0^{ε} and P_1^{ε} in respectively (3.1), (3.6) and (3.21), we deduce that

$$\begin{aligned}
P^{\varepsilon,\delta} \approx \widetilde{P^{\varepsilon,\delta}} &:= P_0 + \sqrt{\varepsilon} P_{1,0} + \sqrt{\delta} P_{0,1} \\
&= P_0 + (T-t) \left(-\sqrt{\varepsilon}\, \mathcal{A} + \frac{\sqrt{\delta}}{2} \langle \mathcal{M}_1 \rangle \right) P_0,
\end{aligned} \tag{3.32}$$

where $\mathcal{M}_1$ and $\mathcal{A}$ were defined in (2.13) and (3.16) respectively. We introduce the *group market parameters* $(V_0^{\delta}, V_1^{\delta}, V_2^{\varepsilon}, V_3^{\varepsilon})$, which depend on z:

$$V_0^{\delta} = \frac{\sqrt{\delta}}{2} g \langle \Gamma \rangle \bar{\sigma} \bar{\sigma}' \tag{3.33}$$

$$V_1^\delta = -\frac{\sqrt{\delta}}{2}\rho_2 g\langle f\rangle\bar\sigma\bar\sigma' \tag{3.34}$$

$$V_2^\varepsilon = -\frac{\sqrt{\varepsilon}}{\sqrt{2}}\nu\left\langle \Lambda\frac{\partial\phi}{\partial y}\right\rangle \tag{3.35}$$

$$V_3^\varepsilon = \frac{\sqrt{\varepsilon}}{\sqrt{2}}\nu\rho_1\left\langle f\frac{\partial\phi}{\partial y}\right\rangle. \tag{3.36}$$

The parametrization $(V_2^\varepsilon, V_3^\varepsilon)$ is convenient to separate the influences of the correlation ρ_1 (contained in V_3^ε) and the market price of risk Λ (contained in V_2^ε). In [5] the parametrization (V_2, V_3) was chosen to separate the second and third order derivatives with respect to x. These parametrizations are related simply by

$$V_3^\varepsilon = V_3, \qquad V_2^\varepsilon = V_2 - 2V_3.$$

Recall from Definition 3.3 that $P_0(t,x,z) = P_{BS}(t,x;\bar\sigma(z))$. Therefore we can write

$$-\frac{\sqrt{\delta}}{2}\langle\mathcal{M}_1\rangle P_0 = \frac{1}{\bar\sigma}\left[V_0^\delta\frac{\partial}{\partial\sigma} + V_1^\delta x\frac{\partial^2}{\partial x\partial\sigma}\right]P_{BS} \tag{3.37}$$

$$\sqrt{\varepsilon}\mathcal{A}P_0 = \left[V_2^\varepsilon x^2\frac{\partial^2}{\partial x^2} + V_3^\varepsilon x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2}{\partial x^2}\right)\right]P_{BS}. \tag{3.38}$$

With this notation, the price approximation in (3.32) reads

$$\begin{aligned}\widetilde{P^{\varepsilon,\delta}} &= P_{BS}\\ &-(T-t)\left\{\frac{1}{\bar\sigma}\left[V_0^\delta\frac{\partial}{\partial\sigma} + V_1^\delta x\frac{\partial^2}{\partial x\partial\sigma}\right] + \left[V_2^\varepsilon x^2\frac{\partial^2}{\partial x^2} + V_3^\varepsilon x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2}{\partial x^2}\right)\right]\right\}P_{BS}.\end{aligned} \tag{3.39}$$

An alternative expression is given in (4.3). We now make precise the accuracy of the approximation.

Theorem 3.6. *When the payoff h is smooth, for fixed (t,x,y,z) and for any $\varepsilon\le 1, \delta\le 1$, there exists a constant $C>0$ such that*

$$|P^{\varepsilon,\delta} - \widetilde{P^{\varepsilon,\delta}}| \le C(\varepsilon + \delta + \sqrt{\varepsilon\delta}).$$

In the case of call and put options, where the payoff is continuous but only piecewise smooth, the accuracy is given by

$$|P^{\varepsilon,\delta} - \widetilde{P^{\varepsilon,\delta}}| \le C(\varepsilon|\log\varepsilon| + \delta + \sqrt{\varepsilon\delta}).$$

Proof We prove the first part of the theorem corresponding to a smooth payoff. The case of a call option can be proven by generalizing the regularization argument introduced in [8]. We discuss this generalization at the end of the proof. In order to establish the accuracy of the approximation we introduce the following higher order approximation for $P^{\varepsilon,\delta}$

$$\begin{aligned}\widehat{P^{\varepsilon,\delta}} &= \widetilde{P^{\varepsilon,\delta}} + \varepsilon(P_{2,0} + \sqrt{\varepsilon}P_{3,0}) + \sqrt{\delta}(\sqrt{\varepsilon}P_{1,1} + \varepsilon P_{2,1})\\ &= P_0 + \sqrt{\varepsilon}P_{1,0} + \varepsilon P_{2,0} + \varepsilon^{3/2}P_{3,0} + \sqrt{\delta}(P_{0,1} + \sqrt{\varepsilon}P_{1,1} + \varepsilon P_{2,1}),\end{aligned} \tag{3.40}$$

where P_0 and $P_{1,0}$ are defined in (3.10 and (3.15), $P_{2,0}$ and $P_{3,0}$ are defined in respectively (3.13) and (3.14). Moreover, $P_{0,1}$ is defined in (3.25), $P_{1,1}$ and $P_{2,1}$ are defined respectively by (3.31) and (3.29).

We next introduce the residual

$$R^{\varepsilon,\delta} = \widehat{P^{\varepsilon,\delta}} - P^{\varepsilon,\delta} \tag{3.41}$$

which satisfies

$$\begin{aligned}
&\mathcal{L}^{\varepsilon,\delta} R^{\varepsilon,\delta} = \\
&\frac{1}{\varepsilon}\left(\mathcal{L}_0 P_0\right) + \frac{1}{\sqrt{\varepsilon}}\left(\mathcal{L}_0 P_{1,0} + \mathcal{L}_1 P_0\right) + \left(\mathcal{L}_0 P_{2,0} + \mathcal{L}_1 P_{1,0} + \mathcal{L}_2 P_0\right) \\
&+\sqrt{\varepsilon}\left(\mathcal{L}_0 P_{3,0} + \mathcal{L}_1 P_{2,0} + \mathcal{L}_2 P_{1,0}\right) \\
&+\sqrt{\delta}\left(\frac{1}{\varepsilon}\left(\mathcal{L}_0 P_{0,1}\right) + \frac{1}{\sqrt{\varepsilon}}\left(\mathcal{L}_0 P_{1,1} + \mathcal{L}_1 P_{0,1} + \mathcal{M}_3 P_0\right)\right) \\
&+\sqrt{\delta}\left(\mathcal{L}_0 P_{2,1} + \mathcal{L}_1 P_{1,1} + \mathcal{L}_2 P_{0,1} + \mathcal{M}_1 P_0 + \mathcal{M}_3 P_{1,0}\right) \\
&+\varepsilon R_1^\varepsilon + \sqrt{\varepsilon\delta} R_2^\varepsilon + \delta R_3^\varepsilon
\end{aligned}$$

where R_1^ε, R_2^ε and R_3^ε are given by

$$R_1^\varepsilon = \mathcal{L}_2 P_{2,0} + \mathcal{L}_1 P_{3,0} + \sqrt{\varepsilon}\mathcal{L}_2 P_{3,0}, \tag{3.42}$$

$$\begin{aligned} R_2^\varepsilon &= \mathcal{L}_2 P_{1,1} + \mathcal{L}_1 P_{2,1} + \mathcal{M}_1 P_{1,0} + \mathcal{M}_3 P_{2,0} \\ &\quad + \sqrt{\varepsilon}(\mathcal{L}_2 P_{2,1} + \mathcal{M}_1 P_{2,0} + \mathcal{M}_3 P_{3,0}) + \varepsilon \mathcal{M}_1 P_{3,0}, \end{aligned} \tag{3.43}$$

$$\begin{aligned} R_3^\varepsilon &= \mathcal{M}_1 P_{0,1} + \mathcal{M}_2 P_0 + \mathcal{M}_3 P_{1,1} \\ &\quad + \sqrt{\varepsilon}(\mathcal{M}_1 P_{1,1} + \mathcal{M}_2 P_{1,0} + \mathcal{M}_3 P_{2,1}) + \varepsilon(\mathcal{M}_1 P_{2,1} + \mathcal{M}_2 P_{2,0}). \end{aligned} \tag{3.44}$$

They are smooth functions of t, x, y and z that are, for $\varepsilon \leq 1$ and $\delta \leq 1$, bounded by smooth functions of t, x, y, z independent of ε and δ, uniformly bounded in t, x, z and at most linearly growing in y through the solution of the Poisson equation (3.18). The term of order $1/\varepsilon$ cancels by (3.7), the term of order $1/\sqrt{\varepsilon}$ cancels by (3.8), the term of order one cancels by (3.9), the term of order $\sqrt{\varepsilon}$ cancels by (3.14). Moreover, the term of order $\sqrt{\delta}/\varepsilon$ cancels by (3.22), the term of order $\sqrt{\delta}/\sqrt{\varepsilon}$ cancels by (3.23), finally, the term of order $\sqrt{\delta}$ cancels by (3.24). Therefore we find

$$\mathcal{L}^{\varepsilon,\delta} R^{\varepsilon,\delta} = \varepsilon R_1^\varepsilon + \sqrt{\varepsilon\delta} R_2^\varepsilon + \delta R_3^\varepsilon. \tag{3.45}$$

Note next that, at the terminal time T, we can write

$$\begin{aligned}
R^{\varepsilon,\delta}(T,x,y,z) &= \widehat{P^{\varepsilon,\delta}}(T,x,y,z) \\
&= \varepsilon(P_{2,0} + \sqrt{\varepsilon} P_{3,0})(T,x,y,z)) + \sqrt{\varepsilon}\sqrt{\delta}(P_{1,1} + \sqrt{\varepsilon} P_{2,1})(T,x,y,z) \\
&:= \varepsilon G_1(x,y,z) + \sqrt{\varepsilon\delta} G_2(x,y,z)
\end{aligned} \tag{3.46}$$

where G_1 and G_2 are independent of t, and have in the other variables the same properties as the functions R's discussed above. It follows from (3.45) and (3.46) that

$$\begin{aligned}
R^{\varepsilon,\delta} &= \varepsilon I\!\!E^\star \left\{ e^{-r(T-t)} G_1(X_T, Y_T, Z_T) - \int_t^T e^{-r(s-t)} R_1^\varepsilon(s, X_s, Y_s, Z_s) ds \mid X_t, Y_t, Z_t \right\} \\
&+ \sqrt{\varepsilon\delta} I\!\!E^\star \left\{ e^{-r(T-t)} G_2(X_T, Y_T, Z_T) - \int_t^T e^{-r(s-t)} R_2^\varepsilon(s, X_s, Y_s, Z_s) ds \mid X_t, Y_t, Z_t \right\}
\end{aligned}$$

$$
(3.47) \qquad + \delta \mathbb{E}^\star \left\{ - \int_t^T e^{-r(s-t)} R_3^\varepsilon(s, X_s, Y_s, Z_s) ds \mid X_t, Y_t, Z_t \right\},
$$

where the process (X, Y, Z) is described in Section 2.3. Combined with (3.40) and (3.41), this establishes the first part of the theorem: $P^{\varepsilon,\delta} - \widetilde{P^{\varepsilon,\delta}} = \mathcal{O}(\varepsilon, \delta, \sqrt{\varepsilon\delta})$.

Finally, we comment on the generalization of the proof to the case with a call option. In this case the payoff h is not continuously differentiable. However, we can extend the proof to this case by introducing a regularized payoff function h^Δ as in [8]. This regularized payoff function corresponds to the Black-Scholes price at the terminal time T assuming that the time of expiration is $T + \Delta$ rather than T. The explicit formula for the Black-Scholes price then allows us to bound the difference between the regularized and unregularized prices in terms of Δ. The difference between the regularized price and the corresponding price approximation can be bounded using a generalization of the argument used in the first part of the proof. The main difficulty is to show how we can let Δ go to zero with ε and δ such that we still can bound the right hand side in (3.47). A straightforward generalization of the proof given in [8], but lengthy due to the additional δ-terms, shows that the choice of $\Delta = \varepsilon$ leads to the bound on the residual $|R^{\varepsilon,\delta}| \leq C(\varepsilon|\log\varepsilon| + \sqrt{\varepsilon\delta} + \delta)$. We omit the details here.

4. Implied volatility. Recall that the implied volatility I for a call option with strike K and maturity T produced by our model (Section 2.3) is obtained by inverting the following equation with respect to I:

$$
(4.1) \qquad C_{BS}(t, x; T, K, I) = P^{\varepsilon,\delta}(t, x, z)
$$

where $P^{\varepsilon,\delta}$ is our model price for a call option and C_{BS} is the Black-Scholes call option price with volatility I. We expand the implied volatility by writing

$$
(4.2) \qquad I = I_0 + I_1^\varepsilon + I_1^\delta + \cdots,
$$

where I_1^ε (respectively I_1^δ) is proportional to $\sqrt{\varepsilon}$ (respectively $\sqrt{\delta}$). By a Taylor expansion of C_{BS} around I_0 and rewriting the approximation $\widetilde{P^{\varepsilon,\delta}}$ given in (3.39) as

$$
(4.3) \qquad \widetilde{P^{\varepsilon,\delta}} = P_{BS} - \frac{1}{\bar\sigma} \left\{ \left(V_2^\varepsilon + V_3^\varepsilon x \frac{\partial}{\partial x} \right) + \tau \left(V_0^\delta + V_1^\delta x \frac{\partial}{\partial x} \right) \right\} \frac{\partial}{\partial \sigma} P_{BS},
$$

where $\tau = T - t$, we find that

$$
\begin{aligned}
(4.4) \quad & C_{BS}(I_0) + (I_1^\varepsilon + I_1^\delta) \frac{\partial}{\partial\sigma} C_{BS}(I_0) + \cdots \\
&= P_{BS} - \frac{1}{\bar\sigma} \left\{ \left(V_2^\varepsilon + V_3^\varepsilon x \frac{\partial}{\partial x} \right) + \tau \left(V_0^\delta + V_1^\delta x \frac{\partial}{\partial x} \right) \right\} \frac{\partial}{\partial \sigma} P_{BS} + \cdots.
\end{aligned}
$$

By matching the $\mathcal{O}(1)$ terms we find that $C_{BS}(I_0) = P_{BS}(\bar\sigma(z))$ and hence that

$$
(4.5) \qquad I_0 = \bar\sigma(z).
$$

Matching the $\sqrt{\varepsilon}$ terms and the $\sqrt{\delta}$ terms gives respectively

$$
(4.6) \qquad I_1^\varepsilon \frac{\partial}{\partial\sigma} C_{BS} = -\frac{1}{\bar\sigma} \left(V_2^\varepsilon + V_3^\varepsilon x \frac{\partial}{\partial x} \right) \frac{\partial}{\partial\sigma} P_{BS}
$$

$$
(4.7) \qquad I_1^\delta \frac{\partial}{\partial\sigma} C_{BS} = -\frac{\tau}{\bar\sigma} \left(V_0^\delta + V_1^\delta x \frac{\partial}{\partial x} \right) \frac{\partial}{\partial\sigma} P_{BS}.
$$

A direct computation based on the Black-Scholes formula shows that for a call option with volatility σ we have

$$\left(x\frac{\partial}{\partial x}\right)\frac{\partial}{\partial \sigma}C_{BS}=\left(1-\frac{d_1}{\sigma\sqrt{\tau}}\right)\frac{\partial}{\partial \sigma}C_{BS}, \tag{4.8}$$

where as usual

$$d_1=\frac{\log(x/K)+(r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}. \tag{4.9}$$

From equations (4.6), (4.7) and (4.8) we find

$$I_1^\varepsilon=-\frac{1}{\bar\sigma}\left\{V_2^\varepsilon+V_3^\varepsilon\left(1-\frac{d_1}{\bar\sigma\sqrt{\tau}}\right)\right\} \tag{4.10}$$

$$I_1^\delta=-\frac{\tau}{\bar\sigma}\left\{V_0^\delta+V_1^\delta\left(1-\frac{d_1}{\bar\sigma\sqrt{\tau}}\right)\right\}. \tag{4.11}$$

Our z-dependent approximation for the term structure of implied volatility, $\bar\sigma+I_1^\varepsilon+I_1^\delta$, can now be written as an affine function of

- "Log-Moneyness to Maturity Ratio" (LMMR): $\log(K/x)/(T-t)$,
- "Log-Moneyness" (LM): $\log(K/x)$,
- and time-to-maturity: $T-t$.

The implied volatility surface, in terms of these composite variables, is given by

$$I_0+I_1^\varepsilon+I_1^\delta=\bar\sigma+b^\varepsilon+a^\varepsilon\frac{\log(K/x)}{T-t}+a^\delta\log(K/x)+b^\delta(T-t), \tag{4.12}$$

where the parameters $\bar\sigma$, $a^\varepsilon, a^\delta, b^\varepsilon$, and b^δ depend on z and are related to the group parameters $(V_0^\delta, V_1^\delta, V_2^\varepsilon, V_3^\varepsilon)$ by

$$\begin{aligned}
a^\varepsilon &= -\frac{V_3^\varepsilon}{\bar\sigma^3}\\
b^\varepsilon &= -\frac{V_2^\varepsilon}{\bar\sigma}+\frac{V_3^\varepsilon}{\bar\sigma^3}(r-\frac{\bar\sigma^2}{2})\\
a^\delta &= -\frac{V_1^\delta}{\bar\sigma^3}\\
b^\delta &= -\frac{V_0^\delta}{\bar\sigma}+\frac{V_1^\delta}{\bar\sigma^3}(r-\frac{\bar\sigma^2}{2}).
\end{aligned}$$

The formula (4.12) can also be viewed as a time-varying LMMR parametrization by re-writing it as

$$I\approx\bar\sigma+\left[a^\varepsilon+a^\delta(T-t)\right]\frac{\log(K/x)}{T-t}+\left[b^\varepsilon+b^\delta(T-t)\right]. \tag{4.13}$$

In practice, as we illustrate with real data in the next section, the parameter $\bar\sigma$ is first estimated from historical data over a period of time of order one, that is from the observation of the price of the underlying in the near past. Then, the parameters a^ε, b^ε, a^δ and b^δ are calibrated to the **observed** term structure of implied volatility by using (4.12). Note that once these parameters have been estimated, then for pricing or hedging purposes, we need the parameters $(V_0^\delta/\bar\sigma, V_1^\delta/\bar\sigma, V_2^\varepsilon, V_3^\varepsilon)$ as can be seen from the formula (3.39). These quantities are given by:

$$
\begin{aligned}
V_0^\delta/\bar\sigma &= -\left(b^\delta + a^\delta(r - \frac{\bar\sigma^2}{2})\right) \\
V_1^\delta/\bar\sigma &= -a^\delta\bar\sigma^2 \\
V_2^\varepsilon &= -\bar\sigma\left(b^\varepsilon + a^\varepsilon(r - \frac{\bar\sigma^2}{2})\right) \\
V_3^\varepsilon &= -a^\varepsilon\bar\sigma^3.
\end{aligned}
\tag{4.14}
$$

One of the strengths of our method is that these are the same parameters which are needed to price path dependent contracts as we will show in Section 6.

5. Calibration to data. In this section, we illustrate the improvement in fit of the model's predicted implied volatility, given by formula (4.13), to market data on a specific day. A more extensive analysis of the stability of estimated parameters over time will be detailed in work in preparation. Of course, it is not too surprising that the two-scale volatility model with its additional parameters performs better than either of the one-scale models. However, the pictures of the in-sample fits show visually how the implied volatilities of options of different maturities are better aligned by the multi-scale theory.

Figure 5.1 shows the fit using only the fast-factor approximation

$$I \approx a^\varepsilon(LMMR) + b^\varepsilon + \bar\sigma.$$

Here, we estimate $b^\varepsilon + \bar\sigma$ together, and in practice, as described above, $\bar\sigma$ would be estimated separately each day using data over a long enough period that the fast factor averages out, but the slow factor is approximately constant. Then b^ε can be obtained by subtraction.

Each strand in Figure 5.1 comes from options of different maturities (with the shortest maturities on the left-most strand, and the maturity increasing going clockwise). Clearly the single-factor theory struggles to capture the range of maturities. In Figure 5.2, we show the result of the calibration using only the slow-factor approximation

$$I \approx a^\delta(LM) + b^\delta\tau + \bar\sigma.$$

(Here the fit as a function of the regressor LM is shown, and the maturities increase going counterclockwise from the top-leftmost strand). Again, the single-factor theory struggles to capture the range of maturities.

Finally, in fitting the two-factor volatility approximation (4.13), we first divide the data into implied volatilities of equal maturities and fit an LMMR approximation across different strikes. This gives us, for each maturity τ, estimates of

$$\alpha(\tau) := a^\varepsilon + a^\delta\tau,$$

and

$$\beta(\tau) := \bar\sigma + b^\varepsilon + b^\delta\tau.$$

These are then fitted to linear functions of τ to give estimates of a^ε, a^δ, $\bar\sigma + b^\varepsilon$ and b^δ. A plot of this second term-structure fit is shown in Figure 5.3. The reason for employing such a two-stage fitting procedure is that there are clearly far fewer points in the τ direction than in the moneyness direction.

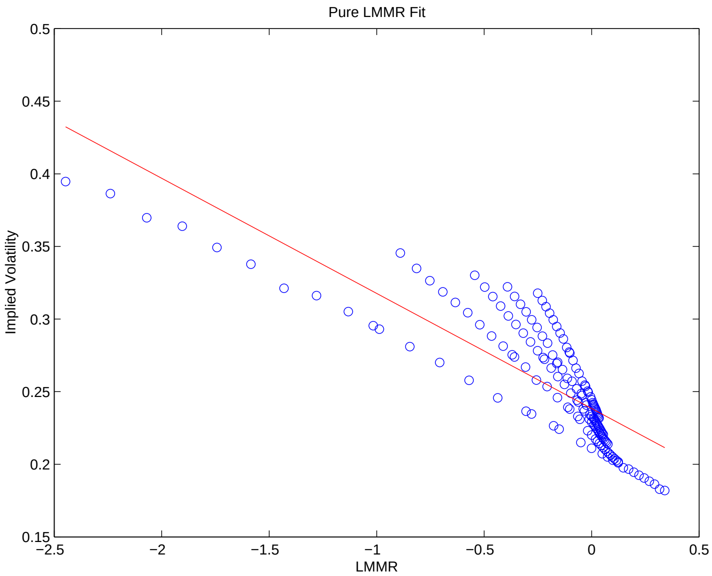

FIG. 5.1. S&P 500 implied volatilities as a function of LMMR on 25 January, 2000, for options with maturities greater than a month and less than 18 months, and moneyness between 0.7 and 1.05. The circles are from S&P 500 data, and the line $a^{\varepsilon}(LMMR) + b^{\varepsilon} + \bar{\sigma}$ shows the result using the estimated parameters from only an LMMR (fast factor) fit.

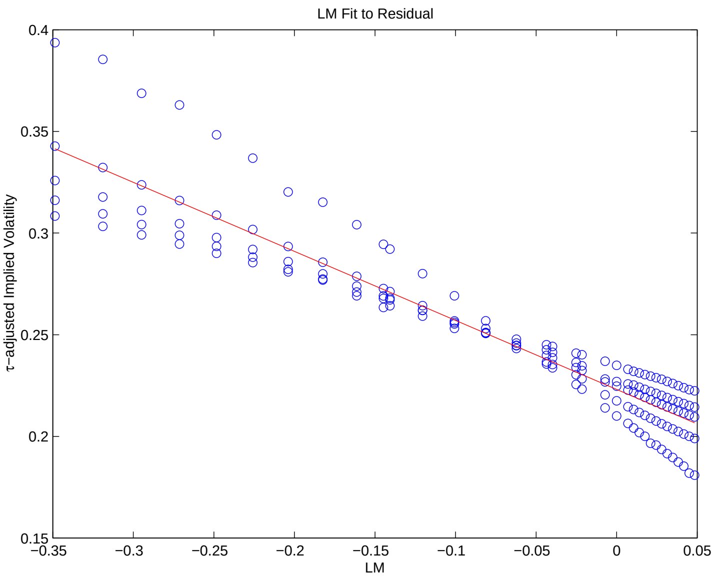

FIG. 5.2. τ-adjusted implied volatility $I - b^{\delta}\tau$ as a function of LM. The circles are from S&P 500 data, and the line $a^{\delta}(LM) + \bar{\sigma}$ shows the fit using the estimated parameters from only a slow factor fit.

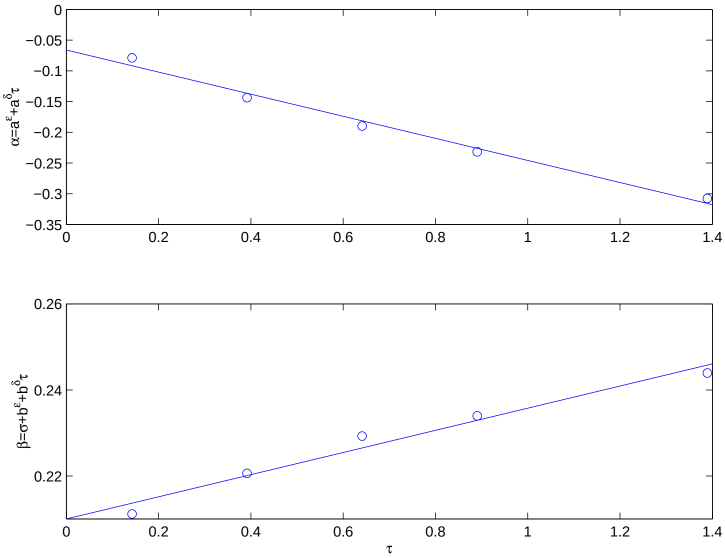

Fig. 5.3. Term-structures fits.

The result of the fit is shown in Figure 5.4. We see the ability to capture the range of maturities is much-improved. The greatest misfitting is at the level of the shortest maturities (the left-most strand). One way to handle these using a periodic scale corresponding to the monthly expiration cycles of traded options is presented in [6].

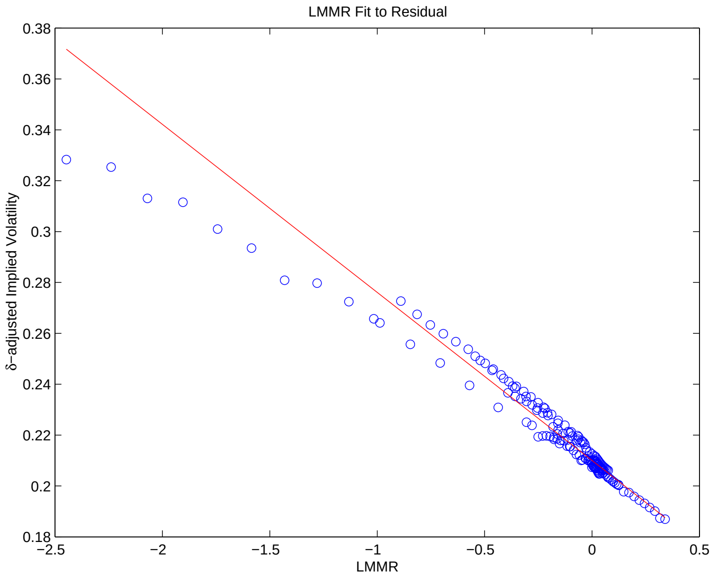

Fig. 5.4. δ-adjusted implied volatility $I - b^{\delta}\tau - a^{\delta}(LM)$ as a function of LMMR. The circles are from S&P 500 data, and the line is $a^{\varepsilon}(LMMR) + b^{\varepsilon} + \bar{\sigma}$ where $(a^{\varepsilon}, b^{\varepsilon} + \bar{\sigma}, a^{\delta}, b^{\delta})$ are the estimated parameters from the full fast & slow factor fit.

6. Pricing with calibrated parameters.

6.1. Vanilla options and the Greeks. We first summarize how the expansion obtained in the previous section is used to approximate the price of a European derivative which pays $h(X_T)$ at maturity time T in the case with the multiscale stochastic volatility model described in (2.4). Note that if the volatility f in this model is constant then the price is P_{BS}, the classical Black-Scholes price at the constant volatility f. In the stochastic case the *leading order* price is P_{BS} evaluated at the effective volatility $\bar{\sigma}(z)$ given in (3.12) where z is the current level of the slow volatility factor. The parameter $\bar{\sigma}(z)$ can be estimated from historical data. We do not discuss the details of this estimator here. The main point regarding the estimator is that $\bar{\sigma}(z)$ is obtained as an average volatility over a period that is long relative to the fast volatility factor, but which is still short relative to the slow volatility factor, thus, a period on the scale of the time to expiration for the contract. The role of the parameter $\bar{\sigma}(z)$ in the pricing equation will be discussed in more detail in forthcoming work. The first step now consists in solving the Black-Scholes equation

$$\frac{\partial P_{BS}}{\partial t} + \frac{1}{2}\bar{\sigma}^2(z)x^2\frac{\partial^2 P_{BS}}{\partial x^2} + r\left(x\frac{\partial P_{BS}}{\partial x} - P_{BS}\right) = 0 \tag{6.1}$$
$$P_{BS}(T,x) = h(x).$$

The next step consists in computing the Greeks of this Black-Scholes price, namely the *Delta*, $\partial P_{BS}/\partial x$, the *Vega*, $\partial P_{BS}/\partial\sigma$ and the *DeltaVega* $\partial^2 P_{BS}/\partial x\partial\sigma$. Recall that the Vega is related to the *Gamma* ($\partial^2 P_{BS}/\partial x^2$) through formula (3.27) with $\sigma = \bar{\sigma}(z)$. Finally, we can then compute the *corrected price* which incorporates the main effects of the fast and slow volatility factors. This corrected price is given as in (4.3) and we write it here in the form

$$\widetilde{P^{\varepsilon,\delta}} = P_{BS} - \frac{1}{\bar{\sigma}(z)}\left[\left(V_2^\varepsilon + \tau V_0^\delta\right)\text{Vega} + \left(V_3^\varepsilon + \tau V_1^\delta\right)x\text{DeltaVega}\right], \tag{6.2}$$

where $V_0^\delta, V_1^\delta, V_2^\varepsilon$ and V_3^ε are given in (4.14) in terms of the quantities $a^\delta, b^\delta, a^\varepsilon$ and b^ε which are calibrated from the term structure of implied volatility as explained in the previous section. Recall that V_0^δ and V_1^δ are small $\mathcal{O}(\sqrt{\delta})$ and that V_0^ε and V_1^ε are small $\mathcal{O}(\sqrt{\varepsilon})$. Note that for vanilla options of the type discussed in this section no arbitrage pricing can then in principle be carried out using directly the "continuum" of call options prices, if these are available. A main advantage of our asymptotic approach comes when we want to price exotic options based on the underlying for which we did the calibration. We discuss this in the next section in the context of path dependent contracts.

6.2. Path dependent contracts. In order to illustrate the strength of our approach we present a particular example, but hasten to add that a large family of exotics can be handled via analogous modifications. The discussion below is a generalization of the one presented in [5] to the case with a multiscale volatility. The main point of our discussion below is to show that the parameters that we calibrated above can be used to price also exotic derivatives on the underlying.

We consider an average-strike option where the strike price depends on the average of the stock price over the lifetime of the option. That is, the payoff function is

$$h = \left(X_T - \frac{1}{T}\int_0^T X_s ds\right)^+. \tag{6.3}$$

This derivative involves the new stochastic process

$$I_t = \int_0^t X_s ds. \tag{6.4}$$

The model under the risk-neutral measure is still (2.5) with the addition of:

$$\begin{aligned} dI_t &= X_t dt \\ I_0 &= 0 \end{aligned}$$

since the equation for I is not affected by the change of measure. The price of the Asian option, which we denote $Q^{\varepsilon,\delta}$, is now given by

$$Q^{\varepsilon,\delta}(t, X_t, Y_t, Z_t, I_t) = I\!E^\star\left\{e^{-r(T-t)}h \mid X_t, Y_t, Z_t, I_t\right\}, \tag{6.5}$$

where we again explicitly show the dependence on the two small parameters ε and δ. As above, an application of the Feynman-Kac formula gives a characterization of $Q^{\varepsilon,\delta}(t,x,y,z,I)$ in (6.5) as the solution of the parabolic PDE with a final condition:

$$\hat{\mathcal{L}}^{\varepsilon,\delta} Q^{\varepsilon,\delta} = 0 \tag{6.6}$$

$$Q^{\varepsilon,\delta}(T,x,y,z,I) = \left(x - \frac{I}{T}\right)^+, \tag{6.7}$$

where the partial differential operator $\hat{\mathcal{L}}^{\varepsilon,\delta}$ is given by

$$\hat{\mathcal{L}}^{\varepsilon,\delta} = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \hat{\mathcal{L}}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3, \tag{6.8}$$

with $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{M}_1$, $\mathcal{M}_2$ and $\mathcal{M}_3$ being defined as in (2.10) and below. The operator $\hat{\mathcal{L}}_2$ is a modification of $\mathcal{L}_2$:

$$\hat{\mathcal{L}}_2 = \mathcal{L}_2 + x\frac{\partial}{\partial I}. \tag{6.9}$$

We can therefore proceed with the asymptotic analysis exactly as in Section 3, the only change being that $\mathcal{L}_2$ is replaced by $\hat{\mathcal{L}}_2$. Note that

$$\hat{\mathcal{L}}_2 - \left\langle \hat{\mathcal{L}}_2 \right\rangle = \mathcal{L}_2 - \langle \mathcal{L}_2 \rangle . \tag{6.10}$$

The leading order price approximation Q_0 solves a modified version of the problem (3.10)

$$\begin{aligned} \langle \hat{\mathcal{L}}_2 \rangle Q_0 &= 0 \\ Q_0(T,x,z,I) &= \left(x - \frac{I}{T}\right)^+, \end{aligned} \tag{6.11}$$

where now

$$\langle \hat{\mathcal{L}}_2 \rangle = \mathcal{L}_{BS}(\bar{\sigma}^2(z)) + x\frac{\partial}{\partial I},$$

with $\mathcal{L}_{BS}$ being the Black-Scholes operator with volatility $\bar{\sigma}(z)$.

We then have that the first correction in the fast scale, $Q_{1,0}$, is given by an expression analogous to the one in (3.15):

$$\begin{aligned} \langle \hat{\mathcal{L}}_2 \rangle Q_{1,0} &= \mathcal{A} Q_0 \\ Q_{1,0}(T,x,z,I) &= 0 \end{aligned} \tag{6.12}$$

where again $\mathcal{A}$ is the operator specified in (3.16). The first correction in the slow scale is similarly determined by

$$\begin{aligned} \langle \hat{\mathcal{L}}_2 \rangle Q_{0,1} &= -\langle \mathcal{M}_1 \rangle Q_0 \\ Q_{0,1}(T,x,z) &= 0. \end{aligned} \tag{6.13}$$

Recall that the operators $\mathcal{A}$ and $\mathcal{M}_2$ can be expressed in terms of the market group parameters $V_0^\delta, V_1^\delta, V_2^\varepsilon$ and V_3^ε. Thus, we can find the price approximation for the average strike option by solving (6.11) for the leading order price and (6.12) and (6.13) for the corrections after having calibrated the market parameters in the manner described above. Observe that the problem (6.11) admits no explicit solution and must be solved numerically. The problems for the corrections must also be solved numerically.

To summarize, putting together the solutions of these linear equations, the price $Q^{\varepsilon,\delta}$ can be approximated by the solution $q^{\varepsilon,\delta}$ of the PDE problem

$$\begin{aligned} \langle \hat{\mathcal{L}}_2 \rangle q^{\varepsilon,\delta} &= \mathcal{L}_s Q_0 \\ q^{\varepsilon,\delta}(T,x,z,I) &= \left(x - \frac{I}{T}\right)^+, \end{aligned} \tag{6.14}$$

where

$$\mathcal{L}_s := \left\{-\sqrt{\delta}\langle\mathcal{M}_1\rangle + \sqrt{\varepsilon}\mathcal{A}\right\}$$
$$= \left\{\frac{2}{\bar{\sigma}}\left[V_0^\delta\frac{\partial}{\partial\sigma} + V_1^\delta x\frac{\partial^2}{\partial x\partial\sigma}\right] + \left[V_2^\varepsilon x^2\frac{\partial^2}{\partial x^2} + V_3^\varepsilon x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2}{\partial x^2}\right)\right]\right\}.$$

6.3. Hedging. The problem of hedging exotic option positions by trading the underlying asset and possibly other vanilla options is an important one, and is less clear-cut in incomplete markets such as described by stochastic volatility models, than in complete markets where it is a by-product of the pricing problem. Often, one might want to introduce a measure of hedging performance and solve an optimal control problem to derive a hedging strategy.

One natural strategy is to extend the analogous hedging rule from the Black-Scholes model to our corrected price. Typically, for example in the case of the Asian option, the strategy is to hold the quantity given by the Delta of the price in stocks,

$$\Delta = \frac{\partial Q_0}{\partial x},$$

and the remainder $Q_0 - \Delta X$ in the bank account. In the Black-Scholes model, this is a self-financing strategy that hedges the option perfectly. In the stochastic volatility market, the hedge defined by holding

$$\Delta = \frac{\partial}{\partial x} q^{\varepsilon,\delta}$$

and the amount $q^{\varepsilon,\delta} - \Delta X$ in the bank account. As discussed in [book, Ch 7], this is not a self-financing portfolio, but its value is close to the price of the option. Another type of strategy described there, that also depends only on the calibrated asymptotic parameters, reduces the bias of the hedging error, as measured by the difference in the terminal payoff of the option and the stock/bank portfolio.

REFERENCES

[1] S. Alizadeh, M. Brandt, and F. Diebold. Range-based estimation of stochastic volatility models. *Journal of Finance*, **57**(3): 1047–91, 2002.

[2] G. Bakshi, C. Cao and Z. Chen. Empirical Performance of Alternative Option Pricing Models. *Journal of Finance*, **52**(5): 2003–2049, 1997.

[3] D. Duffie, *Dynamic Asset Pricing Theory.* 3rd edn., Princeton University Press, 2001.

[4] D. Duffie, J. Pan, and K. Singleton. Transform analysis and option pricing for affine jump-diffusions. *Econometrica*, **68**: 1343-1376, 2000.

[5] J.P. Fouque, G. Papanicolaou, and K.R. Sircar. *Derivatives in Financial Markets with Stochastic Volatility.* Cambridge University Press, 2000.

[6] J.P. Fouque, G. Papanicolaou, K.R. Sircar and K. Solna. *Maturity Cycles in Implied Volatility.* To appear in *Finance & Stochastics.*

[7] J.-P. Fouque, G. Papanicolaou, K. R. Sircar, and K. Solna. Short Time-Scale in S&P 500 Volatility. *Journal of Computational Finance*, **6**(4): 1–23, 2003.

[8] J.P. Fouque, G. Papanicolaou, K.R. Sircar and K. Solna. *Singular Perturbations in Option Pricing.* To appear in SIAM J. Applied Math. 2003.

[9] E. Fournie, J. Lebuchoux, and N. Touzi. *Small Noise Expansion and Importance Sampling.* Asymptotic Analysis, **14**(4): 361–376, 1997.

[10] S. Heston. A closed-form solution for options with Stochastic Volatility with applications to bond and currency options. *Review of Financial Studies* **6**(2): 327–343, 1993.

[11] J. Hull and A. White. The Pricing of Options on Assets with Stochastic Volatilities. *J. Finance* **XLII**(2): 281–300, 1987.

[12] J. Hull and A. White. An analysis of the bias in option pricing caused by a stochastic volatility. *Advances in Futures and Options Research*, **3**: 29–61, 1988.
[13] R. Lee, Local volatilities under stochastic volatility. *International Journal of Theoretical and Applied Finance* **4**(1): 45-89, 1999.
[14] A. Lewis. *Option Valuation under Stochastic Volatility.* Finance Press, Newport Beach, CA, 2000.
[15] B. Øksendal, *Stochastic Differential Equations.* Springer, 1998.
[16] L. Scott. Option Pricing when the Variance changes randomly: Theory, Estimation, and an Application. *J. Financial and Quantitative Analysis* **22**(4): 419–438, 1987.
[17] K. R. Sircar and G. C. Papanicolaou, Stochastic Volatility, Smile and Asymptotics. *Applied Mathematical Finance*, **6** (2): 107–145, 1999.
[18] J. Wiggins. Option Values under Stochastic Volatility. *J. Financial Economics* **19**(2): 351–372, 1987.
[19] Y. Zhu and M. Avellaneda. A risk-neutral stochastic volatility model. *Intl. J. Theor. Appl. Fin.*, 1(2): 289–310, 1998.