Perturbed Gaussian Copula*

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Abstract

Gaussian copula is by far the most popular copula used in the financial industry in default dependency modeling. However, it has a major drawback — it does not exhibit tail dependence, a very important property for copula. The essence of tail dependence is the interdependence when extreme events occur, say, defaults of corporate bonds. In this paper we show that some tail dependence can be restored by introducing stochastic volatility on a Gaussian copula. Using perturbation methods we then derive an approximate copula — called perturbed Gaussian copula in this paper.

A copula is a joint distribution function of uniform random variables. Sklar's Theorem states that for any multivariate distribution, the univariate marginal distributions and the dependence structure can be separated. The dependence structure is completely determined by the copula. It then implies that one can "borrow" the dependence structure, namely the copula, of one set of dependent random variables and exchange the marginal distributions for a totally different set of marginal distributions.

An important property of copula is its invariance under monotonic transformation. More precisely, if g_i is strictly increasing for each i, then $(g_1(X_1), g_2(X_2), \ldots, g_n(X_n))$ have the same copula as $(X_1, X_2, \ldots, X_n)$.

From the above discussion, it is not hard to see that copula comes in default dependency modeling very naturally. For a much detailed coverage on copula, including the precise format of Sklar's Theorem, as well as modeling default dependency by way of copula, the readers are referred to Schonbucher (2003) [5].

Let $(Z_1, \ldots, Z_n)$ be a normal random vector with standard normal marginals and correlation matrix R, and $\Phi(\cdot)$ be the standard normal cumulative distribution function. Then the joint distribution function of $(\Phi(Z_1), \ldots, \Phi(Z_n))$ is called the **Gaussian copula** with correlation matrix R.

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Gaussian copula is by far the most popular copula used in the financial industry in default dependency modeling. This is basically because of two reasons. Firstly it is easy to simulate. Secondly it requires the “right” number of parameters — equal to the number of correlation coefficients among the underlying names. However, Gaussian copula does not exhibit any tail dependence, a very important property for copula. The essence of tail dependence is the interdependence when extreme events occur, say, defaults of corporate bonds. In fact, this is considered as a major drawback of Gaussian copula.

On the other hand, by introducing stochastic volatility into the classic Black-Scholes model, Fouque, Papanicolaou and Sircar (2000) [1], by way of singular perturbation method, gave a satisfactory answer to the “smile curve” problem of implied volatilities in the financial market, leading to a pricing formula which is in the form of a robust simple correction to the classic Black-Scholes constant volatility formula. Furthermore, an application of this perturbation method to defaultable bond pricing has been studied by Fouque, Sircar and Solna (2005) [3]. By fitting real market data, they concluded that the method works fairly well. An extension to multi-name first passage models is proposed by Fouque, Wignall and Zhou (2006) [4].

In this paper we will show the effect of stochastic volatility on a Gaussian copula. Specifically, in Section 1, we first set up the stochastic volatility model and state out the objective — the transition density functions. Then by singular perturbation, we obtain approximate transition density functions. In order to make them true probability density functions, we introduce the transformation $1 + \tanh(\cdot)$. In Section 2, we study this new class of approximate copula density functions, first analytically and then numerically. Section 3 concludes this paper.

1 Asymptotics

1.1 Model Setup

We start with a process $(X_t^{(1)}, X_t^{(2)}, Y_t)$ defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and which follows the dynamics:

$$
\begin{aligned}
\mathrm{d}X_t^{(1)} &= f_1(Y_t)\mathrm{d}W_t^{(1)}, \\
\mathrm{d}X_t^{(2)} &= f_2(Y_t)\mathrm{d}W_t^{(2)}, \\
\mathrm{d}Y_t &= \frac{1}{\epsilon}(m - Y_t)\mathrm{d}t + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\mathrm{d}W_t^{(Y)},
\end{aligned}
$$

where $W_t^{(1)}, W_t^{(2)}$ and $W_t^{(Y)}$ are standard Brownian motions correlated as follows

$$
\mathrm{d}\langle W^{(1)}, W^{(2)}\rangle_t = \rho\mathrm{d}t, \quad \mathrm{d}\langle W^{(1)}, W^{(Y)}\rangle_t = \rho_{1Y}\mathrm{d}t, \quad \mathrm{d}\langle W^{(2)}, W^{(Y)}\rangle_t = \rho_{2Y}\mathrm{d}t,
$$

with $-1 \leq \rho, \rho_{1Y}, \rho_{2Y} \leq 1$ and making the correlation matrix

$$
\begin{bmatrix} 1 & \rho & \rho_{1Y} \\ \rho & 1 & \rho_{2Y} \\ \rho_{1Y} & \rho_{2Y} & 1 \end{bmatrix}
$$

symmetric positive definite, ϵ and ν are positive constant numbers with $\epsilon \ll 1$ being small. The f_i's are real functions for $i = 1, 2$, and are assumed here to be bounded above and below away from 0. It is worth noting that f_i's are not explicit functions of t. They depend on t only through Y_t.

Observe that Y_t is a mean-reverting process and $1/\epsilon$ is the rate of mean-reversion so that Y_t is fast mean-reverting. Furthermore, Y_t admits the unique invariant normal distribution $\mathcal{N}(m, \nu^2)$.

For a fixed time $T > 0$, our objective is to find, for $t < T$, the joint distribution

$$\mathbb{P}\left\{ X_T^{(1)} \leq \xi_1, X_T^{(2)} \leq \xi_2 \middle| \mathbf{X}_t = \mathbf{x}, Y_t = y \right\}$$

and the two marginal distributions

$$\mathbb{P}\left\{ X_T^{(1)} \leq \xi_1 \middle| \mathbf{X}_t = \mathbf{x}, Y_t = y \right\}, \quad \mathbb{P}\left\{ X_T^{(2)} \leq \xi_2 \middle| \mathbf{X}_t = \mathbf{x}, Y_t = y \right\},$$

where $\mathbf{X}_t \equiv (X_t^{(1)}, X_t^{(2)})$, $\mathbf{x} \equiv (x_1, x_2)$, and ξ_1, ξ_2 are two arbitrary numbers. Equivalently, we need to find the following three transition densities:

$$\begin{aligned}
u^\epsilon &\equiv \mathbb{P}\left\{ X_T^{(1)} \in \mathrm{d}\xi_1, X_T^{(2)} \in \mathrm{d}\xi_2 \middle| \mathbf{X}_t = \mathbf{x}, Y_t = y \right\}, \\
v_1^\epsilon &\equiv \mathbb{P}\left\{ X_T^{(1)} \in \mathrm{d}\xi_1 \middle| \mathbf{X}_t = \mathbf{x}, Y_t = y \right\}, \\
v_2^\epsilon &\equiv \mathbb{P}\left\{ X_T^{(2)} \in \mathrm{d}\xi_2 \middle| \mathbf{X}_t = \mathbf{x}, Y_t = y \right\},
\end{aligned}$$

where we show the dependence on the small parameter ϵ.

1.2 PDE Representation

Let us consider u^ϵ first. In terms of partial differential equation (PDE), u^ϵ satisfies the following Kolmogorov backward equation

$$\begin{aligned}
\mathcal{L}^\epsilon u^\epsilon(t, x_1, x_2, y) &= 0, \\
u^\epsilon(T, x_1, x_2, y) &= \delta(\xi_1; x_1)\delta(\xi_2; x_2),
\end{aligned}$$

where $\delta(\xi_i; x_i)$ is the Dirac delta function of ξ_i with spike at $\xi_i = x_i$ for $i = 1, 2$, and operator $\mathcal{L}^\epsilon$ has the following decomposition:

$$\mathcal{L}^\epsilon = \frac{1}{\epsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathcal{L}_1 + \mathcal{L}_2,$$

with the notations:

$$\mathcal{L}_0 = (m - y)\frac{\partial}{\partial y} + \nu^2 \frac{\partial^2}{\partial y^2}, \tag{1}$$

$$\mathcal{L}_1 = \nu\sqrt{2}\,\rho_{1Y} f_1(y)\frac{\partial^2}{\partial x_1 \partial y} + \nu\sqrt{2}\,\rho_{2Y} f_2(y)\frac{\partial^2}{\partial x_2 \partial y}, \tag{2}$$

$$\mathcal{L}_2 = \frac{\partial}{\partial t} + \frac{1}{2}f_1^2(y)\frac{\partial^2}{\partial x_1^2} + \frac{1}{2}f_2^2(y)\frac{\partial^2}{\partial x_2^2} + \rho f_1(y) f_2(y)\frac{\partial^2}{\partial x_1 \partial x_2}. \tag{3}$$

As in Fouque, Papanicolaou and Sircar (2000) [1], we expand the solution u^ϵ in powers of $\sqrt{\epsilon}$:

$$u^\epsilon = u_0 + \sqrt{\epsilon}\, u_1 + \epsilon u_2 + \epsilon^{3/2} u_3 + \cdots.$$

In the following, we will determine the first few terms appearing on the right hand side of the above expansion. Specifically, we will retain

$$\bar{u} \equiv u_0 + \sqrt{\epsilon}\, u_1, \tag{4}$$

as an approximation to u^ϵ (later we will propose another approximation in order to restore positiveness.)

1.3 Leading Order Term u_0

Following Fouque, Papanicolaou and Sircar (2000) [1], the leading order term u_0, which is independent of variable y, is characterized by:

$$\begin{aligned} \langle \mathcal{L}_2 \rangle u_0(t, x_1, x_2) &= 0, \\ u_0(T, x_1, x_2) &= \delta(\xi_1; x_1)\delta(\xi_2; x_2), \end{aligned} \tag{5}$$

where $\langle \cdot \rangle$ denotes the average with respect to the invariant distribution $\mathcal{N}(m, \nu^2)$ of Y_t, i.e.,

$$\langle g \rangle \equiv \int_{-\infty}^{\infty} g(y) \frac{1}{\nu\sqrt{2\pi}} \exp\left\{ -\frac{(y-m)^2}{2\nu^2} \right\} \mathrm{d}y$$

for a general function g of y.

We define the effective volatilities $\bar{\sigma}_1$ and $\bar{\sigma}_2$, and the effective correlation $\bar{\rho}$ by:

$$\bar{\sigma}_1 \equiv \sqrt{\langle f_1^2 \rangle}, \quad \bar{\sigma}_2 \equiv \sqrt{\langle f_2^2 \rangle}, \quad \bar{\rho} \equiv \frac{\rho \langle f_1 f_2 \rangle}{\bar{\sigma}_1 \bar{\sigma}_2}. \tag{6}$$

Using the definition (3) and the notations (6), equation (5) becomes

$$\begin{aligned} \frac{\partial u_0}{\partial t} + \frac{1}{2}\bar{\sigma}_1^2 \frac{\partial^2 u_0}{\partial x_1^2} + \frac{1}{2}\bar{\sigma}_2^2 \frac{\partial^2 u_0}{\partial x_2^2} + \bar{\rho}\bar{\sigma}_1\bar{\sigma}_2 \frac{\partial^2 u_0}{\partial x_1 \partial x_2} &= 0, \\ u_0(T, x_1, x_2) &= \delta(\xi_1; x_1)\delta(\xi_2; x_2). \end{aligned}$$

It can be verified that u_0 is the transition density of two correlated scaled Brownian motions with instantaneous correlation $\bar{\rho}$ and scale factors $\bar{\sigma}_1$ and $\bar{\sigma}_2$, respectively. That is,

$$\begin{aligned} u_0(t, x_1, x_2) = & \frac{1}{2\pi\bar{\sigma}_1\bar{\sigma}_2(T-t)\sqrt{1-\bar{\rho}^2}} \\ & \exp\left\{ -\frac{1}{2(1-\bar{\rho}^2)} \left[\frac{(\xi_1 - x_1)^2}{\bar{\sigma}_1^2(T-t)} - 2\bar{\rho}\frac{(\xi_1 - x_1)(\xi_2 - x_2)}{\bar{\sigma}_1\bar{\sigma}_2(T-t)} + \frac{(\xi_2 - x_2)^2}{\bar{\sigma}_2^2(T-t)} \right] \right\}. \end{aligned} \tag{7}$$

1.4 Correction Term $\sqrt{\epsilon}\, u_1$

Again, similar to Fouque, Papanicolaou and Sircar (2000) [1], the correction term u_1, which is also independent of variable y, is characterizd by:

$$\begin{aligned} \langle \mathcal{L}_2 \rangle u_1(t, x_1, x_2) &= \mathcal{A} u_0, \\ u_1(T, x_1, x_2) &= 0, \end{aligned} \tag{8}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A} = \langle \mathcal{L}_1 \mathcal{L}_0^{-1} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) \rangle,$$

and the inverse $\mathcal{L}_0^{-1}$ is taken on the centered quantity $\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle$.

From the definition (3) of $\mathcal{L}_2$, it is straightforward to obtain that

$$\begin{aligned} & \mathcal{L}_2 - \langle \mathcal{L}_2 \rangle \\ = \;& \frac{1}{2}(f_1^2(y) - \langle f_1^2 \rangle) \frac{\partial^2}{\partial x_1^2} + \frac{1}{2}(f_2^2(y) - \langle f_2^2 \rangle) \frac{\partial^2}{\partial x_2^2} + \rho (f_1(y) f_2(y) - \langle f_1 f_2 \rangle) \frac{\partial^2}{\partial x_1 \partial x_2}. \end{aligned}$$

Let us denote by $\phi_1(y), \phi_2(y)$ and $\phi_{12}(y)$ solutions of the following Poisson equations respectively

$$\begin{aligned} \mathcal{L}_0 \phi_1(y) &= f_1^2(y) - \langle f_1^2 \rangle, \\ \mathcal{L}_0 \phi_2(y) &= f_2^2(y) - \langle f_2^2 \rangle, \\ \mathcal{L}_0 \phi_{12}(y) &= f_1(y) f_2(y) - \langle f_1 f_2 \rangle. \end{aligned}$$

Their existence (with at most polynomial growth at infinity) is guarantied by the centering property of the right hand sides and the Fredholm alternative for the infinitesimal generator $\mathcal{L}_0$. They are defined up to additive constants in y which will play no role after applying the operator $\mathcal{L}_1$ which takes derivatives with respect to y. It then follows that

$$\mathcal{L}_0^{-1} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) = \frac{1}{2} \phi_1(y) \frac{\partial^2}{\partial x_1^2} + \frac{1}{2} \phi_2(y) \frac{\partial^2}{\partial x_2^2} + \rho \phi_{12}(y) \frac{\partial^2}{\partial x_1 \partial x_2}.$$

Now by the definition (2) of $\mathcal{L}_1$, we have

$$\begin{aligned} \mathcal{L}_1 \mathcal{L}_0^{-1} (\mathcal{L}_2 - \langle \mathcal{L}_2 \rangle) &= \nu \sqrt{2}\, \rho_{1Y} f_1(y) \left[\frac{1}{2} \phi_1'(y) \frac{\partial^3}{\partial x_1^3} + \frac{1}{2} \phi_2'(y) \frac{\partial^3}{\partial x_1 \partial x_2^2} + \rho \phi_{12}'(y) \frac{\partial^3}{\partial x_1^2 \partial x_2} \right] \\ &\quad + \nu \sqrt{2}\, \rho_{2Y} f_2(y) \left[\frac{1}{2} \phi_1'(y) \frac{\partial^3}{\partial x_1^2 \partial x_2} + \frac{1}{2} \phi_2'(y) \frac{\partial^3}{\partial x_2^3} + \rho \phi_{12}'(y) \frac{\partial^3}{\partial x_1 \partial x_2^2} \right]. \end{aligned}$$

Therefore the operator $\sqrt{\epsilon} \mathcal{A}$ can be written

$$\sqrt{\epsilon} \mathcal{A} = R_1 \frac{\partial^3}{\partial x_1^3} + R_2 \frac{\partial^3}{\partial x_2^3} + R_{12} \frac{\partial^3}{\partial x_1 \partial x_2^2} + R_{21} \frac{\partial^3}{\partial x_1^2 \partial x_2}$$

where the constant parameters R_1, R_2, R_{12} and R_{21} are defined as follows:

$$R_1 \equiv \frac{\nu \rho_{1Y} \sqrt{\epsilon}}{\sqrt{2}} \langle f_1 \phi_1' \rangle,$$

$$
\begin{aligned}
R_2 &\equiv \frac{\nu\rho_{2Y}\sqrt{\epsilon}}{\sqrt{2}}\langle f_2\phi_2'\rangle, \\
R_{12} &\equiv \frac{\nu\rho_{1Y}\sqrt{\epsilon}}{\sqrt{2}}\langle f_1\phi_2'\rangle + \nu\sqrt{2\epsilon}\,\rho\rho_{2Y}\langle f_2\phi_{12}'\rangle, \\
R_{21} &\equiv \frac{\nu\rho_{2Y}\sqrt{\epsilon}}{\sqrt{2}}\langle f_2\phi_1'\rangle + \nu\sqrt{2\epsilon}\,\rho\rho_{1Y}\langle f_1\phi_{12}'\rangle.
\end{aligned}
$$

Note that they are all small of order $\sqrt{\epsilon}$.

It can be checked directly that u_1 is given explicitly by

$$
u_1 = -(T-t)\mathcal{A}u_0,
$$

and therefore

$$
\sqrt{\epsilon}\,u_1 = -(T-t)\left[R_1\frac{\partial^3}{\partial x_1^3} + R_2\frac{\partial^3}{\partial x_2^3} + R_{12}\frac{\partial^3}{\partial x_1\partial x_2^2} + R_{21}\frac{\partial^3}{\partial x_1^2\partial x_2}\right]u_0. \tag{9}
$$

Explicit formulas for the third order partial derivatives of u_0 are given in Appendix A.

1.5 Regularity Conditions for Density Functions

Since

$$
1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u_0(t,x_1,x_2;\xi_1,\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2,
$$

by Lebesgue dominated convergence theorem, we then have

$$
0 = \frac{\partial^{k_1+k_2}1}{\partial x_1^{k_1}\partial x_2^{k_2}} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial^{k_1+k_2}}{\partial x_1^{k_1}\partial x_2^{k_2}}u_0(t,x_1,x_2;\xi_1,\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2,
$$

for integers $k_1, k_2 \geq 0$. It follows that

$$
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sqrt{\epsilon}\,u_1(t,x_1,x_2;\xi_1,\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2 = 0,
$$

and hence

$$
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\bar{u}(t,x_1,x_2;\xi_1,\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2 = 1,
$$

where $\bar{u} = u_0 + \sqrt{\epsilon}\,u_1$ is the approximation introduced in (4).

In order to guarantee that our approximated transition density function is always non-negative, which is the other regularity condition for a density function, we seek a *multiplicative perturbation* of the form

$$
\tilde{u} \equiv \hat{u}_0(1+\tanh(\sqrt{\epsilon}\,\hat{u}_1)),
$$

where $\hat{u}_0$ and $\hat{u}_1$ are defined such that

$$u_0 + \sqrt{\epsilon}\, u_1 = \hat{u}_0(1 + \sqrt{\epsilon}\, \hat{u}_1)$$

for any $\epsilon > 0$. It can be easily seen that this is achieved with the choice:

$$\hat{u}_0 = u_0, \quad \hat{u}_1 = u_1/u_0.$$

Now instead of using $\bar{u}$ as our approximation for u^ϵ, we use

$$\begin{aligned} \tilde{u} &= u_0[1 + \tanh(\sqrt{\epsilon}\, u_1/u_0)] \qquad (10) \\ &= u_0 \left\{ 1 + \tanh\left(-(T-t)\frac{1}{u_0}\left[R_1 \frac{\partial^3 u_0}{\partial x_1^3} + R_2 \frac{\partial^3 u_0}{\partial x_2^3} + R_{12} \frac{\partial^3 u_0}{\partial x_1 \partial x_2^2} + R_{21} \frac{\partial^3 u_0}{\partial x_1^2 \partial x_2} \right] \right) \right\}. \end{aligned}$$

Before proving that $\tilde{u}$ given in (10) is indeed a probability density function, we clarify a definition first.

Definition 1 *Let g be a function of n variables $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. The function g is called an n-dimensional even function if*

$$g(-x_1, -x_2, \ldots, -x_n) = g(x_1, x_2, \ldots, x_n)$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, and an n-dimensional odd function if

$$g(-x_1, -x_2, \ldots, -x_n) = -g(x_1, x_2, \ldots, x_n)$$

for all $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

With this definition, we can state the following proposition.

Proposition 1 *Let $g(\mathbf{x})$ be a probability density function on $\mathbb{R}^n$ for $n \geq 1$, and $\varphi(\mathbf{x})$ be an odd function. If g is an even function, then the function f defined by*

$$f(\mathbf{x}) = (1 + \tanh(\varphi(\mathbf{x})))\, g(\mathbf{x})$$

is also a probability density function on $\mathbb{R}^n$.

Proof We need to prove that f is globally non-negative and its integral over $\mathbb{R}^n$ is equal to one. Observe that $\tanh(\cdot)$ is strictly between -1 and 1, and this together with the non-negativity of g justifies that f is always non-negative. On the other hand, $\tanh(\cdot)$ is a (1-dimensional) odd function, and hence $\tanh(\varphi(\mathbf{x}))$ is an (n-dimensional) odd function. Now by change of variables $\mathbf{y} = -\mathbf{x}$, we have

$$\begin{aligned} I &\equiv \int_{\mathbb{R}^n} \tanh(\varphi(\mathbf{x})) g(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} \tanh(\varphi(-\mathbf{y})) g(-\mathbf{y}) \mathrm{d}\mathbf{y} \\ &= -\int_{\mathbb{R}^n} \tanh(\varphi(\mathbf{y})) g(\mathbf{y}) \mathrm{d}\mathbf{y} = -I, \end{aligned}$$

which implies that $I = 0$. Therefore

$$\int_{\mathbb{R}^n} f(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^n} g(\mathbf{x}) \mathrm{d}\mathbf{x} + I = 1 + 0 = 1.$$

The proof is complete. ■

Now observe that u_0 is a probability density function with respect to the variables (ξ_1, ξ_2), and is even on $(\xi_1 - x_1, \xi_2 - x_2)$. In addition, $\sqrt{\epsilon}\, u_1/u_0$ is an odd function on $(\xi_1 - x_1, \xi_2 - x_2)$. By Proposition 1 we know that $\tilde{u}$ given in (10) is indeed a probability density function.

As for the approximation accuracy $|\tilde{u} - u^\epsilon|$, we first note that

$$\tanh(x) \approx x - \frac{x^3}{3},$$

when x is close to 0. Now for fixed (t, x_1, x_2), when ϵ is small, we have

$$\begin{aligned} \tilde{u} &= u_0[1 + \tanh(\sqrt{\epsilon}\, u_1/u_0)] \\ &\approx u_0 \left[1 + \frac{\sqrt{\epsilon}\, u_1}{u_0} - \frac{1}{3} \left(\frac{\sqrt{\epsilon}\, u_1}{u_0} \right)^3 \right] \\ &= u_0 + \sqrt{\epsilon}\, u_1 - \epsilon^{3/2} \left(\frac{u_1^3}{3u_0^2} \right) \;=\; \bar{u} - \epsilon^{3/2} \left(\frac{u_1^3}{3u_0^2} \right). \end{aligned}$$

Therefore $|\tilde{u} - \bar{u}|$ is small of order $\epsilon^{3/2}$, while $|\bar{u} - u^\epsilon|$ is small of order ϵ (see Fouque et al. (2003) [2]). Thus $|\tilde{u} - u^\epsilon|$ is small of the same order of ϵ as $|\bar{u} - u^\epsilon|$, i.e., the approximation accuracy remains unchanged when replacing $\bar{u}$ by $\tilde{u}$.

1.6 Marginal Transition Densities

For the marginal transition density function

$$v_1^\epsilon \;\equiv\; \mathbb{P}\left\{ X_T^{(1)} \in \mathrm{d}\xi_1 \middle| \mathbf{X}_t = \mathbf{x}, Y_t = y \right\},$$

the above argument goes analogously, and we obtain

$$v_1^\epsilon \approx \bar{v}_1 \equiv p_1(t, x_1; T, \xi_1 | \bar{\sigma}_1) - (T - t) R_1 \frac{\partial^3}{\partial x_1^3} p_1(t, x_1; T, \xi_1 | \bar{\sigma}_1),$$

where $p_1(t, x_1; T, \xi_1 | \bar{\sigma}_1)$ is the transition density of the scaled Brownian motion with scale factor $\bar{\sigma}_1$, that is,

$$p_1(t, x_1; T, \xi_1 | \bar{\sigma}_1) = \frac{1}{\sqrt{2\pi(T - t)}\, \bar{\sigma}_1} \exp\left\{ -\frac{(\xi_1 - x_1)^2}{2\bar{\sigma}_1^2 (T - t)} \right\}.$$

A straightforward calculation shows that

$$\frac{\partial^3 p_1}{\partial x_1^3} \;=\; \left[-\frac{3(\xi_1 - x_1)}{\sqrt{2\pi}\, \bar{\sigma}_1^5 (T - t)^{5/2}} + \frac{(\xi_1 - x_1)^3}{\sqrt{2\pi}\, \bar{\sigma}_1^7 (T - t)^{7/2}} \right] \exp\left\{ -\frac{(\xi_1 - x_1)^2}{2\bar{\sigma}_1^2 (T - t)} \right\}.$$

Note again that

$$\int_{-\infty}^{\infty} \bar{v}_1(t, x_1; T, \xi_1) \mathrm{d}\xi_1 = 1.$$

To guarantee the non-negativity of the approximated density function, we, again, use instead

$$\tilde{v}_1 \equiv p_1 \left[1 + \tanh \left(-(T-t) R_1 \frac{1}{p_1} \frac{\partial^3 p_1}{\partial x_1^3} \right) \right]$$

as our approximation to v_1^ϵ.

By symmetry we have

$$\begin{aligned}
v_2^\epsilon &\equiv \mathbb{P}\left\{ X_T^{(2)} \in \mathrm{d}\xi_2 \middle| \mathbf{X}_t = \mathbf{x}, Y_t = y \right\} \\
&\approx \bar{v}_2 \equiv p_2(t, x_2; T, \xi_2 | \bar{\sigma}_2) - (T-t) R_2 \frac{\partial^3}{\partial x_2^3} p_2(t, x_2; T, \xi_2 | \bar{\sigma}_2) \\
&\approx \tilde{v}_2 \equiv p_2 \left[1 + \tanh \left(-(T-t) R_2 \frac{1}{p_2} \frac{\partial^3 p_2}{\partial x_2^3} \right) \right],
\end{aligned}$$

where

$$\begin{aligned}
p_2(t, x_2; T, \xi_2 | \bar{\sigma}_2) &= \frac{1}{\sqrt{2\pi(T-t)}\,\bar{\sigma}_2} \exp\left\{ -\frac{(\xi_2 - x_2)^2}{2\bar{\sigma}_2^2 (T-t)} \right\}, \\
\frac{\partial^3 p_2}{\partial x_2^3} &= \left[-\frac{3(\xi_2 - x_2)}{\sqrt{2\pi}\,\bar{\sigma}_2^5 (T-t)^{5/2}} + \frac{(\xi_2 - x_2)^3}{\sqrt{2\pi}\,\bar{\sigma}_2^7 (T-t)^{7/2}} \right] \exp\left\{ -\frac{(\xi_2 - x_2)^2}{2\bar{\sigma}_2^2 (T-t)} \right\},
\end{aligned}$$

and $\tilde{v}_2$ is our approximation to v_2^ϵ.

By exactly the same argument used for $\tilde{u}$, one can show that $\tilde{v}_1$ and $\tilde{v}_2$ are indeed probability density functions of ξ_1 and ξ_2, respectively. Furthermore, the approximation accuracies remain unchanged when switching from $\bar{v}_1$ to $\tilde{v}_1$, and from $\bar{v}_2$ to $\tilde{v}_2$.

2 Density of the Perturbed Copula

2.1 Approximated Copula Density

Now suppose that conditional on $\{\mathbf{X}_t = \mathbf{x}, Y_t = y\}$, $(X_T^{(1)}, X_T^{(2)})$ admits the copula $\Psi(\cdot, \cdot)$, then, by Sklar's Theorem, its density function $\psi(\cdot, \cdot)$ can be represented as

$$\psi(z_1, z_2) = \frac{u^\epsilon(t, x_1, x_2, y; T, \xi_1, \xi_2)}{v_1^\epsilon(t, x_1, y; T, \xi_1)\; v_2^\epsilon(t, x_2, y; T, \xi_2)},$$

where

$$\begin{aligned}
z_1 &= \mathbb{P}\left\{ X_T^{(1)} \le \xi_1 \middle| \mathbf{X}_t = \mathbf{x}, Y_t = y \right\}, \\
z_2 &= \mathbb{P}\left\{ X_T^{(2)} \le \xi_2 \middle| \mathbf{X}_t = \mathbf{x}, Y_t = y \right\}.
\end{aligned}$$

Observe that if the volatility terms $(f_1(\cdot), f_2(\cdot))$ for $(X_t^{(1)}, X_t^{(2)})$ were constant numbers, say, the process $\{Y_t\}_{t\leq T}$ was constant or the f_i's were both identically constant, then Ψ would be a Gaussian copula.

Using our approximations to u^ϵ, v_1^ϵ and v_2^ϵ, we have

$$\psi(\zeta_1, \zeta_2) \approx \tilde{\psi}(\zeta_1, \zeta_2) \equiv \frac{\tilde{u}(t, x_1, x_2; T, \xi_1, \xi_2)}{\tilde{v}_1(t, x_1; T, \xi_1)\, \tilde{v}_2(t, x_2; T, \xi_2)}, \tag{11}$$

where

$$\begin{aligned}
\zeta_1 &= \int_{-\infty}^{\xi_1} \tilde{v}_1(t, x_1; T, \xi_1)\mathrm{d}\xi_1 \\
&= \int_{-\infty}^{\xi_1} p_1(t, x_1; T, \xi_1) \\
&\qquad \left[1 + \tanh\left(-(T-t)R_1 \frac{1}{p_1(t, x_1; T, \xi_1)} \frac{\partial^3 p_1(t, x_1; T, \xi_1)}{\partial x_1^3}\right)\right] \mathrm{d}\xi_1, \\
\zeta_2 &= \int_{-\infty}^{\xi_2} \tilde{v}_2(t, x_2; T, \xi_2)\mathrm{d}\xi_2 \\
&= \int_{-\infty}^{\xi_2} p_2(t, x_2; T, \xi_2) \\
&\qquad \left[1 + \tanh\left(-(T-t)R_2 \frac{1}{p_2(t, x_2; T, \xi_2)} \frac{\partial^3 p_2(t, x_2; T, \xi_2)}{\partial x_2^3}\right)\right] \mathrm{d}\xi_2.
\end{aligned}$$

The function $\tilde{u}$ is given by (10), and the marginals (p_1, p_2) and their derivatives $\frac{\partial^3 p_1}{\partial x_1^3}, \frac{\partial^3 p_2}{\partial x_2^3}$ are given explicitly in the previous section 1.6.

Before justifying that $\tilde{\psi}$ is a probability density function defined on the unit square $[0,1]^2$, we need the following proposition.

Proposition 2 *Suppose function $\Theta(x_1, x_2, \ldots, x_n)$ is an n-dimensional probability density function on $\mathbb{R}^n$ for $n \geq 2$, and $h_1(x_1), h_2(x_2), \ldots, h_n(x_n)$ are 1-dimensional strictly positive probability density functions. Then the function c defined on the unit hyper-square $[0,1]^n$ by*

$$c(z_1, z_2, \ldots, z_n) = \frac{\Theta(x_1, x_2, \ldots, x_n)}{\prod_{i=1}^n h_i(x_i)}$$

with $z_i \in [0,1]$ given by

$$z_i = \int_{-\infty}^{x_i} h_i(y_i)\mathrm{d}y_i$$

is a probability density function on $[0,1]^n$. Furthermore, c is a copula density function if and only if $h_1(x_1), h_2(x_2), \ldots, h_n(x_n)$ are the marginal density functions of $\Theta(x_1, x_2, \ldots, x_n)$, meaning that

$$h_i(x_i) = \int_{\mathbb{R}^{n-1}} \Theta(x_1, x_2, \ldots, x_n)\mathrm{d}x_1\mathrm{d}x_2 \cdots \mathrm{d}x_{i-1}\mathrm{d}x_{i+1} \cdots \mathrm{d}x_n$$

for every $i = 1, 2, \ldots, n$.

PROOF Let H_i be the cumulative distribution function of h_i. Then H_i is strictly increasing, implying the existence of its inverse function, and

$$z_i = H_i(x_i), \quad \text{or equivalently,} \quad x_i = H_i^{-1}(z_i)$$

for each i. Since Θ is non-negative, and h_i's are strictly positive, the function c is non-negative. On the other hand,

$$\begin{aligned}
& \int_{[0,1]^n} c(z_1, z_2, \ldots, z_n) \mathrm{d}z_1 \mathrm{d}z_2 \cdots \mathrm{d}z_n \\
= & \int_{\mathbb{R}^n} c\left(H_1(x_1), H_2(x_2), \ldots, H_n(x_n)\right) \prod_{i=1}^{n} h_i(x_i)\, \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n \\
= & \int_{\mathbb{R}^n} \Theta(x_1, x_2, \ldots, x_n)\, \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n \;=\; 1.
\end{aligned}$$

Therefore $c(z_1, z_2, \ldots, z_n)$ is a probability density function on $[0,1]^n$.

Now if the additional condition is satisfied, then we have

$$\begin{aligned}
& \int_{[0,1]^{n-1}} c(z_1, z_2, \ldots, z_n) \mathrm{d}z_2 \cdots \mathrm{d}z_n \\
= & \int_{\mathbb{R}^{n-1}} c\left(z_1, H_2(x_2), \ldots, H_n(x_n)\right) \prod_{i=2}^{n} h_i(x_i)\, \mathrm{d}x_2 \cdots \mathrm{d}x_n \\
= & \frac{1}{h_1(x_1)} \int_{\mathbb{R}^{n-1}} \Theta(x_1, x_2, \ldots, x_n)\, \mathrm{d}x_2 \cdots \mathrm{d}x_n \;=\; 1.
\end{aligned}$$

This is to say that the marginal density function for the variable z_1 is one, and hence the marginal distribution for the variable z_1 is uniform. Similarly, we can show that the marginal distributions for the variables $z_2, \ldots, z_n$ are also uniform. By definition of copula, we know that function c is a copula density function. The converse can be obtained by reversing the above argument. The proof is complete. ■

Now from definition (11) of $\tilde{\psi}$, by combining the fact that $\tilde{u}$, $\tilde{v}_1$ and $\tilde{v}_2$ are all probability density functions, one can see that $\tilde{\psi}$ is a density function on $[0,1]^2$ by applying Proposition 2. However, $\tilde{\psi}$ is not a copula density function in general, because the additional condition required in Proposition 2 is not satisfied in general in our case, and hence $\tilde{\Psi}$, the "copula" corresponding to density function $\tilde{\psi}$, is not an exact copula in general.

Asymptotically, when ϵ goes to 0, for fixed (t, x_1, x_2), the density $\tilde{\psi}$ converges to

$$\phi(z_1, z_2) \equiv \frac{u_0(t, x_1, x_2; T, \xi_1, \xi_2)}{p_1(t, x_1; T, \xi_1)\; p_2(t, x_2; T, \xi_2)}$$

with

$$z_i = \int_{-\infty}^{\xi_i} p_i(t, x_i; T, \xi_i) \mathrm{d}\xi_i = \mathrm{N}\left(\frac{\xi_i - x_i}{\bar{\sigma}_i \sqrt{T-t}}\right)$$

for $i = 1, 2$, where $\mathrm{N}(\cdot)$ denotes the univariate standard normal cumulative distribution function. One should observe that $\phi(\cdot,\cdot)$ is the two-dimensional Gaussian copula density function with correlation parameter $\bar{\rho}$, and that it **depends only on the parameter** $\bar{\rho}$, independent of any other variables/parameters, including $x_1, x_2, t, T, \bar{\sigma}_1, \bar{\sigma}_2$, etc.

As a consequence, $\tilde{\Psi}$ **converges** to the Gaussian copula Φ with correlation parameter $\bar{\rho}$. Since the method used in this paper is a perturbation method, we call $\tilde{\Psi}$ a **perturbed Gaussian copula**.

2.2 Numerical Results

In this subsection, we illustrate the effectiveness of our approximation method by showing some numerical results.

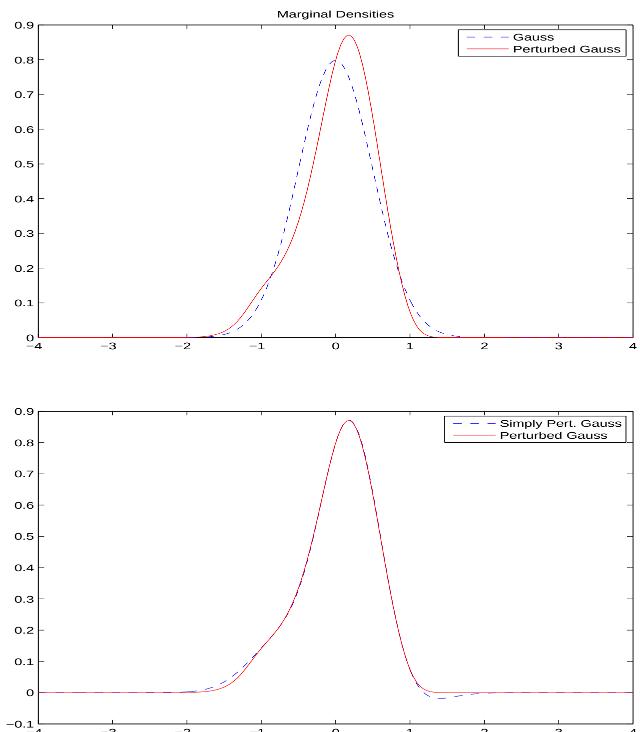

Figure 1: Perturbed Gaussian densities

In Figure 1, we plot $p_1(t, x_1; T, \xi_1)$, $\bar{v}_1(t, x_1; T, \xi_1)$ and $\tilde{v}_1(t, x_1; T, \xi_1)$ as functions of ξ_1. Note that $p_1(t, x_1; T, \xi_1)$ is a standard Gaussian density without any perturbation. The upper graph demonstrates the difference between $p_1(t, x_1; T, \xi_1)$ (standard Gaussian) and $\tilde{v}_1(t, x_1; T, \xi_1)$ (perturbed Gaussian), and the lower one between $\bar{v}_1(t, x_1; T, \xi_1)$ (simply perturbed Gaussian) and $\tilde{v}_1(t, x_1; T, \xi_1)$ (perturbed Gaussian).

It can be seen from Figure 1 that

- $\bar{v}_1(t, x_1; T, \xi_1)$ (simply perturbed Gaussian) takes on negative values at some places;
- $\tilde{v}_1(t, x_1; T, \xi_1)$ (perturbed Gaussian), however, does not take on negative values, which is guaranteed by its formation;
- $\bar{v}_1(t, x_1; T, \xi_1)$ and $\tilde{v}_1(t, x_1; T, \xi_1)$ are almost globally identical, which justifies the modification of the form $1 + \tanh(\cdot)$;

- $\tilde{v}_1(t, x_1; T, \xi_1)$ is considerably different from $p_1(t, x_1; T, \xi_1)$ (standard Gaussian); specifically, it shifts to the right from $p_1(t, x_1; T, \xi_1)$;

- Despite the difference between $\tilde{v}_1(t, x_1; T, \xi_1)$ and $p_1(t, x_1; T, \xi_1)$, the areas under them do seem to be of the same size, which is justified by the fact that both are probability density functions and hence the overall integrals should both be one.

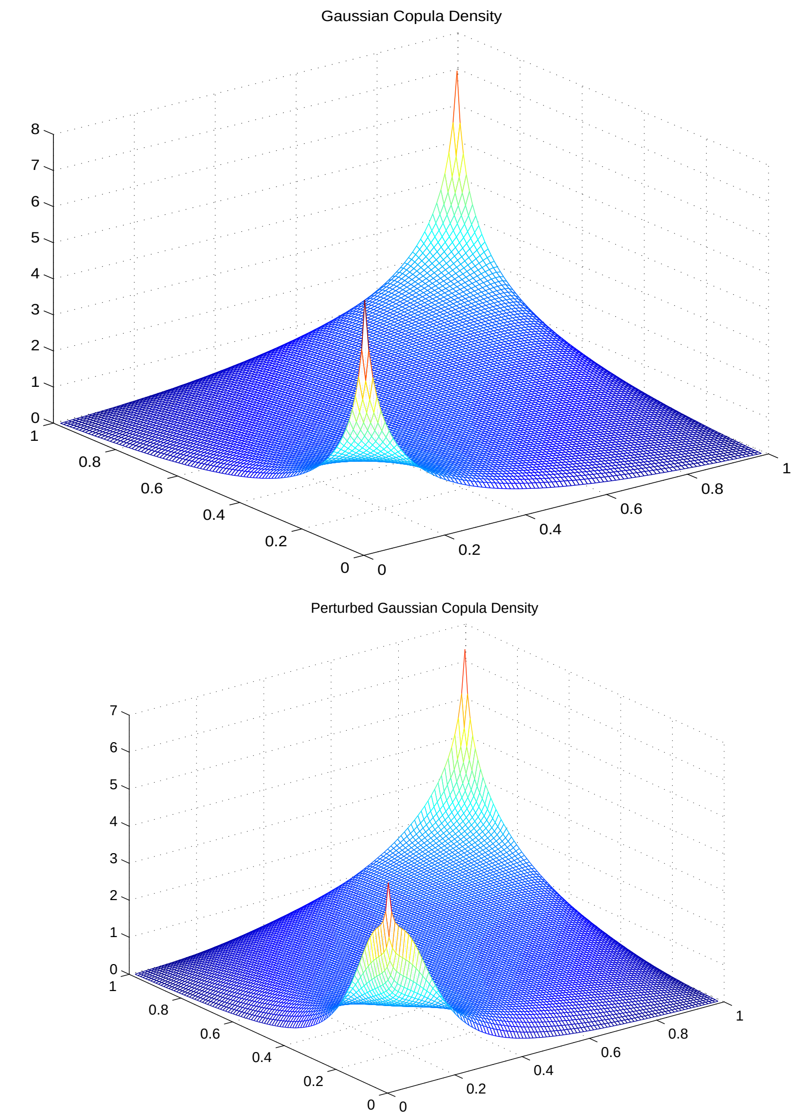

Figure 2: Gaussian copula and perturbed Gaussian copula densities

In Figure 2, we plot $\tilde{\psi}(\cdot, \cdot)$ in the lower graph and $\phi(\cdot, \cdot)$ in the upper graph, the Gaussian copula density that $\tilde{\psi}$ converges to when ϵ goes to 0. It can be seen from Figure 2 that the standard Gaussian copula density (upper graph) and the perturbed Gaussian copula density (lower graph) both present singularities at $(0, 0)$ and $(1, 1)$ but the perturbed one has more tail dependence at $(0, 0)$. Our numerous numerical experiments show that this picture is extremely sensitive to the choice of parameters and gives a lot of flexibility to the shape of the perturbed Gaussian copula density (the Matlab code is available on demand).

Tail dependence is a very important property for a copula, especially when this copula is to be used in modeling default correlation. The essence of tail dependence is the interdependence when extreme events occur, say, defaults of corporate bonds. The lack of tail dependence has been for years a major criticism on standard Gaussian copula.

Throughout the computation, we used the following parameters:

$$\begin{aligned} R_1 = 0.02, &\quad R_2 = 0.02, &\quad R_{12} = 0.03, \\ R_{21} = 0.03, &\quad \bar{\rho} = 0.5, &\quad T - t = 1, \\ \bar{\sigma}_1 = 0.5, &\quad \bar{\sigma}_2 = 0.5, &\quad x_1 = 0,\, x_2 = 0. \end{aligned}$$

3 Conclusion

In summary, based on a stochastic volatility model, we derived an approximate copula function by way of singular perturbation that was introduced by Fouque, Papanicolaou and Sircar (2000) [1]. During the approximation, however, in order to make the candidate probability density functions globally non-negative, instead of directly using the obtained perturbation result as in [1], we introduced a multiplicative modification, namely the $1+\tanh(\cdot)$ form. It turns out that this modification is both necessary (to restore positiveness) and sufficient to guarantee the resulting functions to be density functions. Finally the resulting approximate copula — the so-called **perturbed Gaussian copula** in this paper — has a very desirable property compared to standard Gaussian copula: tail dependence at point $(0,0)$. Some numerical results were provided and they strongly supported the methods described above, both the singular perturbation and the modification.

References

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A Explicit Formulas

As obtained in Section 1.3:

$$
\begin{aligned}
u_0(t,x_1,x_2) &= \frac{1}{2\pi\bar\sigma_1\bar\sigma_2(T-t)\sqrt{1-\bar\rho^2}} \\
&\quad \exp\left\{-\frac{1}{2(1-\bar\rho^2)}\left[\frac{(\xi_1-x_1)^2}{\bar\sigma_1^2(T-t)} - 2\bar\rho\frac{(\xi_1-x_1)(\xi_2-x_2)}{\bar\sigma_1\bar\sigma_2(T-t)} + \frac{(\xi_2-x_2)^2}{\bar\sigma_2^2(T-t)}\right]\right\}.
\end{aligned}
$$

By a straightforward calculation and symmetry, we obtain

$$
\begin{aligned}
\frac{\partial^3 u_0}{\partial x_1^3} &= \exp\left\{-\frac{1}{2(1-\bar\rho^2)}\left[\frac{(\xi_1-x_1)^2}{\bar\sigma_1^2(T-t)} - 2\bar\rho\frac{(\xi_1-x_1)(\xi_2-x_2)}{\bar\sigma_1\bar\sigma_2(T-t)} + \frac{(\xi_2-x_2)^2}{\bar\sigma_2^2(T-t)}\right]\right\} \\
&\quad \times\left\{\left[-\frac{2(\xi_1-x_1)}{\bar\sigma_1^2} + \frac{2\bar\rho(\xi_2-x_2)}{\bar\sigma_1\bar\sigma_2}\right]\frac{3}{4\pi\bar\sigma_1^3\bar\sigma_2(T-t)^3(1-\bar\rho^2)^{5/2}}\right. \\
&\qquad \left. - \left[-\frac{2(\xi_1-x_1)}{\bar\sigma_1^2} + \frac{2\bar\rho(\xi_2-x_2)}{\bar\sigma_1\bar\sigma_2}\right]^3\frac{1}{16\pi\bar\sigma_1\bar\sigma_2(T-t)^4(1-\bar\rho^2)^{7/2}}\right\}, \\
\frac{\partial^3 u_0}{\partial x_1^2\partial x_2} &= \exp\left\{-\frac{1}{2(1-\bar\rho^2)}\left[\frac{(\xi_1-x_1)^2}{\bar\sigma_1^2(T-t)} - 2\bar\rho\frac{(\xi_1-x_1)(\xi_2-x_2)}{\bar\sigma_1\bar\sigma_2(T-t)} + \frac{(\xi_2-x_2)^2}{\bar\sigma_2^2(T-t)}\right]\right\} \\
&\quad \times\left\{\left[\frac{2\bar\rho(\xi_1-x_1)}{\bar\sigma_1\bar\sigma_2} - \frac{2(\xi_2-x_2)}{\bar\sigma_2^2}\right]\frac{1}{4\pi\bar\sigma_1^3\bar\sigma_2(T-t)^3(1-\bar\rho^2)^{5/2}}\right. \\
&\qquad - \left[-\frac{2(\xi_1-x_1)}{\bar\sigma_1^2} + \frac{2\bar\rho(\xi_2-x_2)}{\bar\sigma_1\bar\sigma_2}\right]\frac{\bar\rho}{2\pi\bar\sigma_1^2\bar\sigma_2^2(T-t)^3(1-\bar\rho^2)^{5/2}} \\
&\qquad - \left[-\frac{2(\xi_1-x_1)}{\bar\sigma_1^2} + \frac{2\bar\rho(\xi_2-x_2)}{\bar\sigma_1\bar\sigma_2}\right]^2\left[\frac{2\bar\rho(\xi_1-x_1)}{\bar\sigma_1\bar\sigma_2} - \frac{2(\xi_2-x_2)}{\bar\sigma_2^2}\right] \\
&\qquad \left.\frac{1}{16\pi\bar\sigma_1\bar\sigma_2(T-t)^4(1-\bar\rho^2)^{7/2}}\right\}.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^3 u_0}{\partial x_2^3} &= \exp\left\{-\frac{1}{2(1-\bar\rho^2)}\left[\frac{(\xi_2-x_2)^2}{\bar\sigma_2^2(T-t)} - 2\bar\rho\frac{(\xi_2-x_2)(\xi_1-x_1)}{\bar\sigma_2\bar\sigma_1(T-t)} + \frac{(\xi_1-x_1)^2}{\bar\sigma_1^2(T-t)}\right]\right\} \\
&\quad \times\left\{\left[-\frac{2(\xi_2-x_2)}{\bar\sigma_2^2} + \frac{2\bar\rho(\xi_1-x_1)}{\bar\sigma_2\bar\sigma_1}\right]\frac{3}{4\pi\bar\sigma_2^3\bar\sigma_1(T-t)^3(1-\bar\rho^2)^{5/2}}\right. \\
&\qquad \left. - \left[-\frac{2(\xi_2-x_2)}{\bar\sigma_2^2} + \frac{2\bar\rho(\xi_1-x_1)}{\bar\sigma_2\bar\sigma_1}\right]^3\frac{1}{16\pi\bar\sigma_2\bar\sigma_1(T-t)^4(1-\bar\rho^2)^{7/2}}\right\}, \\
\frac{\partial^3 u_0}{\partial x_1\partial x_2^2} &= \exp\left\{-\frac{1}{2(1-\bar\rho^2)}\left[\frac{(\xi_2-x_2)^2}{\bar\sigma_2^2(T-t)} - 2\bar\rho\frac{(\xi_2-x_2)(\xi_1-x_1)}{\bar\sigma_2\bar\sigma_1(T-t)} + \frac{(\xi_1-x_1)^2}{\bar\sigma_1^2(T-t)}\right]\right\} \\
&\quad \times\left\{\left[\frac{2\bar\rho(\xi_2-x_2)}{\bar\sigma_2\bar\sigma_1} - \frac{2(\xi_1-x_1)}{\bar\sigma_1^2}\right]\frac{1}{4\pi\bar\sigma_2^3\bar\sigma_1(T-t)^3(1-\bar\rho^2)^{5/2}}\right. \\
&\qquad - \left[-\frac{2(\xi_2-x_2)}{\bar\sigma_2^2} + \frac{2\bar\rho(\xi_1-x_1)}{\bar\sigma_2\bar\sigma_1}\right]\frac{\bar\rho}{2\pi\bar\sigma_2^2\bar\sigma_1^2(T-t)^3(1-\bar\rho^2)^{5/2}} \\
&\qquad - \left[-\frac{2(\xi_2-x_2)}{\bar\sigma_2^2} + \frac{2\bar\rho(\xi_1-x_1)}{\bar\sigma_2\bar\sigma_1}\right]^2\left[\frac{2\bar\rho(\xi_2-x_2)}{\bar\sigma_2\bar\sigma_1} - \frac{2(\xi_1-x_1)}{\bar\sigma_1^2}\right] \\
&\qquad \left.\frac{1}{16\pi\bar\sigma_2\bar\sigma_1(T-t)^4(1-\bar\rho^2)^{7/2}}\right\}.
\end{aligned}
$$