Time reversal detection in one-dimensional random media

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Abstract

In this paper, we study time reversal in reflection of an acoustic wave in a one-dimensional random medium with an embedded reflector. The main result shows that time reversal is an effective tool for imaging a reflector even in the absence of a coherent reflection. We carry out the analysis in the regime of separation of scales, where the probing pulse is large compared to the medium inhomogeneities but small relative to the depth of the reflector. The limiting quantities of interest are given as a solution to a system of transport equations, which is solved by using Monte-Carlo simulations.

1 Introduction

Imaging of an object embedded into a random medium is a long standing, important and complicated problem. Its relevance to various fields ranging from seismic to radar and medical applications is hard to overemphasize. The experiment is typically designed as follows. An acoustic pulse is sent into the medium and the reflections generated by the inhomogeneities of the latter

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as well as by the object are recorded. The recorded signal is then analyzed with the purpose of identifying a foreign inclusion whose acoustic properties are clearly distinct from the those of the background.

A distinctive feature of all imaging methods proposed up to date has been the availability of a direct reflection from the object of interest. (See [1] for the case of imaging in clutter and [2] for the case of imaging in a finely layered medium.) A coherent reflection is physically produced by a significant contrast in the acoustic impedance between the object and its background. While such a contrast alone does not guarantee a quality image, it helps significantly to:

- solve the detection problem, i.e. establish the presence of an object;
- time the reflections and thus approximate the distance from the object to the receiver.

The imaging method proposed in this paper relies on acoustic time reversal. Time reversal of an acoustic wave has been observed, studied and used in various contexts including imaging [3]. A time reversal mirror is a device capable of recording a signal, reversing it in time and sending the resulting signal back into the medium. A celebrated result of the time reversal experiment is the refocusing of the scattered signal. Even more surprisingly, in certain regimes, the shape of the refocused signal depends only on statistics of the medium but not on its particular realization. By comparing the initial pulse with its refocused version, one can then study the part of the medium traveled by the pulse.

In this paper we consider a situation where an object embedded into a one-dimensional random medium has the same or a similar impedance as the rest of the medium. We show that while a traveling wave produces no coherent reflection when meeting the object, we can detect its presence and identify its location by relying solely on incoherent reflections.

We start by looking at the detection problem in a constant medium having in mind to set up the context as well as introduce the relevant machinery and work our way through to more interesting cases of a strong and weak interfaces in a random environment.

2 Reflector in a homogenous medium

In order to introduce notations, we briefly review in this section the mathematical setup of an acoustic pulse scattered by an interface between two homogeneous media. We consider a one-dimensional medium characterized by its *density* ρ and *bulk modulus* K, as follows:

$$\rho(z) = \begin{cases} \rho_1, z \in [-L, -L_1] \\ \rho_2, z \in (-L_1, 0] \end{cases}, \; K(z) = \begin{cases} K_1, z \in [-L, -L_1] \\ K_2, z \in (-L_1, 0] \end{cases}, \tag{1}$$

where $\rho_j, K_j,\ j = 1, 2$ are positive constants (Figure 1).

An acoustic wave that propagates through this medium is then governed by the equations:

$$\begin{cases} \rho(z)\dfrac{\partial u}{\partial t} + \dfrac{\partial p}{\partial z} = 0 \\ K^{-1}(z)\dfrac{\partial p}{\partial t} + \dfrac{\partial u}{\partial z} = 0, \end{cases} \tag{2}$$

supplemented by appropriate boundary conditions at $z = 0$ and $z = -L$ to be specified later. We look for a solution $\{p = p(t, z),\ u = u(t, z)\}$ that is continuous everywhere on $[-L, 0]$ and continuously differentiable on $[-L, 0] \setminus \{-L_1\}$.

The medium acoustic *impedance* I and the wave *speed of propagation* c are defined by

$$I(z) = \sqrt{K(z)\rho(z)},\ c(z) = \sqrt{\frac{K(z)}{\rho(z)}}, \tag{3}$$

and we denote $I_j = \sqrt{K_j \rho_j},\ c_j = \sqrt{K_j/\rho_j},\ j = 1, 2$.

The wave is decomposed into its *left-* and *right-going* components, A and B respectively, defined by:

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} I^{-1/2} & I^{1/2} \\ -I^{-1/2} & I^{1/2} \end{bmatrix} \begin{bmatrix} p \\ u \end{bmatrix}, \tag{4}$$

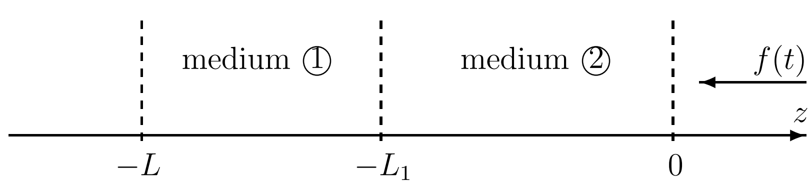


Figure 1: A pulse is impinging onto a stack of two homogeneous slabs.

so that

$$\begin{cases} \dfrac{\partial A}{\partial z} + \dfrac{1}{c}\dfrac{\partial A}{\partial t} = 0 \\ \dfrac{\partial B}{\partial z} - \dfrac{1}{c}\dfrac{\partial B}{\partial t} = 0 \end{cases}, \quad z \neq -L_1. \tag{5}$$

Assume that a pulse $f(t)$ is sent into the medium at $z = 0$ and no energy comes into the slab from the left, i.e. at $z = -L$. These assumptions result in the following boundary conditions:

$$\begin{aligned} A(t, -L) &= 0 \\ B(t, 0) &= f(t), \end{aligned} \tag{6}$$

and the problem (2) is now well-posed. Observe that these boundary conditions are equivalent to having an infinite medium on the left matching medium ① at $-L$ and an infinite medium on the right matching medium ② at the surface, $z = 0$. In other words, the depth L does not play a particular role in the problem. Our quantity of interest is the *reflected wave* $A(t, 0)$.

At the interface $z = -L_1$ we have

$$B(t, -L_1^+) = f\left(t - \frac{L_1}{c_2}\right) \tag{7}$$

and

$$A(t, -L_1^-) = 0. \tag{8}$$

The continuity of u and p at the interface $z = -L_1$ gives, after a classical computation, that

$$A(t, -L_1^+) = R_{21}\, f\left(t - \frac{L_1}{c_2}\right), \tag{9}$$

where

$$R_{21} = \frac{I_2 - I_1}{I_2 + I_1}. \tag{10}$$

Consequently, for $z > -L_1$ we have

$$A(t, z) = R_{21}\, f\left(t - \frac{L_1}{c_2} - \frac{z + L_1}{c_2}\right). \tag{11}$$

Therefore, our quantity of interest, the reflected wave at the surface $z = 0$ is given by

$$A(t, 0) = R_{21}\, f\left(t - \frac{2L_1}{c_2}\right). \tag{12}$$

Thus if there is a contrast of impedances ($I_1 \neq I_2 \Longrightarrow R_{21} \neq 0$), sending a wave into a piecewise constant medium results in its part coming back after twice the time it takes to reach the interface. In the case of matched impedances ($I_1 = I_2 \Longrightarrow R_{21} = 0$), the entire wave passes the interface (with a change in velocity), and no reflection is generated. It is clear that the latter situation makes detection of the interface impossible, since no information is available at the surface ($z = 0$) for analysis.

With a random medium, reflections are generated not only by the interface between the media, but also by the inhomogeneities of the medium. The main goal of this paper is to show that incoherent reflections due to the inhomogeneities in the medium provide us information that can be used to detect the presence of an interface.

3 Reflectors in a random medium

As in the previous section, we will consider a medium with a piecewise constant background, but this time the bulk modulus is randomly fluctuating around its constant local background level (Figure 2). For simplicity of presentation, we will assume that the density remains piecewise constant without any fluctuations:

$$\rho(z) = \bar{\rho}(z),\ K^{-1}(z) = \bar{K}^{-1}(z)\left(1 + \nu\left(\frac{z}{\varepsilon^2}\right)\right), \tag{13}$$

where the background density, $\bar{\rho}(z)$, and homogenized bulk modulus, $\bar{K}(z)$ are as in the previous section:

$$\bar{\rho}(z) = \begin{cases} \bar{\rho}_1, z \in [-L, -L_1] \\ \bar{\rho}_2, z \in (-L_1, 0] \end{cases},\ \bar{K}^{-1}(z) = \begin{cases} \bar{K}_1^{-1}, z \in [-L, -L_1] \\ \bar{K}_2^{-1}, z \in (-L_1, 0] \end{cases}. \tag{14}$$

The process ν is assumed to be centered, stationary, and having strong mixing properties given later. In addition, we consider the regime of *separation of*

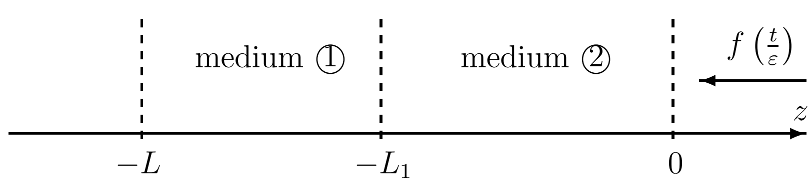


Figure 2: A random medium with an interface inside is probed with a pulse.

scales introduced in [4]. The correlation length of the inhomogeneties is represented by ε^2, where $\varepsilon \ll 1$. We further assume that a pulse $f^\varepsilon(t) \equiv f\left(\frac{t}{\varepsilon}\right)$ impinges on the medium at $z = 0$ which gives rise to an acoustic wave propagating according to (2).

The parameter ε is dimensionless and serves to separate scales in the problem. The correlation length ε^2 (size of a typical inhomogeneity) of the medium is much smaller than the typical wavelength ε of the pulse, which in turn is much smaller than the distance of propagation L_1 to the interface.

3.1 Propagator and the reflected wave

We first introduce the propagator in the case of a constant background medium corresponding to $\bar{\rho}_1 = \bar{\rho}_2 \equiv \bar{\rho}$ and $\bar{K}_1 = \bar{K}_2 \equiv \bar{K}$. We decompose the wave into left- and right-going parts and look at the frequency modes, $\hat{a}^\varepsilon(\omega, z)$ and $\hat{b}^\varepsilon(\omega, z)$, along the characteristics of the homogenized system. These modes satisfy the following ordinary differential equation:

$$\frac{d}{dz}\begin{bmatrix} \hat{a}^\varepsilon \\ \hat{b}^\varepsilon \end{bmatrix}(\omega, z) = Q^\varepsilon(\omega, z)\begin{bmatrix} \hat{a}^\varepsilon \\ \hat{b}^\varepsilon \end{bmatrix}(\omega, z), \tag{15}$$

with

$$Q^\varepsilon(\omega, z) = \frac{i\omega}{2\varepsilon\bar{c}}\nu\left(\frac{z}{\varepsilon^2}\right)\begin{bmatrix} -1 & e^{\frac{2i\omega\vartheta(z)}{\varepsilon}} \\ -e^{-\frac{2i\omega\vartheta(z)}{\varepsilon}} & 1 \end{bmatrix}, \tag{16}$$

where

$$\bar{c} = \sqrt{\bar{K}/\bar{\rho}},\ \vartheta(z) = \frac{z}{\bar{c}}. \tag{17}$$

With the boundary conditions

$$\begin{aligned} \hat{a}^\varepsilon(\omega, -L) &= 0 \\ \hat{b}^\varepsilon(\omega, 0) &= \hat{f}(\omega), \end{aligned} \tag{18}$$

corresponding to a pulse coming from the right and a *radiation condition* on the left, the problem is well-posed. Here $\hat{f}(\omega)$ is the usual unscaled Fourier transform of $f(t)$.

In order to solve the two-point boundary value problem defined above, it is convenient to turn it into an initial value problem by introducing a *propagator* [4], $P^\varepsilon(\omega,-L,z)$. The propagator is a 2×2 matrix, which is the solution to the following initial value problem:

$$\begin{cases} \dfrac{d}{dz}P^\varepsilon(\omega,-L,z) = Q^\varepsilon(\omega,z)P^\varepsilon(\omega,-L,z) \\ P^\varepsilon(\omega,-L,-L) = \mathrm{Id} \end{cases} \tag{19}$$

so that

$$P^\varepsilon(\omega,-L,z)\begin{bmatrix} \hat{a}^\varepsilon(\omega,-L) \\ \hat{b}^\varepsilon(\omega,-L) \end{bmatrix} = \begin{bmatrix} \hat{a}^\varepsilon(\omega,z) \\ \hat{b}^\varepsilon(\omega,z) \end{bmatrix}. \tag{20}$$

One can check that

$$P^\varepsilon(\omega,-L,z) = \begin{bmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{bmatrix}, \tag{21}$$

with $|\alpha|^2 - |\beta|^2 = 1$, since the trace of the matrix Q^ε is zero.

The *transmission* coefficient $T^\varepsilon(\omega,-L,z)$, and the *reflection* coefficient $R^\varepsilon(\omega,-L,z)$ for the slab $[-L,z]$ (Figure 3)are defined by:

$$P^\varepsilon(\omega,-L,z)\begin{bmatrix} 0 \\ T^\varepsilon(\omega,-L,z) \end{bmatrix} = \begin{bmatrix} R^\varepsilon(\omega,-L,z) \\ 1 \end{bmatrix}, \tag{22}$$

so that

$$T^\varepsilon(\omega,-L,z) = \frac{1}{\overline{\alpha(\omega,-L,z)}},\ R^\varepsilon(\omega,-L,z) = \frac{\overline{\beta(\omega,-L,z)}}{\overline{\alpha(\omega,-L,z)}}. \tag{23}$$

The reflection coefficient $R^\varepsilon \equiv R^\varepsilon(\omega,-L,z)$ satisfies the following Riccati equation:

$$\begin{cases} \dfrac{dR^\varepsilon}{dz} = -\dfrac{i\omega}{2\bar{c}\varepsilon}\nu\left(e^{-\frac{2i\omega\vartheta(z)}{\varepsilon}} - 2R^\varepsilon + (R^\varepsilon)^2 e^{\frac{2i\omega\vartheta(z)}{\varepsilon}}\right) \\ R^\varepsilon(\omega,-L,-L) = 0. \end{cases} \tag{24}$$

Figure 3: Transmission and reflection coefficients.

The quantity of interest is the reflected wave $A(t,0)$, which admits the following representation:

$$A(t,0) = = \frac{1}{2\pi}\int \hat{a}^\varepsilon(\omega,0)e^{-\frac{i\omega t}{\varepsilon}}\,d\omega \\ = \frac{1}{2\pi}\int R^\varepsilon(\omega,-L,0)\hat{f}(\omega)e^{-\frac{i\omega t}{\varepsilon}}\,d\omega. \tag{25}$$

3.2 Strong reflector

We show first that if there is a contrast of average impedances ($\bar{I}_1 \equiv \sqrt{\bar{K}_1/\bar{\rho}_1} \neq \sqrt{\bar{K}_2/\bar{\rho}_2} \equiv \bar{I}_2$) then once the wave hits the interface, a coherent reflection is generated. It will then propagate back to the surface and be recorded there after twice the travel time to the interface.

The coefficients of the medium are given by (13) and (14). The propagator $P^\varepsilon(\omega,-L,0)$ can then be split into a product of three propagators that correspond to medium 1, the interface, and medium 2. More precisely,

$$P^\varepsilon(\omega,-L,0) = P_2^\varepsilon(\omega,-L_1,0)J^\varepsilon(\omega,-L_1)P_1^\varepsilon(\omega,-L,-L_1), \tag{26}$$

where

$$P_1^\varepsilon(\omega,-L,-L_1)\begin{bmatrix}\hat{a}_1^\varepsilon(\omega,-L)\\ \hat{b}_1^\varepsilon(\omega,-L)\end{bmatrix} = \begin{bmatrix}\hat{a}_1^\varepsilon(\omega,-L_1^-)\\ \hat{b}_1^\varepsilon(\omega,-L_1^-)\end{bmatrix}, \tag{27}$$

$$J^\varepsilon(\omega,-L_1)\begin{bmatrix}\hat{a}_1^\varepsilon(\omega,-L_1^-)\\ \hat{b}_1^\varepsilon(\omega,-L_1^-)\end{bmatrix} = \begin{bmatrix}\hat{a}_2^\varepsilon(\omega,-L_1^+)\\ \hat{b}_2^\varepsilon(\omega,-L_1^+)\end{bmatrix} \tag{28}$$

and

$$P_2^\varepsilon(\omega,-L_1,0)\begin{bmatrix}\hat{a}_2^\varepsilon(\omega,-L_1^+)\\ \hat{b}_2^\varepsilon(\omega,-L_1^+)\end{bmatrix} = \begin{bmatrix}\hat{a}_2^\varepsilon(\omega,0)\\ \hat{b}_2^\varepsilon(\omega,0)\end{bmatrix}. \tag{29}$$

Here the interface propagator is given by

$$J^\varepsilon(\omega,-L_1) = \begin{bmatrix} r^{(+)} & -r^{(-)}\,e^{-2\frac{i\omega L_1}{\varepsilon\bar{c}_2}} \\ -r^{(-)}\,e^{2\frac{i\omega L_1}{\varepsilon\bar{c}_2}} & r^{(+)}\end{bmatrix}, \tag{30}$$

where

$$r^{(\pm)} = \frac{1}{2}\left(\sqrt{\frac{\bar{I}_1}{\bar{I}_2}} \pm \sqrt{\frac{\bar{I}_2}{\bar{I}_1}}\right),\ \bar{c}_j = \sqrt{\bar{K}_j/\bar{\rho}_j},\ j=1,2. \tag{31}$$

Note that if $\bar{I}_1 = \bar{I}_2$, the interface propagator becomes diagonal. The propagators P_1^ε and P_2^ε satisfy the following equations:

$$\begin{cases} \dfrac{d}{dz} P_1^\varepsilon(\omega, -L, z) = Q_1^\varepsilon(\omega, z) P_1^\varepsilon(\omega, -L, z), \ z \in [-L, -L_1] \\ P_1^\varepsilon(\omega, -L, -L) = \mathrm{Id} \end{cases} \tag{32}$$

$$\begin{cases} \dfrac{d}{dz} P_2^\varepsilon(\omega, -L_1, z) = Q_2^\varepsilon(\omega, z) P_2^\varepsilon(\omega, -L_1, z), \ z \in [-L_1, 0] \\ P_2^\varepsilon(\omega, -L_1, -L_1) = \mathrm{Id} \end{cases}, \tag{33}$$

where

$$Q_j^\varepsilon(\omega, z) = \frac{i\omega}{2\varepsilon \bar{c}_j} \nu\left(\frac{z}{\varepsilon^2}\right) \begin{bmatrix} -1 & e^{\frac{2i\omega\vartheta(z)}{\varepsilon}} \\ -e^{-\frac{2i\omega\vartheta(z)}{\varepsilon}} & 1 \end{bmatrix}, \tag{34}$$

and

$$\vartheta(z) = \int_0^z \frac{ds}{\bar{c}(s)} = \begin{cases} \dfrac{z + L_1}{\bar{c}_1} - \dfrac{L_1}{\bar{c}_2}, & z \in [-L, -L_1], \\ \dfrac{z}{\bar{c}_2}, & z \in [-L_1, 0]. \end{cases} \tag{35}$$

The propagators defined above admit the representations:

$$P_1^\varepsilon(\omega, -L, -L_1) = \begin{bmatrix} \alpha_1 & \bar{\beta}_1 \\ \beta_1 & \bar{\alpha}_1 \end{bmatrix}, \tag{36}$$

$$P_2^\varepsilon(\omega, -L_1, 0) = \begin{bmatrix} \alpha_2 & \bar{\beta}_2 \\ \beta_2 & \bar{\alpha}_2 \end{bmatrix}, \tag{37}$$

$$J^\varepsilon(\omega, -L_1) = \begin{bmatrix} \alpha_I & \bar{\beta}_I \\ \beta_I & \bar{\alpha}_I \end{bmatrix}, \tag{38}$$

with $|\alpha_j|^2 - |\beta_j|^2 = 1, \ j = 1, 2, I$. It then follows from Equation (26) that we can define the transmission, $T^\varepsilon(\omega, -L, 0)$, and reflection, $R^\varepsilon(\omega, -L, 0)$, coefficients over the entire space $-L \leq z \leq 0$, which will satisfy the following equation:

$$\begin{bmatrix} R^\varepsilon(\omega, -L, 0) \\ 1 \end{bmatrix} = P^\varepsilon(\omega, -L, 0) \begin{bmatrix} 0 \\ T^\varepsilon(\omega, -L, 0) \end{bmatrix}, \tag{39}$$

where

$$P^\varepsilon(\omega, -L, 0) = \begin{bmatrix} \alpha_2 & \bar{\beta}_2 \\ \beta_2 & \bar{\alpha}_2 \end{bmatrix} \begin{bmatrix} \alpha_I & \bar{\beta}_I \\ \beta_I & \bar{\alpha}_I \end{bmatrix} \begin{bmatrix} \alpha_1 & \bar{\beta}_1 \\ \beta_1 & \bar{\alpha}_1 \end{bmatrix}. \tag{40}$$

Solving for $R^\varepsilon(\omega,-L,0)$, we get

$$R^\varepsilon(\omega,-L,0) = \frac{\alpha_2\alpha_I\overline{\beta_1} + \overline{\beta_2}\beta_I\overline{\beta_1} + \alpha_2\overline{\beta_I}\overline{\alpha_1} + \overline{\beta_2}\overline{\alpha_I\alpha_1}}{\alpha_I\beta_2\overline{\beta_1} + \overline{\alpha_2}\beta_I\overline{\beta_1} + \beta_2\overline{\beta_I}\overline{\alpha_1} + \overline{\alpha_2\alpha_I\alpha_1}}. \tag{41}$$

Introducing the notations

$$\begin{aligned} T_j &= \tfrac{1}{\bar\alpha_j}, \\ R_j &= \tfrac{\bar\beta_j}{\bar\alpha_j}, \\ \tilde R_j &= -\tfrac{\beta_j}{\bar\alpha_j}, \\ D_j &\equiv T_j^2 - R_j\tilde R_j = \tfrac{\alpha_j}{\bar\alpha_j},\ j = 1,2,I, \end{aligned} \tag{42}$$

we obtain

$$\begin{aligned} R^\varepsilon(\omega,-L,0) &= \frac{D_2D_IR_1 - R_1R_2\tilde R_I + D_2R_I + R_2}{1 - \left(\tilde R_2D_IR_1 + \tilde R_IR_1 - \tilde R_2R_I\right)} \\ &= \left(D_2D_IR_1 - R_1R_2\tilde R_I + D_2R_I + R_2\right) \\ &\times \sum_{k=0}^{\infty}\left[\tilde R_2D_IR_1 + \tilde R_IR_1 - \tilde R_2R_I\right]^k \end{aligned} \tag{43}$$

In order to study the reflected wave

$$A(t,0) = \frac{1}{2\pi}\int R^\varepsilon(\omega,-L,0)\hat f(\omega)e^{-\frac{i\omega t}{\varepsilon}}\,d\omega \tag{44}$$

we use the approach taken in [5]. The moments of $A(t,0)$ involve moments of the form

$$\mathbb{E}[R^\varepsilon(\omega_1,-L_1,0)\ldots R^\varepsilon(\omega_n,-L_1,0)], \tag{45}$$

for n distinct frequencies $\omega_1,\ldots,\omega_n$. These moments involve sums of expectations of products of reflections and transmission coefficients. These expectations can be factorized because the coefficients associated with the medium 1 are asymptotically independent from the coefficients associated to the medium 2. An application of Itô's formula establishes that an expectation involving a product of reflection and transmission coefficients vanishes as $\varepsilon \to 0$ as soon as the product contains reflection coefficients. Only one term of the expansion of R^ε does not involve a reflection coefficient, and it is given by $-\frac{r^{(-)}}{r^{(+)}}e^{\frac{2i\omega L_1}{\varepsilon \bar{c}_2}} T_2^2(\omega, -L_1, 0)$, coming from the $D_2 R_I$ term with $k=0$ in the series. As a result the problem is reduced to the identification of the limits of the moments

$$\mathbb{E}\left[T_2^2(\omega_1, -L_1, 0)\dots T_2^2(\omega_n, -L_1, 0)\right] \tag{46}$$

for n distinct frequencies $(\omega_j)_{1\leq j\leq n}$. This study follows the same lines as the one performed in [5]. We get that the phase compensated moments

$$\mathbb{E}\left[e^{-\frac{2i\omega_1 L_1}{\varepsilon \bar{c}_2}} R^{\varepsilon}(\omega_1, -L_1, 0)\dots e^{-\frac{2i\omega_n L_1}{\varepsilon \bar{c}_2}} R^{\varepsilon}(\omega_n, -L_1, 0)\right], \tag{47}$$

converge to limits given by.

$$\left(-\frac{r^{(-)}}{r^{(+)}}\right)^n \mathbb{E}\left[\tilde{T}(\omega_1)^2\dots \tilde{T}(\omega_n)^2\right], \tag{48}$$

Here $\tilde{T}(\omega)$ is a random variable given by

$$\tilde{T}(\omega) = \exp\left(i\omega\frac{\sqrt{\gamma}}{2\bar{c}_2}W(L_1) - \omega^2\frac{\gamma}{8\bar{c}_2^2}L_1\right), \tag{49}$$

where the coefficient γ is the positive integrated autocorrelation

$$\gamma = \int_{-\infty}^{\infty} \mathbb{E}[\nu(0)\nu(z)]\, dz, \tag{50}$$

and $W(z)$ is a standard Brownian motion. Substituting into the integral representation (44) of the reflected wave, this shows that the following coherent reflected pulse can be observed around the time $t_0 = 2L_1/\bar{c}_2$:

$$A(2L_1/\bar{c}_2 + \varepsilon s, 0) \xrightarrow{\varepsilon\to 0} a(s) \equiv \left(-\frac{r^{(-)}}{r^{(+)}}\right)\frac{1}{2\pi}\int e^{-i\omega s}\tilde{T}(\omega)^2\hat{f}(\omega)d\omega. \tag{51}$$

At any other observation times $t_0 \neq 2L_1/\bar{c}_2$ the reflected wave vanishes in the limit $\varepsilon \to 0$ because of the remaining rapid phase in the integral representation. In particular this implies that, even with random inhomogeneities in the medium, the arrival time of the coherent reflection can be used to identify the depth of the jump in the background parameters with a precision of order ε due to the random time shift.

The limiting *coherent reflected front* can be written

$$a(s) = R_{21} f \star \mathcal{N}_{D_{2L_1}}(s - 2\Theta_{L_1}), \tag{52}$$

where D_z and Θ_z are defined as

$$D_z^2 = \frac{\gamma z}{4\bar{c}_2^2}, \qquad \Theta_z = \frac{\gamma^{1/2}}{2\bar{c}_2} W(z), \tag{53}$$

and $\mathcal{N}_D$ is the centered Gaussian distribution with variance D^2. The reflection coefficient

$$R_{21} = -\frac{r^{(-)}}{r^{(+)}} = \frac{\bar{I}_2 - \bar{I}_1}{\bar{I}_2 + \bar{I}_1},$$

corresponds to the one introduced in (10). It is the reflection coefficient corresponding to the case where the interface separates two homogeneous media with the impedances $\bar{I}_2$ and $\bar{I}_1$. The reflected pulse front has a deterministic shape imposed by the convolution with the Gaussian kernel $\mathcal{N}_{D_{2L_1}}$, and it is random only through the random time shift $2\Theta_{L_1}$. The result that we obtain in the random case is not surprising once the behavior of a transmitted pulse front is understood. Indeed the reflected front does a round trip in the random medium to go from the surface $z = 0$ to the interface $z = -L_1$ and come back. The deterministic spreading thus corresponds to a travel distance of $2L_1$, and the random time shift is simply twice the one-way shift because the wave travels in the same medium.

In the case of no contrast of average impedances ($R_{21} = 0$), this analysis shows that there is no coherent front reflected to the surface $z = 0$. However, we will show that the incoherent reflected wave due to scattering by the inhomogeneities contains information about the change in the medium, in this case, a jump in the average sound speed without a contrast of impedance. In order to extract this information, we use a time reversal technique.

4 Time reversal in reflection for a constant background

In this section we briefly present the time reversal analysis for one-dimensional medium with constant background [6]. As before, we consider a pulse of the form $f\left(\frac{t}{\varepsilon}\right)$ impinging upon a slab $[-L,0]$ at $z=0$. As the wave propagates through the random medium, it get scattered by inhomogeneities. The reflected wave, $A(t,0)$, admits the representation:

$$A(t,0)=\frac{1}{2\pi}\int R^{\varepsilon}(\omega,-L,0)\hat{f}(\omega)e^{-\frac{i\omega t}{\varepsilon}}\,d\omega, \tag{54}$$

where $R^{\varepsilon}(\omega,-L,0)$ is the reflection coefficient over $[-L,0]$. A piece of the reflected wave is then recorded at the surface to yield

$$y(t)=A(t,0)\,G(t), \tag{55}$$

where $G(t)$ is a cut-off function, e.g. $G(t)=\mathbf{1}_{[0,t_1]}(t)$. The recorded signal is reversed in time to form a new pulse

$$f_{\rm new}(t)=y(t_1-t)=A(t_1-t,0)\,G(t_1-t), \tag{56}$$

which is then sent back into the medium. Its propagation through the slab gives rise to a new reflected wave at the surface, $A_{\rm new}(t,0)$. We observe these reflections around time $t_{\rm obs}$ on the scale ε, i.e. we introduce the quantity

$$\begin{aligned} S^{\varepsilon}(t_{\rm obs}+\varepsilon\sigma) &\equiv A_{\rm new}(t_{\rm obs}+\varepsilon\sigma,0)\\ &=\frac{1}{2\pi\varepsilon}\int e^{-i\omega_1\left(\sigma+\frac{t_{\rm obs}}{\varepsilon}\right)}R^{\varepsilon}(\omega_1,-L,0)\hat{f}^{\varepsilon}_{\rm new}(\omega_1)\,d\omega_1, \end{aligned} \tag{57}$$

where

$$\hat{f}^{\varepsilon}_{\rm new}(\omega_1)=\frac{1}{2\pi}\int\overline{R^{\varepsilon}(\omega_2,-L,0)}\,\overline{\hat{f}(\omega_2)}\,\overline{\hat{G}\left(\frac{\omega_1-\omega_2}{\varepsilon}\right)}e^{\frac{i\omega t}{\varepsilon}}\,d\omega_2. \tag{58}$$

A change of variables $\omega_1=\omega+\frac{\varepsilon h}{2}$, $\omega_2=\omega-\frac{\varepsilon h}{2}$ yields

$$\begin{aligned} S^{\varepsilon}(t_{\rm obs}+\varepsilon\sigma)=\frac{1}{(2\pi)^2}\iint e^{-i\omega\sigma}e^{\frac{i\omega(t_1-t_{\rm obs})}{\varepsilon}}e^{\frac{ih(t_1-t_{\rm obs})-i\varepsilon h\sigma}{2}}\overline{\hat{f}\left(\omega-\frac{\varepsilon h}{2}\right)}\\ \times\overline{\hat{G}(h)}\,\overline{R^{\varepsilon}\left(\omega-\frac{\varepsilon h}{2},-L,0\right)}R^{\varepsilon}\left(\omega+\frac{\varepsilon h}{2},-L,0\right)\,dh\,d\omega. \end{aligned} \tag{59}$$

Note that if $t_{\rm obs}\neq t_1$ the integral vanishes as $\varepsilon\to 0$ because of the highly oscillating exponent inside. If, on the other hand, $t_{\rm obs}=t_1$, we observe a *refocused pulse*

$$\begin{aligned} S^\varepsilon(t_1+\varepsilon\sigma) = \frac{1}{(2\pi)^2} \iint e^{-i\omega\sigma - \frac{i\varepsilon h\sigma}{2}} \overline{\hat{f}\left(\omega - \frac{\varepsilon h}{2}\right)} \overline{\hat{G}(h)} \\ \times \overline{R^\varepsilon\left(\omega - \frac{\varepsilon h}{2}, -L, 0\right)} R^\varepsilon\left(\omega + \frac{\varepsilon h}{2}, -L, 0\right) dh\, d\omega. \end{aligned} \tag{60}$$

It follows from (60) that $S^\varepsilon(t_1+\varepsilon\sigma)$ is characterized by the behavior of the cross moments of $R^\varepsilon(\omega,-L,0)$ at close frequencies $\omega \pm \frac{\varepsilon h}{2}$.

4.1 Moments of the reflected coefficient

We follow the analysis of [4]. Define $U^\varepsilon_{p,q}(\omega,h,z) = R^\varepsilon\left(\omega+\frac{\varepsilon h}{2}\right)^p \overline{R^\varepsilon\left(\omega+\frac{\varepsilon h}{2}\right)^q}$. We are particularly interested in $U^\varepsilon_{1,1}(\omega,h,z)$. Note, however, that $U^\varepsilon_{1,1}(\omega,h,z)$ does not satisfy a closed form differential equation. We therefore have to consider the whole family $\left\{U^\varepsilon_{p,q}\right\}_{p,q}$ simultaneously.

We use the Riccati equation satisfied by $R^\varepsilon(\omega,-L,z)$ to derive that

$$\left\{ \begin{aligned} \frac{\partial U^\varepsilon_{p,q}}{\partial z} &= \frac{i\omega}{\bar{c}\varepsilon}\nu(p-q)U^\varepsilon_{p,q} + \frac{i\omega}{2\bar{c}\varepsilon}\nu e^{\frac{2i\omega z}{\bar{c}\varepsilon}}\left(qe^{-\frac{ihz}{\bar{c}}}U^\varepsilon_{p,q-1} - pe^{\frac{ihz}{\bar{c}}}U^\varepsilon_{p+1,q}\right) \\ &\quad + \frac{i\omega}{2\bar{c}\varepsilon}e^{-\frac{2i\omega z}{\varepsilon}}\left(qe^{\frac{ihz}{\bar{c}}}U^\varepsilon_{p,q+1} - pe^{-\frac{ihz}{\bar{c}}}U^\varepsilon_{p-1,q}\right) \\ U^\varepsilon_{p,q}&(\omega,h,-L) = \mathbf{1}_0(p)\mathbf{1}_0(q). \end{aligned} \right. \tag{61}$$

We now remove the fast phase by introducing the following shifted Fourier transform with respect to the variable h.

$$V^\varepsilon_{p,q}(\omega,\tau,z) = \frac{1}{2\pi}\int e^{-ih\left(\tau - \frac{p+q}{\bar{c}}z\right)}U^\varepsilon(\omega,h,z)\,dh. \tag{62}$$

Then a direct computation reveals that $\left\{V^\varepsilon_{p,q}(\omega,\tau,z)\right\}$ satisfy

$$\left\{ \begin{aligned} \frac{\partial V^\varepsilon_{p,q}}{\partial z} &= -\frac{p+1}{\bar{c}}\frac{\partial V^\varepsilon_{p,q}}{\partial \tau} + \frac{i\omega}{\bar{c}\varepsilon}\nu(p-q)V^\varepsilon_{p,q} \\ &\quad + \frac{i\omega}{2\bar{c}\varepsilon}\nu e^{\frac{2i\omega z}{\bar{c}\varepsilon}}(qV^\varepsilon_{p,q-1} - pV^\varepsilon_{p+1,q}) \\ &\quad + \frac{i\omega}{2\bar{c}\varepsilon}\nu e^{-\frac{2i\omega z}{\bar{c}\varepsilon}}(qV^\varepsilon_{p,q+1} - pV^\varepsilon_{p-1,q}) \\ V^\varepsilon_{p,q}&(\omega,\tau,-L) = \delta_0(\tau)\mathbf{1}_0(p)\mathbf{1}_0(q). \end{aligned} \right. \tag{63}$$

We now look at the limiting behavior of $V^\varepsilon_{p,q}(\omega,\tau,z)$ as $\varepsilon \to 0$. The assumptions on the stochastic process ν imply that the ratio $\frac{\nu(t/\varepsilon^2)}{\varepsilon}$ will behave like a white noise. A formal application of the diffusion approximation theorem reveals that $V^\varepsilon_{p,q} \to V_{p,q}$ as $\varepsilon \to 0$, where $V_{p,q}$ satisfies a stochastic differential equation containing drift and diffusion terms. Taking the expectation $\mathbb{E}\left[V_{p,q}\right]$ removes the diffusion part, while what remains reads:

$$\begin{aligned}\frac{\partial \mathbb{E}[V_{p,q}]}{\partial z} = &- \frac{q+p}{\bar{c}}\frac{\partial \mathbb{E}[V_{p,q}]}{\partial \tau} - \frac{3\gamma\omega^2}{4\bar{c}^2}(p-q)^2\mathbb{E}[V_{p,q}] \\ &+ \frac{\gamma\omega^2}{4\bar{c}^2}pq\big(\mathbb{E}[V_{p+1,q+1}] + \mathbb{E}[V_{p-1,q-1}] - 2\mathbb{E}[V_{p,q}]\big),\end{aligned} \tag{64}$$

where γ is as in the previous section. We now proceed with computing the moments.

I. Consider a family of moments $\breve{W}_p(\omega,\tau,z) = \mathbb{E}[V_{p+1,p}(\omega,\tau,z)]$. This family satisfies a closed system of transport equations with the zero initial condition:

$$\begin{cases}\dfrac{\partial \breve{W}_p}{\partial z} = - \dfrac{2p+1}{\bar{c}}\dfrac{\breve{W}_p}{\partial \tau} \\ \qquad\qquad + \dfrac{\gamma\omega^2}{4\bar{c}^2}\left[p(p+1)(\breve{W}_{p+1} + \breve{W}_{p-1} - 2\breve{W}_p) - 3\breve{W}_p\right] \\ \breve{W}_p(\omega,\tau,-L) = 0.\end{cases} \tag{65}$$

As such, $\breve{W}_p \equiv 0,\ \forall p$. It means that $\mathbb{E}\left[R^\varepsilon(\omega,-L,0)\right] \to 0$ as $\varepsilon \to 0$, and in particular that the first reflected wave is completely incoherent (noise-like). This result can be generalized to show that *all* off-diagonal moments, $\mathbb{E}[V_{p,q}],\ p \neq q$, are zero.

II. Consider now a family of *diagonal* moments, $W_p(\omega,\tau,z) = \mathbb{E}\left[V_{p,p}(\omega,\tau,z)\right]$. It also satisfies a closed system of transport equations:

$$\begin{cases}\dfrac{\partial W_p}{\partial z} + \dfrac{2p}{\bar{c}}\dfrac{\partial W_p}{\partial \tau} = (\mathcal{L}W)_p \\ (\mathcal{L}W)_p = \dfrac{\gamma\omega^2 p^2}{4\bar{c}^2}(W_{p+1} + W_{p-1} - 2W_p) \\ W_p(\omega,\tau,-L) = \delta_0(\tau)\mathbf{1}_0(p)\end{cases} \tag{66}$$

In particular the second moment of the reflection coefficient has the following limit as $\varepsilon \to 0$:

$$\begin{aligned}\mathbb{E}\left[R^\varepsilon\left(\omega+\frac{\varepsilon h}{2},-L,0\right)\overline{R^\varepsilon\left(\omega-\frac{\varepsilon h}{2},-L,0\right)}\right]&\\ =\mathbb{E}\left[U^\varepsilon_{1,1}(\omega,h,0)\right]&\\ =\int\mathbb{E}\left[V^\varepsilon_{1,1}(\omega,\tau,0)\right]e^{ih\tau}\,d\tau&\\ \stackrel{\varepsilon\to 0}{\to}\int W_1(\omega,\tau,0)\,e^{ih\tau}\,d\tau.&\end{aligned}\tag{67}$$

4.2 Probabilistic representation of W_p

The solution to the infinite-dimensional system of transport equations may be given a neat probabilistic interpretation. The latter gives rise to an efficient numerical solution in general, and even an explicit formula in some particular cases.

We begin by introducing a Markovian jump process $(N_z)_{z\geq -L}$ constructed as follows. The jump times are distributed exponentially with the intensity $\frac{p^2\gamma\omega^2}{2\bar{c}^2}$, where $p\in\mathbb{N}^*$ is the current state of the process. In particular, $p=0$ is an absorbing state. At a jump time, the process then changes its state to $p\pm 1$ with probability $\frac{1}{2}$.

Construct another process, $(T_z)_{z\geq -L}$, so that

$$T_z - T_{-L} = -\frac{2}{\bar{c}}\int_{-L}^{z} N_\zeta\,d\zeta.\tag{68}$$

The pair (N_z, T_z) is then a Markov process by itself. Its *infinitesimal gener-*

ator is given by

$$
\begin{aligned}
\mathcal{L}_N f(p,\tau) &= \lim_{h\to 0} \frac{\mathbb{E}\left[f\left(N_{z+h}, T_{z+h}\right) \mid (N_z, T_z) = (p,\tau)\right] - f(p,\tau)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\Big[f(p+1, \tau + O(h)) \frac{p^2\gamma\omega^2}{4\bar{c}^2} h \\
&\qquad + f(p-1, \tau + O(h)) \frac{p^2\gamma\omega^2}{4\bar{c}^2} h \\
&\qquad + f\left(p, -\frac{2}{\bar{c}} ph\right) \left(1 - \frac{p^2\gamma\omega^2}{2\bar{c}^2} h\right) + o(h) \\
&\qquad - f(p,\tau)\Big] \\
&= \left(\mathcal{L} - \frac{2p}{\bar{c}} \frac{\partial}{\partial\tau}\right) f(p,\tau)
\end{aligned}
\tag{69}
$$

The Feynman-Kac formula [7] then gives that the solution to (66) can be written as

$$
\begin{aligned}
W_p(\omega,\tau,z) &= \mathbb{E}\big[W_{N_z}(\omega, T_z, -L) \mid (N_{-L}, T_{-L}) = (p,\tau)\big] \\
&= \mathbb{E}\left[\delta_0\left(\tau - \frac{2}{\bar{c}} \int_{-L}^{z} N_\zeta d\zeta\right) \mathbf{1}_0(N_z) \mid N_{-L} = p\right].
\end{aligned}
\tag{70}
$$

Integrating with respect to τ and setting $z=0$, we have:

$$
\int_{\tau_0}^{\tau_1} W_p(\omega,\tau,0) d\tau = \mathbb{P}\left[N_0 = 0, \frac{2}{\bar{c}} \int_{-L}^{0} N_\zeta d\zeta \in [\tau_0, \tau_1] \mid N_{-L} = p\right]. \tag{71}
$$

A couple of observations are in order about W_p in light of its probabilistic representation. By construction of (N_z), $W_p(\omega,\tau,z) = 0$ when $\tau < 0$. Also any realization of (N_z) that contributes to the right hand side of (71) satisfies $\frac{2}{\bar{c}} \int_{-L}^{0} N_\zeta d\zeta < \tau_1$. If $\tau_1 < \frac{2L}{\bar{c}}$ then it follows that $\int_{-L}^{0} N_\zeta d\zeta < L$, and hence N_z has to be absorbed by 0 at some $z_0 < 0$. It the follows that $W_p(\omega,\tau,0)$ does not depend on L so long as $\tau < \frac{2L}{\bar{c}}$.

Recall that L was introduced as an artificial non-physical parameter in order to help set up the problem mathematically. As τ is related in practice to the time of recording, it remains finite. Thus for any practical τ, the formula (71) holds true if L is chosen *a priori* large enough. We show now that by taking $L \to \infty$ (half-space approximation), we may write an explicit formula for $W_p(\omega, \tau, 0)$. The formula remains true for a finite medium so long as τ is appropriately bounded.

We first use the homogeneity of (N_z) to shift it in z, that is define

$$\tilde{N}_z = N_{z-L}, \; z \in [0, L]. \tag{72}$$

Upon continuing trajectories of $(\tilde{N}_z)$ to $z > L$ we note that this process is recurrent. In particular, it will reach the absorbing state 0 at a large enough (although random) z, and hence the following random variable is well-defined:

$$\mu^p = \frac{2}{\bar{c}} \int_0^\infty \tilde{N}_z dz. \tag{73}$$

By taking $L \to \infty$ we then obtain

$$\begin{aligned} \int_{\tau_0}^{\tau_1} W_p(\omega, \tau, 0) d\tau &= \mathbb{P}\left[\tilde{N}_L = 0, \frac{2}{\bar{c}} \int_0^L \tilde{N}_z dz \in [\tau_0, \tau_1] \mid \tilde{N}_0 = p\right] \\ &\stackrel{L\to\infty}{\to} \mathbb{P}\left[\mu^p \in [\tau_0, \tau_1] \mid \tilde{N}_0 = p\right] \\ &= \mathbb{P}\left[\int_{\tau_0}^{\tau_1} f_{\mu^p}(\tau) d\tau \mid N_0 = p\right], \end{aligned} \tag{74}$$

where $f_{\mu^p}(z)$ stands for the probability density function of μ^p. An application of Feynman-Kac formula similar to the case of W_p reveals that the collection of functions $\{f_{\mu^p}\}$ satisfies (66) but with $\frac{\partial f_{\mu^p}}{\partial z} = 0$ since μ_p does not depend on z:

$$\begin{cases} \dfrac{\partial f_{\mu^p}}{\partial \tau} = \dfrac{\gamma \omega^2 p}{8\bar{c}} (f_{\mu^{p+1}} + f_{\mu^{p-1}} - 2 f_{\mu^p}) \\ f_{\mu^p}(\omega, \tau) = \delta_0(\tau) \mathbf{1}_0(p). \end{cases} \tag{75}$$

For the cumulative distribution functions of μ^p, $F_{\mu^p}(\tau)$, we then have

$$\begin{cases} \dfrac{\partial F_{\mu^p}}{\partial \tau} = \dfrac{\gamma \omega^2 p}{8\bar{c}} (F_{\mu^{p+1}} + F_{\mu^{p-1}} - 2 F_{\mu^p}) \\ F_{\mu^p}(\omega, 0) = \mathbf{1}_0(p). \end{cases} \tag{76}$$

By direct verification one obtains that the solution to (76) is given by

$$F_{\mu^p}(\tau) = \left(\frac{4\gamma\omega^2\tau}{8\bar{c} + \gamma\omega^2\tau}\right)^p \mathbf{1}_{[0,\infty)}(\tau), \tag{77}$$

and thus

$$\begin{aligned} f_{\mu^p}(\tau) &= \frac{\partial}{\partial\tau}\left(\frac{4\gamma\omega^2\tau}{8\bar{c} + \gamma\omega^2\tau}\right)^p \mathbf{1}_{[0,\infty)}(\tau) \\ &= \begin{cases} \delta_0(\tau), & p = 0 \\ \dfrac{8p\bar{c}\left(\gamma\omega^2\right)^p \tau^{p-1}}{\left(8\bar{c} + \gamma\omega^2\tau\right)^{p+1}} \mathbf{1}_{[0,\infty)}(\tau), & p \neq 0. \end{cases} \end{aligned} \tag{78}$$

4.3 Characterization of the refocused pulse

It follows from (60) that

$$\begin{aligned} \mathbb{E}\left[S^\varepsilon(t_1 + \varepsilon\sigma)\right] = \frac{1}{(2\pi)^2} \iint e^{-i\omega\sigma} e^{-\frac{i\varepsilon h\sigma}{2}} \overline{\hat{f}\left(\omega - \frac{\varepsilon h}{2}\right)}\, \overline{\hat{G}(h)} \\ \times \mathbb{E}\left[R^\varepsilon\left(\omega + \frac{\varepsilon h}{2}, -L, 0\right) \overline{R^\varepsilon\left(\omega - \frac{\varepsilon h}{2}, -L, 0\right)}\right] dh\, d\omega \end{aligned} \tag{79}$$

By taking the limit as $\varepsilon \to 0$, we obtain

$$\mathbb{E}\left[S^\varepsilon(t_1 + \varepsilon\sigma)\right] \overset{\varepsilon \to 0}{\to} \frac{1}{2\pi} \iint e^{-i\omega\sigma} \overline{\hat{f}(\omega)} G(\tau) W_1(\omega, \tau, 0)\, d\tau\, d\omega. \tag{80}$$

This defines a deterministic shape, and one can show that the higher moments of S^ε converge to the respective powers of that shape. By combining that together with the tightness of S^ε in the space of continuous functions endowed with the usual sup-norm, one obtains that

$$S^\varepsilon(t_1 + \varepsilon\sigma) \xrightarrow{\mathbb{P}} s(\sigma) = \frac{1}{2\pi} \iint \Lambda(\omega, \tau) \overline{\hat{f}(\omega)} e^{-i\omega\sigma} G(\tau) d\omega\, d\tau, \tag{81}$$

where $\Lambda(\omega, \tau) \equiv W_1(\omega, \tau, 0)$. We can equivalently write that

$$s(\sigma) = \big(f(-\cdot) \star K(\cdot)\big)(\sigma), \tag{82}$$

where the Fourier transform of K satisfies:

$$\hat{K}(\omega) = \int G(\tau)\Lambda(\omega, \tau)\, d\tau. \tag{83}$$

K is called the *refocusing kernel* and Λ is its *density* in the Fourier domain.

5 Weak reflector

We consider now the case studied in Section 3, i.e. an interface between two random media with constant background parameters but now with no contrast of impedance so that there is no reflected coherent front as explained in the end of Section 3. We call such a reflector weak because a wave hitting passes through with a change in velocity while generating no coherent reflection. We therefore cannot rely upon a direct arrival for the purposes of detection and location of such an interface. Instead we will show that the incoherent data can be used for these purposes.

5.1 Reflection coefficient in a medium with a reflector

We assume the medium is given by (13), (14) with $I_1 = I_2$ and $c_1 \neq c_2$. We now consider the propagator, $P^\varepsilon(\omega, -L, z)$, over $[-L, z]$ and introduce the corresponding transmission and reflection coefficients defined by

$$P^\varepsilon(\omega, -L, z) \begin{bmatrix} 0 \\ T^\varepsilon(\omega, -L, z) \end{bmatrix} = \begin{bmatrix} R^\varepsilon(\omega, -L, z) \\ 1 \end{bmatrix}. \tag{84}$$

We consider the following cases:

I. If $z \in [-L, -L_1)$ then $P^\varepsilon(\omega, -L, z) = P_1^\varepsilon(\omega, -L, z)$ defined by equation (32). Consequently, on $[-L, -L_1]$ the reflection coefficient $R^\varepsilon(\omega, -L, z)$ satisfies the Riccati equation:

$$\begin{cases} \dfrac{dR^\varepsilon}{dz} = -\dfrac{i\omega}{2\bar{c}_1\varepsilon}\nu\left(\dfrac{z}{\varepsilon^2}\right)\left(e^{-\frac{2i\omega\vartheta(z)}{\varepsilon}} - 2R^\varepsilon + (R^\varepsilon)^2 e^{\frac{2i\omega\vartheta(z)}{\varepsilon}}\right) \\ R^\varepsilon(\omega, -L, -L) = 0. \end{cases} \tag{85}$$

II. If $z \in [-L_1, 0]$ then the propagator $P^\varepsilon(\omega, -L, z)$ can be written as

$$P^\varepsilon(\omega, -L, z) = P_2^\varepsilon(\omega, -L_1, z) J^\varepsilon(\omega, -L_1) P_1^\varepsilon(-L, -L_1), \tag{86}$$

where $P_2^\varepsilon(\omega, -L_1, z)$ satisfies equation (33), and $J^\varepsilon(\omega, -L_1)$ is simply the identity matrix since $r^{(+)} = 1$ and $r^{(-)} = 0$. The reflection coefficient can now be written as

$$
\begin{aligned}
R^\varepsilon(\omega,-L,z) &= \frac{\alpha_2\bar\beta_1+\bar\beta_2\bar\alpha_1}{\beta_2\bar\beta_1+\bar\alpha_2\bar\alpha_1}\\
&= \frac{R_2^\varepsilon+\left((T_2^\varepsilon)^2-R_2^\varepsilon\tilde R_2^\varepsilon\right)R_1^\varepsilon}{1-\tilde R_2^\varepsilon R_1^\varepsilon}\\
&= R_2^\varepsilon+(T_2^\varepsilon)^2\sum_{k=0}^\infty\left[(R_1^\varepsilon)^{k+1}\left(\tilde R_2^\varepsilon\right)^k\right].
\end{aligned}
\tag{87}
$$

Here

$$
P_j^\varepsilon=\begin{bmatrix}\alpha_j & \bar\beta_j\\ \beta_j & \bar\alpha_j\end{bmatrix},\ R_j^\varepsilon=\frac{\bar\beta_j}{\bar\alpha_j},\ T_j^\varepsilon=\frac{1}{\bar\alpha_j},\ \tilde R_j^\varepsilon=-\frac{\beta_j}{\bar\alpha_j},\ j=1,2, \tag{88}
$$

and we have used the geometric series expansion as well as the identity $|\alpha_j|^2-|\beta_j|^2=1$. The reflection coefficient $R_2^\varepsilon\equiv R_2^\varepsilon(\omega,-L_1,z)$ satisfies the Riccati equation

$$
\frac{dR_2^\varepsilon}{dz}=-\frac{i\omega}{2\bar c_2\varepsilon}\nu\left(\frac{z}{\varepsilon^2}\right)\left(e^{-\frac{2i\omega\vartheta(z)}{\varepsilon}}-2R_2^\varepsilon+(R_2^\varepsilon)^2e^{\frac{2i\omega\vartheta(z)}{\varepsilon}}\right). \tag{89}
$$

In addition, a straightforward computation yields

$$
\begin{aligned}
\frac{dT_2^\varepsilon}{dz} &= \frac{i\omega}{2\bar c_2\varepsilon}\nu\left(\frac{z}{\varepsilon^2}\right)\left(T_2^\varepsilon-T_2^\varepsilon R_2^\varepsilon e^{\frac{2i\omega\vartheta(z)}{\varepsilon}}\right),\\
\frac{d\tilde R_2^\varepsilon}{dz} &= -\frac{i\omega}{2\bar c_2\varepsilon}\nu\left(\frac{z}{\varepsilon^2}\right)(T_2^\varepsilon)^2\,e^{\frac{2i\omega\vartheta(z)}{\varepsilon}}.
\end{aligned}
\tag{90}
$$

By differentiating (87) and using (89), (90) along with the identity

$$
\left(\sum_{k=0}^\infty a^{k+1}b^k\right)^2=\sum_{k=0}^\infty(k+1)a^{k+2}b^k=\frac{a^2}{(1-ab)^2}, \tag{91}
$$

we get that $R^\varepsilon(\omega,-L,z)$ satisfies the Riccati equation on $[-L_1,z]$:

$$
\frac{dR^\varepsilon}{dz}=-\frac{i\omega}{2\bar c_2\varepsilon}\nu\left(\frac{z}{\varepsilon^2}\right)\left(e^{-\frac{2i\omega\vartheta(z)}{\varepsilon}}-2R^\varepsilon+(R^\varepsilon)^2e^{\frac{2i\omega\vartheta(z)}{\varepsilon}}\right). \tag{92}
$$

By combining I and II, we obtain that $R^\varepsilon(\omega, -L, z)$ satisfies the Riccati equation on the entire domain $[-L, 0]$:

$$\frac{dR^\varepsilon}{dz} = -\frac{i\omega}{2\bar{c}(z)\varepsilon}\nu\left(\frac{z}{\varepsilon^2}\right)\left(e^{-\frac{2i\omega\vartheta(z)}{\varepsilon}} - 2R^\varepsilon + (R^\varepsilon)^2 e^{\frac{2i\omega\vartheta(z)}{\varepsilon}}\right), \tag{93}$$

where

$$\bar{c}(z) = \begin{cases} \bar{c}_1, & z \in [-L, -L_1) \\ \bar{c}_2, & z \in [-L_1, 0] \end{cases}, \quad \vartheta(z) = \int_0^z \frac{ds}{\bar{c}(s)}. \tag{94}$$

The analysis of time reversal follows closely the case of a constant background. Because all transformations introduced in Section 4.1 are local in z, it follows that the same analysis will carry through. Time reversal experiment therefore results in a pulse which converges to a deterministic shape as $\varepsilon \to 0$. That shape can be written in the integral form (81), where $\Lambda(\omega, \tau) \equiv W_1(\omega, \tau, 0)$ can be obtained as a solution to the transport equations (66) with the background velocity $\bar{c}(z)$ given by (94).

5.2 Probabilistic representation

We know use the probabilistic representation of the solution to (66) to solve the same equations but with a jump in the coefficient $\bar{c}$. We first solve the equations in the medium ①. Since the solution for finite τ does not depend on L so long as the latter is large enough, we may set $L \to \infty$. Denoting the solution by $W_p^{(1)}$, we obtain

$$W_p^{(1)}(\omega, \tau, -L_1) = \begin{cases} \delta_0(\tau), & p = 0 \\ \dfrac{4p\bar{c}_1 (\gamma\omega^2)^p \tau^{p-1}}{(4\bar{c}_1 + \gamma\omega^2\tau)^{p+1}} \mathbf{1}_{[0,\infty)}(\tau), & p \neq 0. \end{cases} \tag{95}$$

We next solve the transport equations in medium ② with the background velocity $\bar{c}_2$ and the initial condition $W_p^{(1)}(\omega, \tau, -L_1)$. The Feynman-Kac formula yields:

$$\Lambda(\omega, \tau) = \mathbb{E}\left[W^{(1)}_{N_0^{(2)}}\left(\omega, \tau - \frac{2}{\bar{c}_2}\int_{-L_1}^0 N_s^{(2)}\, ds, -L_1\right) \mid N_{-L_1}^{(2)} = 1\right]. \tag{96}$$

Here $N_z^{(2)}$ is a Markov process constructed as before with $\bar{c} = \bar{c}_2$.

We would like to compare the solution to the transport equation with a jump in the coefficient $\bar{c}$, corresponding to a medium with a reflector to the equations with a constant background velocity $\bar{c}_2$ that reflect an empty medium. Towards that end, we write our solutions as

$$\Lambda(\omega,\tau) = \Lambda^{(2)}(\omega,\tau) + \mathbb{E}\left[\left(W^{(1)}_{N_0^{(2)}} - W^{(2)}_{N_0^{(2)}}\right)\left(\omega, \tau - \frac{2}{\bar{c}_2}\int_{-L_1}^{0} N_s^{(2)}\, ds, -L_1\right) \mid N^{(2)}_{-L_1} = 1\right], \tag{97}$$

where $W_p^{(2)}(\omega,\tau,-L_1)$ (and in particular $\Lambda^{(2)}(\omega,\tau)$) is a solution to the constant background problem with $\bar{c}_2$. Consider trajectories of $N^{(2)}$ that are factored into the expectation in (97)

i. If a trajectory $N_z^{(2)}$ is such that $N_0^{(2)} = 0$, it does not contribute to the expectation because $W_0^{(1)} - W_0^{(2)} \equiv 0$;

ii. If $N_0^{(2)} \neq 0$, it follows that $\int_{-L_1}^{0} N_s^{(2)}\, ds \geq L_1$. Since $W^{(i)}_{N_0^{(2)}}(\omega,\tau',-L_1) = 0, \forall \tau' < 0, i = 1,2$, the difference inside the expectation is non-zero only when $\tau - \frac{2}{\bar{c}_2}\int_{-L_1}^{0} N_s^{(2)}\, ds > 0$, or equivalently $\tau > \frac{2}{\bar{c}_2}\int_{-L_1}^{0} N_s^{(2)}\, ds \geq \frac{2L_1}{\bar{c}_2}$.

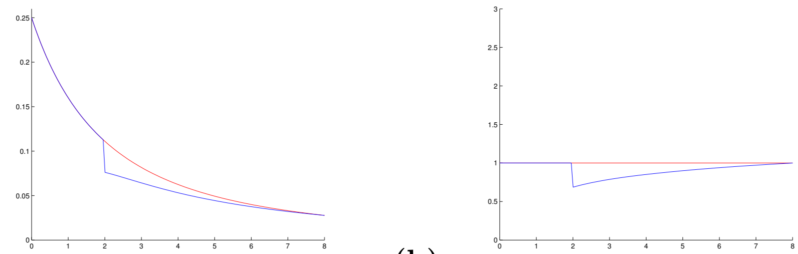
   More precisely,

   ii.1 Trajectories $N_z^{(2)}$, such that $N_z^{(2)} \equiv 1, \forall z \in [-L_1, 0]$ will create a jump in $\Lambda(\omega,\tau)$ *exactly* at $\tau = \frac{2L_1}{\bar{c}_2}$ (Figures 4,5);

   ii.2 All other trajectories will manifest themselves in the expextation for $\tau > \frac{2L_1}{\bar{c}_2}$.

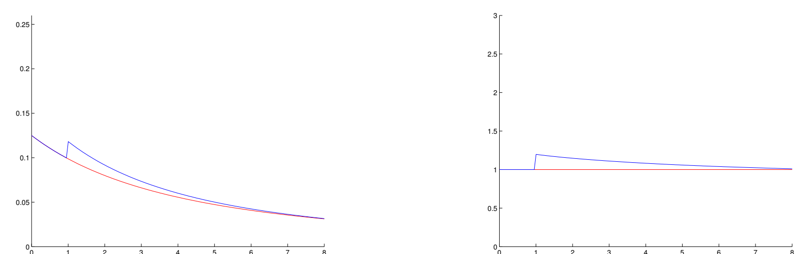
We can use ii.1 to explicitly compute the size of the jump in Λ at $\tau = \frac{2L_1}{\bar{c}_2}$. It is given by

$$\begin{aligned}\mathbb{E}\left[\left(W_1^{(1)} - W_1^{(2)}\right)(\omega, 0, -L_1)\cdot \mathbf{1}_{\left\{N_z^{(2)}\equiv 1\right\}} \mid N^{(2)}_{-L_1} = 1\right] \\ = \frac{\gamma\omega^2}{4}\left(\frac{1}{\bar{c}_1} - \frac{1}{\bar{c}_2}\right) e^{-\frac{\gamma\omega^2 L_1}{\bar{c}_2^2}}\end{aligned} \tag{98}$$



(a) **(b)**

Figure 4: Plot (a): Densities $\Lambda(\omega,\tau)$ and $\Lambda^{(2)}(\omega,\tau)$. Plot (b): Ratio of the densities $\frac{\Lambda(\omega,\tau)}{\Lambda^{(2)}(\omega,\tau)}$. Here we assume $\gamma\omega^2 = 2$, $L_1 = 1$, $\bar{c}_1 = 1.3$, and $\bar{c}_2 = 1$,



(a) **(b)**

Figure 5: Plot (a): Densities $\Lambda(\omega,\tau)$ and $\Lambda^{(2)}(\omega,\tau)$. Plot (b): Ratio of the densities $\frac{\Lambda(\omega,\tau)}{\Lambda^{(2)}(\omega,\tau)}$. Here we assume $\gamma\omega^2 = 2$, $L_1 = 1$, $\bar{c}_1 = 1.7$, and $\bar{c}_2 = 2$,

We finally recall that the presence of a dissipative layer results in a jump in the *derivative* of Λ [8]. A weak reflector introduced here creates a jump in Λ directly. The latter is clearly easier to detect in practice when $\varepsilon \approx 0$ but still positive, as has been observed numerically in [9].

5.3 Expanding window time reversal experiment

Following [8], we consider a family of expanding cut-off window functions

$$\left\{G^\tau(\cdot) = \mathbf{1}_{[0,\tau]}(\cdot)\right\}_{\tau\geq 0}, \tag{99}$$

For the same initial pulse, this family corresponds to a continuous family of time reversal experiments, which in turn gives rise to a collection of refocused signals:

$$s^\tau(\sigma) = \left(f(-\cdot) \star K(\cdot)\right)(\sigma). \tag{100}$$

The refocused signal s^τ will contain no reflections from the embedded layer if $\tau < \frac{2L_1}{\bar{c}_2}$. If, on the other hand, $\tau > \frac{2L_1}{\bar{c}_2}$, then the recorded coda contains information about the reflector introduced through the refocusing kernel K.

Taking the Fourier transform of (100), we obtain

$$\hat{K}(\omega,\tau) = \frac{\widehat{s^\tau}(\omega)}{\overline{\hat{f}(\omega)}}. \tag{101}$$

Also, since

$$\begin{aligned} \hat{K}(\omega,\tau) &= \int G^\tau(\tau')\Lambda(\omega,\tau')\,d\tau' \\ &= \int_{t_1}^{\tau} \Lambda(\omega,\tau')\,d\tau', \end{aligned} \tag{102}$$

we have

$$\frac{\partial}{\partial\tau}\hat{K}(\omega,\tau) = \Lambda(\omega,\tau).$$

Thus by performing a sufficient number of time reversal experiments with an increasingly large cut-off window, we are able to extract the density $\Lambda(\omega,\tau)$ to a satisfactory resolution.

5.4 Summary of detection algorithm

The above discussion on detection of a weak reflector embedded into a random medium characterized by the speed of sound propagation $\bar{c}_2$ can be summarized in the form of the following algorithm.

1. We first perform a set of expanding window time reversal experiments and record corresponding deterministic refocused pulses.

2. We use those pulses to retrieve the density $\Lambda(\omega, \tau)$.

3. For a fixed $\omega = \omega_0$, an observed jump in $\Lambda(\omega_0, \tau)$ at a time $\tau = \tau^\star$ would indicate the presence of weak reflector at the depth given by $L_1 = \frac{\tau^\star \bar{c}_2}{2}$.

4. The speed of sound propagation inside the reflector, $\bar{c}_1$, can then be retrieved from the formula (98).

We finally note here that a generalization to the case of multiple weak reflectors is possible. The locations of jumps in the density $\Lambda(\omega_0, \tau)$ will correspond to two way travel times of an acoustic wave to the physical locations of the reflectors. The formulas for the corresponding speeds, however, are not explicit.

6 Jump in medium statistics

We now consider the case of two media with the same background (macroscopic) parameters separated by an interface. The noises in the bulk modulus in each medium are denoted as $\nu_j,\ j = 1, 2$ and assumed to be independent, centered, stationary and strongly mixing, so that the respective integrated autocorrelations are well defined:

$$\gamma_j = \int_{-\infty}^{\infty} \mathbb{E}\big[\nu_j(0)\nu_j(s)\big]\, ds,\ j = 1, 2. \tag{103}$$

The average impedance is constant, and a wave impinging on the interface does not generate a coherent reflection. Proceeding as before, we derive that the reflection coefficient $R^\varepsilon \equiv R^\varepsilon(\omega, -L, z)$ satisfies a Riccati equation:

$$\begin{cases} \dfrac{dR^\varepsilon}{dz} = -\dfrac{i\omega}{2\bar{c}\varepsilon}\nu\left(z, \dfrac{z}{\varepsilon^2}\right)\left(e^{-\frac{2i\omega z}{\bar{c}\varepsilon}} - 2R^\varepsilon + (R^\varepsilon)^2 e^{\frac{2i\omega}{\bar{c}\varepsilon}}\right) \\ R^\varepsilon(\omega, -L, -L) = 0, \end{cases} \tag{104}$$

where the fluctuation process $\nu(\cdot,\cdot)$ is defined as follows:

$$\nu\left(z,\frac{z}{\varepsilon^2}\right)=\begin{cases}\nu_1\left(\frac{z}{\varepsilon^2}\right), z\in[-L,-L_1)\\ \nu_2\left(\frac{z}{\varepsilon^2}\right), z\in[-L_1,0].\end{cases} \tag{105}$$

The analysis of the time reversal experiment follows the lines of the previous sections. The refocused pulse converges to a deterministic shape as $\varepsilon\to 0$, and the refocusing density Λ is obtained through a system of transport equations similar to (66), where now the constant coefficient γ is replaced with a piecewise constant function $\gamma(z)$ given by

$$\gamma(z)=\begin{cases}\gamma_1,\, z\in[-L,-L_1)\\ \gamma_2,\, z\in[-L_1,0].\end{cases} \tag{106}$$

We now continue by letting $L\to\infty$, which enables us to obtain an explicit formula for $\Lambda(\omega,\tau)$ by solving the transport equations () in the medium ① with the constant noise autocorrelation γ_1. Denoting the solution by $W_p^{(1)}$, we have

$$W_p^{(1)}(\omega,\tau,-L_1)=\begin{cases}\delta_0(\tau), & p=0\\ \dfrac{8p\gamma_1\omega^2}{\bar{c}}\dfrac{(\gamma_1\omega^2\tau/\bar{c})^{p-1}}{(8+\gamma_1\omega^2\tau/\bar{c})^{p+1}}\,\mathbf{1}_{[0,\infty)}(\tau), & p\neq 0.\end{cases} \tag{107}$$

Next we solve the same system of transport equations in the medium ② with the autocorrelation constant γ_2 and the initial condition $W_p^{(1)}(\omega,\tau,-L_1)$ obtained above. The probabilistic interpretation derived in () for the resulting solution yields:

$$\begin{aligned}\Lambda(\omega,\tau) &\equiv W_1(\omega,\tau,0)\\ &=\mathbb{E}\left[W^{(1)}_{N_0^{(2)}}\left(\omega,\tau-\frac{2}{\bar{c}}\int_{-L_1}^{0}N_s^{(2)}ds,-L_1\right)\mid N_{-L_1}^{(2)}=1\right].\end{aligned} \tag{108}$$

$N^{(2)}$ is here a jump process with the state dependent intensity $\frac{N^2\gamma_2\omega^2}{2\bar{c}^2}$ and jumps of size ±1 with equal probabilities. Denote as $W_p^{(2)}$ (and consequently $\Lambda^{(2)}$) the solution to the transport equations corresponding to no interface, i.e. $\gamma(z) \equiv \gamma_2,\ z \in [-L, 0]$. Then

$$\begin{aligned}
\Lambda(\omega,\tau) &= \Lambda^{(2)}(\omega,\tau) + \left(\Lambda(\omega,\tau) - \Lambda^{(2)}(\omega,\tau)\right) \\
&= \Lambda^{(2)}(\omega,\tau) + \mathbb{E}\left[\left(W^{(1)}_{N_0^{(2)}} - W^{(2)}_{N_0^{(2)}}\right) \left(\omega, \tau - \frac{2}{\bar{c}} \int_{-L_1}^{0} N_s^{(2)} ds, -L_1\right)\right. \\
&\qquad \left. \mid N^{(2)}_{-L_1} = 1\right]
\end{aligned} \tag{109}$$

Using the trajectory analysis presented previously, we deduce that

$$\Lambda(\omega,\tau) = \Lambda^{(2)}(\omega,\tau), \tag{110}$$

when $\tau < \frac{2L_1}{\bar{c}}$, and that there is a jump in $\Lambda(\omega,\tau)$ for any fixed frequency ω at $\tau = \frac{2L_1}{\bar{c}}$. The size of the jump is given by

$$\begin{aligned}
&\mathbb{E}\left[\left(W_1^{(1)} - W_1^{(2)}\right)(\omega, 0, -L_1)\mathbf{1}_{\left\{N_z^{(2)}=1,\, z\in[-L_1,0]\right\}} \mid N^{(2)}_{-L_1} = 1\right] \\
&= \left(W_1^{(1)} - W_1^{(2)}\right)(\omega, 0, -L_1)\,\mathbb{P}\left[N_z^{(2)} = 1,\ z \in [-L_1, 0] \mid N^{(2)}_{-L_1} = 1\right] \\
&= \frac{\omega^2}{8\bar{c}}\left(\gamma_1 - \gamma_2\right) e^{-\frac{\gamma_2 \omega^2 L_1}{2\bar{c}^2}}.
\end{aligned} \tag{111}$$

That implies a sudden change in in the statistics of random fluctuations can be detected and located precisely. Furthermore, if $\bar{c}$ and γ_2 are known, we can characterize the statistics of the noise below the interface. We also note that the above computations cover the case of a region with no noise, i.e. $\gamma_1 = 0$.

A comparison of the results of this section with those obtained previously for the case of a jump in $\bar{c}$ reveals that the defining quantity is the ratio $\gamma/\bar{c}$. If both characteristics, γ and $\bar{c}$, change at the same depth $-L_1$, then the weak reflector mainfests itself in a jump in the density $\Lambda(\omega,\tau)$ at $\tau^\star = 2L_1/\bar{c}_2$. The size of the jump is given by

$$\frac{\omega^2}{8}\left(\frac{\gamma_1}{\bar{c}_1} - \frac{\gamma_2}{\bar{c}_2}\right) e^{-\frac{\gamma_2 \omega^2 L_1}{2\bar{c}_2^2}}, \tag{112}$$

which implies that only the ratio $\gamma_1/\bar{c}_1$ can be recovered from the surface.

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