Diffusion-Approximation for the Advection-Diffusion of a Passive Scalar by a Space-Time Gaussian Velocity Field [1]

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Abstract

We study the asymptotic behavior, as ϵ goes to 0, of a passive scalar $T^\epsilon(x,t)$ solution of the following advection-diffusion equation:

$$\frac{\partial T^\epsilon}{\partial t} = \frac{\nu}{2}\triangle T^\epsilon + \frac{1}{\epsilon}V(x,\frac{t}{\epsilon^2})\cdot\nabla T^\epsilon,\ t>0$$
$$T^\epsilon(x,0) = T_0(x),\ x\in\mathbb{R}^d$$

where ν is a strictly positive diffusion constant and $\{V(x,t), x\in\mathbb{R}^d, t\geq 0\}$ is a mean zero homogeneous Gaussian field. We assume that the covariance is of the form:

$$\mathbb{E}\{V(x,t)V^*(y,s)\} = \Gamma(x-y)\exp(-a|t-s|)$$

and under some regularity assumption on Γ, we prove that $T^\epsilon(x,t)$ converges in distribution to the solution of a stochastic partial differential equation. We derive the effective diffusion coefficient from this result. This work is a generalization of previous works by Bouc-Pardoux [3] and Kushner-Huang [8] where the velocity field is of the form $\frac{1}{\epsilon}V(x,Z_{t/\epsilon^2})$ for some finite dimensional ergodic noise process Z. Our situation is an example of infinite dimensional noise.

1 Introduction

The advection-diffusion of a passive scalar $T^\epsilon(x,t)$ by a velocity field $V_\epsilon(x,t)$ on a small scale ϵ $(0<\epsilon\ll 1)$ is given by the equation:

$$\left\{\begin{array}{l}\frac{\partial T}{\partial t} = \frac{\nu}{2}\triangle T + V_\epsilon\cdot\nabla T\ ,\ t>0\\ T(x,0)=T_0(\epsilon x),\ x\in\mathbb{R}^d.\end{array}\right. \tag{1}$$

We assume that the diffusion constant ν is strictly positive and that V_ϵ is a d-dimensional Gaussian velocity field. Ajusting the scaling to the nature of the singularity of the spectral density of this Gaussian field Avellaneda and Majda have studied several renormalization procedures for this equation. We refer to

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their works [1] and [2] for details. They obtained different limiting regimes: among them, the Mean Field and the so-called Anomalous regimes. In this paper we study the diffusion-approximation regime which takes place when $V_\epsilon(x,t)$ is given by rescaling of a single (nonsingular) field $V(x,t)$. More precisely, we assume that $V_\epsilon(x,t)$ is given by $V(\epsilon x, t)$ with the scaling law $(\frac{x}{\epsilon}\ ,\ \frac{t}{\epsilon^2})$. Denoting $T(\frac{x}{\epsilon}, \frac{t}{\epsilon^2})$ by $T^\epsilon(x,t)$ or $T^\epsilon_t(x)$, our problem (1) becomes:

$$\frac{\partial T^\epsilon}{\partial t} = \frac{1}{2}\nu \triangle T^\epsilon + \frac{1}{\epsilon} V(x, \frac{t}{\epsilon^2}) \cdot \nabla T^\epsilon \tag{2}$$

$$T^\epsilon(x,0) = T_0(x) \tag{3}$$

where we suppose the initial condition T_0 in $L^2(\mathbb{R}^d)$ and where we choose, as a Markovian model, a very simple covariance given by:

$$\mathbb{E}\{V(x,s)V^*(y,t)\} = \Gamma(x-y)\exp\left(-a|t-s|\right) \tag{4}$$

with a constant $a > 0$ and a C^∞ space covariance Γ. In the divergence free case (i.e. in the incompressible case for which $(\mathrm{div} V = 0)$), the problem (1) can be used in the two-dimensional case $d = 2$ as a model for the temperature field at the surface of the ocean. This problem is of great importance. In particular, the diffusion approximation result which we prove for the law of T^ϵ as $\epsilon \searrow 0$ gives the justification for a way to close the equations for the moments of T^ϵ. Such closed equations can be found for example in the paper [9] which was one of the motivating factors of our study.

In the next section we show how the solution of (2) can be represented by means of stochastic flows and we deduce *a priori* estimates from such an expression. The results of [6] on the convergence of flows, as used in [10], cannot be used directly in the present situation because of the unboundedness of the velocity field. In the third section we prove the convergence of $(T^\epsilon_t)_{t\geq 0}$ as processes taking their values in the space of distributions $\mathcal{D}'(\mathbb{R}^d)$. In the last section we study the particular case of incompressible velocity fields for which we have stronger *a priori* estimates. The present work should be regarded as a generalization of the works of Bouc-Pardoux [3] and Kushner-Huang [8] which deal with velocity fields of the form $\frac{1}{\epsilon}V(x, Z_{t/\epsilon^2})$ for some finite dimensional ergodic noise process Z. The main thrust of the present paper is to give a genuine example of an *infinite dimensional randomness* for which diffusion approximation holds.

2 Representation of the Solution and a Simple *a priori* Estimate

We shall denote the $L^2(\mathbb{R}^d)$-norm by $|\cdot|_2$. Let us multiply both sides of equation (2) by T^ϵ_t and let us integrate over $\mathbb{R}^d \times [0,t]$. We get:

$$
\begin{aligned}
|T_t^\epsilon|_2^2 - |T_0|_2^2 &= \nu \int_0^t \int_{\mathbb{R}^d} (\triangle T_s^\epsilon(x))\, T_s^\epsilon(x) dx ds \\
&+ \frac{2}{\epsilon} \int_0^t \int_{\mathbb{R}^d} \left(V(x, \frac{s}{\epsilon^2}) \cdot \nabla T_s^\epsilon(x) \right) T_s^\epsilon(x) dx ds.
\end{aligned}
$$

Integration by parts and substitution in the previous equation give:

$$
\begin{aligned}
|T_t^\epsilon|_2^2 - |T_0|_2^2 &= -\nu \int_0^t |\nabla T_s^\epsilon|_2^2 ds \\
&\qquad - \frac{1}{\epsilon} \int_0^t \int_{\mathbb{R}^d} (T_s^\epsilon(x))^2 \mathrm{div} V(x, \frac{s}{\epsilon^2}) dx ds
\end{aligned} \tag{5}
$$

In order to compensate the last term of this equation one may try a perturbation method as in [8]. Unfortunately $\mathrm{div} V(x, \frac{s}{\epsilon^2})$, as a homogeneous Gaussian field is unbounded and the method fails. Nevertheless we shall use this approach in the incompressible case ($\mathrm{div} V = 0$) to get strong estimates (last section).

We now describe another way to represent the solution $T_t^\epsilon(x)$. Given a d-dimensional standard Brownian motion $\{B_t\}_t$ independent of V we consider the following stochastic differential equation:

$$
\left\{ \begin{array}{l} d\Phi_s^{t,\epsilon}(x) = -\frac{1}{\epsilon} V(\Phi_s^{t,\epsilon}(x), \frac{s}{\epsilon^2}) ds + \nu^{1/2} dB_s \ , \ 0 \le s \le t \\ \Phi_t^{t,\epsilon}(x) = x \end{array} \right. \tag{6}
$$

We shall denote by $\mathbb{E}_B$ and $\mathbb{E}_V$ the expectations with respect to the Brownian motion B and the velocity field V respectively. Since the Gaussian random field V is homogeneous and since its covariance is continuous and tends to 0 at infinity, the growth of the sample realizations of this field V is sublinear. This implies that equation (6) defines a unique stochastic flow Φ. If we suppose that the initial data T_0 is smooth and with compact support then we have:

$$
T^\epsilon(x,t) = \mathbb{E}_B \{ T_0(\Phi_0^{t,\epsilon}(x)) \} \tag{7}
$$

We refer to [5] or [6] for details concerning this representation formula. The following lemma shows that T_t^ϵ is almost surely a function in $L^2(\mathbb{R}^d)$:

Lemma 2.1 *For every $t > 0$ and $\epsilon > 0$, $\mathbb{E}\{|T_t^\epsilon|_2^2\} \le |T_0|_2^2$.*

Proof:
Setting $\Psi_s^{t,\epsilon}(x) = \Phi_s^{t,\epsilon}(x) - x$, we get:

$$
\left\{ \begin{array}{l} d\Psi_s^{t,\epsilon}(x) = -\frac{1}{\epsilon} V(\Psi_s^{t,\epsilon}(x) + x, \frac{s}{\epsilon^2}) ds + \nu^{1/2} dB_s \\ \Psi_t^{t,\epsilon}(x) = 0 \end{array} \right. \tag{8}
$$

which implies, by the stationarity of V, that for fixed x the law of $(\Phi_s^{t,\epsilon}(x), 0 \leq s \leq t)$ is the same as the law of $(\Phi_s^{t,\epsilon}(0)+x, 0 \leq s \leq t)$. Now:

$$
\begin{aligned}
\mathbb{E}_V\{|T_t^\epsilon|_2^2\} &= \mathbb{E}_V\left\{\int_{\mathbb{R}^d} \mathbb{E}_B\{T_0(\Phi_0^{t,\epsilon}(x)\}^2 dx\right\} \\
&\leq \int_{\mathbb{R}^d} \mathbb{E}_B\{\mathbb{E}_V\{T_0^2(\Phi_0^{t,\epsilon}(x))\}\}dx \\
&= \int_{\mathbb{R}^d} \mathbb{E}_B\{\mathbb{E}_V\{T_0^2(\Phi_0^{t,\epsilon}(0)+x\}\} \\
&= \mathbb{E}_B\{\mathbb{E}_V\{\int_{\mathbb{R}^d} T_0^2(\Phi_0^{t,\epsilon}(0)+x)dx\}\} \\
&= |T_0|_2^2
\end{aligned}
$$

$\square$

A possible approach would be to try to generalize the convergence results of stochastic flows given in Kunita [6] and to use the representation (7) to prove the convergence of $\{T_t^\epsilon\}$. This would require the generalization of some of the results of [6] because of the unboundedness of the velocity field V in the present situation. It would also give convergence of (T_t^ϵ) as continuous processes in spaces like C^k. Such an approach has been used in [10] for a class of velocity fields which do not contain our model.

Instead, we have chosen a more functional analytic approach which relates to the parabolic nature of the problem. It consists in studying the weak form of (2) for a well suited set of test functions.

3 Convergence as Distribution-Valued Processes

Let θ be an element of the Schwartz space $\mathcal{D}(\mathbb{R}^d)$ of real-valued smooth C^∞ test functions with compact supports. $\theta \in \mathcal{D}(\mathbb{R}^d)$. We denote by $(\cdot,\cdot)$ the scalar product in $L^2(\mathbb{R}^d)$ as well as the canonical pairing $(\mathcal{D}', \mathcal{D})$ between the space $\mathcal{D}(\mathbb{R}^d)$ and its dual $\mathcal{D}'(\mathbb{R}^d)$. With these notations, the weak form of equation (2) can be written as:

$$
\begin{aligned}
(T_t^\epsilon, \theta) - (T_0, \theta) &= \frac{\nu}{2}\int_0^t (T_s^\epsilon, \triangle\theta)ds \\
&- \frac{1}{\epsilon}\int_0^t \left(T_s^\epsilon, \operatorname{div}\left(\theta V(\cdot, \frac{s}{\epsilon^2})\right)\right) ds \qquad (9)
\end{aligned}
$$

We chose the smallest space of test functions. This is bound to weaken some of the results. Here are the reasons for our choice. On one hand, the regularity of θ

is needed to give a meaning to the equation: we have chosen C^∞ test functions in order to consider our $L^2(\mathbb{R}^d)$-valued process (T_t^ϵ) as a process with values in a nuclear space. This will enable us to use the simple tightness criteria of [4] which we recall in the next section. On the other hand the compactness of the support of the test fuctions will be essential when it comes to handle the unboundedness of the Gaussian field V. As a consequence, we shall consider (T_t^ϵ) as a $\mathcal{D}'(\mathbb{R}^d)$-valued process. Note nevertheless that, in the incompressible case, we are able to prove the convergence in a stronger sense. See the last section for details.

3.1 Tightness

According to [4], a family of $\mathcal{D}'(\mathbb{R}^d)$-valued continuous (resp. right continuous with left limits) processes $\{(T_\cdot^\epsilon), 0 < \epsilon \le 1\}$ is tight if and only if, for every test function θ in $\mathcal{D}(\mathbb{R}^d)$, the family of real-valued continuous (resp. right continuous with left limit) processes $\{(T_\cdot^\epsilon, \theta), 0 < \epsilon \le 1\}$ is tight. As in [8], we prove the tightness of the right continuous processes with left limits $\{f((T_\cdot^\epsilon, \theta)), 0 < \epsilon \le 1\}$ for all the possible choices of twice differentiable compactly supported real valued function f. This is done by a perturbation method and an adequate tightness criterium developped in [7] (see also [8]). From (9) we deduce an equation for $f(T_t^\epsilon, \theta)$:

$$\begin{aligned} f(T_t^\epsilon, \theta) - f(T_0, \theta) &= \frac{\nu}{2} \int_0^t f'(T_s^\epsilon, \theta)(T_s^\epsilon, \triangle\theta) ds \\ &\quad - \frac{1}{\epsilon} \int_0^t f'(T_s^\epsilon, \theta) \left(T_s^\epsilon, \operatorname{div} \left(\theta V(\cdot, \frac{s}{\epsilon^2}) \right) \right) ds. \quad (10) \end{aligned}$$

Denoting by $\mathcal{F}_t^\epsilon$ the σ-algebra generated by $\{V(x,u), x \in \mathbb{R}^d, u \le t/\epsilon^2\}$ and by $\mathbb{E}_t^\epsilon$ the conditional expectation with respect to $\mathcal{F}_t^\epsilon$, we define the *first pertubation* of equation (10) by:

Definition 3.1

$$f_1^\epsilon(t) = \frac{1}{\epsilon} \int_t^{+\infty} \mathbb{E}_t^\epsilon f'(T_t^\epsilon, \theta) \left(T_t^\epsilon, \operatorname{div} \left(\theta V(\cdot, \frac{s}{\epsilon^2}) \right) \right) ds \tag{11}$$

Equation (2) implies that T_t^ϵ is $\mathcal{F}_t^\epsilon$-measurable. Moreover, the form of the covariance given in (4) implies that $\mathbb{E}_t^\epsilon V(x, \frac{s}{\epsilon^2}) = V(x, \frac{t}{\epsilon^2}) \exp(-a(s-t)/\epsilon^2)$. From these facts one easily gets:

$$f_1^\epsilon(t) = \frac{\epsilon}{a} f'(T_t^\epsilon, \theta) \left(T_t^\epsilon, \operatorname{div} \left(\theta V(\cdot, \frac{t}{\epsilon^2}) \right) \right) \tag{12}$$

The next lemma tells us how small the perturbation $f_1^\epsilon(t)$ is:

Lemma 3.2 *We have:*

a) $\sup_t \mathbb{E}|f_1^\epsilon(t)| = \epsilon \cdot O(1)$

b) For every $t_0 \geq 0$, $\sup_{t\leq t_0} |f_1^\epsilon(t)| \longrightarrow 0$ *in probability as* $\epsilon \searrow 0$

Proof:

a) From (12) and lemma 2.1:

$$\begin{aligned} \mathbb{E}\{|f_1^\epsilon(t)|\} &\leq \frac{\epsilon}{a}|f'|_\infty \mathbb{E}\left\{\left|\left(T_t^\epsilon, \mathrm{div}\left(\theta V(\cdot, \frac{t}{\epsilon^2})\right)\right)\right|\right\} \\ &\leq \frac{\epsilon}{a}|f'|_\infty \cdot |T_0|_2 \mathbb{E}\left\{\left|\theta \mathrm{div} V(\cdot, \frac{t}{\epsilon^2}) + V(\cdot, \frac{t}{\epsilon^2}) \cdot \nabla\theta\right|_2^2\right\}^{1/2}. \end{aligned}$$

Using the notation $M = \sup_{x\in supp\theta} |x|$ we have:

$$\begin{aligned} \mathbb{E}\{|\theta \mathrm{div} V(\cdot, \frac{t}{\epsilon^2})|_2^2\} &\leq |\theta|_2^2 \mathbb{E}\left\{\sup_{|x|\leq M} |\mathrm{div} V(x, \frac{t}{\epsilon^2})|^2\right\} \\ &= |\theta|_2^2 \mathbb{E}\left\{\sup_{|x|\leq M} |\mathrm{div} V(x, 0)|^2\right\}. \end{aligned}$$

Similarly:

$$\mathbb{E}\{|V(\cdot, \frac{t}{\epsilon^2}) \cdot \nabla\theta|_2^2\} \leq |\nabla\theta|_2^2 \cdot \mathbb{E}\left\{\sup_{|x|\leq M} |V(x, 0)|^2\right\}.$$

Therefore:

$$\mathbb{E}\{|f_1^\epsilon(t)|\} \leq \frac{\epsilon}{a}|f'|_\infty |T_0|_2 \left(C_1|\theta|_2 + C_2|\nabla\theta|_2\right)$$

where the constants C_1 and C_2 depend on θ or more precisely on its support.

b) From (12), (7) and the boundedness assumption on T_0, we get:

$$\begin{aligned} \sup_{t\leq t_0} |f_1^\epsilon(t)| \leq\ & C(M)\frac{\epsilon}{a}|f'|_\infty |T_0|_\infty \\ & \left(|\theta|_2 \sup_{|x|\leq M, t\leq t_0} |\mathrm{div} V(x, \frac{t}{\epsilon^2})| + |\nabla\theta|_2 \sup_{|x|\leq M, t\leq t_0} |V(x, \frac{t}{\epsilon^2})|\right). \end{aligned}$$

Since the homogeneity of the velocity filed implies that one can find a (random) constant C such that:

$$\sup_{|x|\leq M, t\leq t_0} |V(x, \frac{t}{\epsilon^2})| \leq C \cdot \log\left(\frac{(2M)^d t_0}{\epsilon^2}\right)$$

and a similar inequality for div$V$, we can conclude that $\sup_{t\leq t_0}|f_1^\epsilon(t)| \longrightarrow 0$ almost surely as $\epsilon \searrow 0$. ∎

Following Kushner [7] (Chapter 3), we define the *pseudo-generator* A^ϵ acting on any $\mathcal{F}_t^\epsilon$-measurable function $f(t)$ such that $\sup_t |f(t)| < \infty$ as the L^1-limit of $\delta^{-1}(\mathbb{E}_t^\epsilon f(t+\delta) - f(t))$ whenever this limit exists. Th latter will be denoted by $A^\epsilon f$. The main property of this pseudo-generator is that:

$$f(t) - \int_0^t A^\epsilon f(s) ds$$

is a martingale with respect to the filtration $(\mathcal{F}_t^\epsilon)$ ([7], Chapter 3, Th.1). In the present situation we choose $f(t) = f^\epsilon(t) = f(T_t^\epsilon, \theta) - f_1^\epsilon(t)$ in order to cancel the terms in $1/\epsilon$ in the pseudo-generator:

Proposition 3.3 . *If we use the notations:*

$$f^\epsilon(t) = f(T_t^\epsilon, \theta) - f_1^\epsilon(t) \qquad \textit{and} \qquad \Psi_t^\epsilon(\theta) = \mathrm{div}\left(\theta V(\cdot, \frac{t}{\epsilon^2})\right),$$

then we have:

$$\begin{aligned} A^\epsilon f^\epsilon(t) &= \frac{\nu}{2} f'(T_t^\epsilon, \theta)(T_t^\epsilon, \triangle\theta) + \frac{1}{a} f''(T_t^\epsilon, \theta)(T_t^\epsilon, \Psi_t^\epsilon(\theta))^2 \\ &\quad + \frac{1}{a} f'(T_t^\epsilon, \theta)\,(T_t^\epsilon, \Psi_t^\epsilon(\Psi_t^\epsilon(\theta))) \\ &\quad - \frac{\epsilon\nu}{2a}\{f''(T_t^\epsilon, \theta)(T_t^\epsilon, \triangle\theta) + f'(T_t^\epsilon, \theta)(T_t^\epsilon, \triangle\Psi_t^\epsilon(\theta))\}. \quad (13) \end{aligned}$$

Proof.
The form (10) implies that $f(T_t^\epsilon, \theta)$ is differentiable in t. Notice that A^ϵ acts on the latter as a differentiation operator:

$$A^\epsilon f(T_t^\epsilon, \theta) = \frac{\nu}{2} f'(T_t^\epsilon, \theta)(T_t^\epsilon, \triangle\theta) - \frac{1}{\epsilon} f'(T_t^\epsilon, \theta)(T_t^\epsilon, \Psi_t^\epsilon(\theta)) \qquad (14)$$

The first term on the right hand side of (13) is the first term of the right hand side of (14). Using (12) and $\mathbb{E}_t^\epsilon \Psi_{t+\delta}^\epsilon(\theta) = \Psi_t^\epsilon(\theta)\exp(-a\delta/\epsilon^2)$, we get:

$$\begin{aligned} \mathbb{E}_t^\epsilon f_1^\epsilon(t+\delta) &= \frac{\epsilon}{a}\mathbb{E}_t^\epsilon\left(f'(T_{t+\delta}^\epsilon, \theta) - f'(T_t^\epsilon, \theta)\right)(T_{t+\delta}^\epsilon, \Psi_{t+\delta}^\epsilon(\theta)) \\ &\quad + \frac{\epsilon}{a} f'(T_t^\epsilon, \theta)\mathbb{E}_t^\epsilon(T_{t+\delta}^\epsilon - T_t^\epsilon, \Psi_{t+\delta}^\epsilon(\theta)) \\ &\quad + \frac{\epsilon}{a} f'(T_t^\epsilon, \theta)(T_t^\epsilon, \Psi_t^\epsilon(\theta))\exp(-\frac{a\delta}{\epsilon^2}) \qquad (15) \end{aligned}$$

Substracting $f_1^\epsilon(t)$, dividing by δ and taking a limit as $\delta \searrow 0$ gives:

$$\begin{aligned} A^\epsilon f_1^\epsilon(t) &= -f''(T_t^\epsilon,\theta)(T_t^\epsilon,\Psi_t^\epsilon(\theta))^2 - f'(T_t^\epsilon,\theta)\,(T_t^\epsilon,\Psi_t^\epsilon(\Psi_t^\epsilon(\theta))) \\ &\quad + \frac{\epsilon\nu}{2a}\{f''(T_t^\epsilon,\theta)(T_t^\epsilon,\triangle\theta) + f'(T_t^\epsilon,\theta)(T_t^\epsilon,\Psi_t^\epsilon(\theta))\} \\ &\quad - \frac{1}{\epsilon}f'(T_t^\epsilon,\theta)(T_t^\epsilon,\Psi_t^\epsilon(\theta)) \end{aligned} \tag{16}$$

where we have used the continuity of $\Psi_t^\epsilon(\theta)$ and $(T_t^\epsilon,\Psi_t^\epsilon(\theta))$. The fact that these limits are in L^1 is justified by the same type of estimates that we have obtained in the proof of lemma 3.2.
The equality $f^\epsilon(t) = f(T_t^\epsilon,\theta) - f_1^\epsilon(t)$ together with (14) and (16) give (13) because of the cancellation of the $1/\epsilon$ terms. ∎

Theorem 3.4 *The laws of* $(T_\cdot^\epsilon)_{0<\epsilon\le 1}$ *are tight on* $D\left([0,+\infty);\mathcal{D}'(I\!R^d)\right)$.

Proof:
As explained at the beginning of this section, it is enough to show that the laws of $(f(T_\cdot^\epsilon,\theta))_{0<\epsilon\le 1}$ are tight on $D([0,+\infty);\mathbb{R})$ for each fixed $\theta \in \mathcal{D}(\mathbb{R}^d)$ and each C^2 function f with compact support. We apply the tightness criterium given in Kushner [7] (Chapter 3, Th.4) based on the martingale property of $f^\epsilon(t) - \int_0^t A^\epsilon f^\epsilon(s)ds$ and the fact that the term $f(T_t^\epsilon,\theta) - f^\epsilon(t) = f_1^\epsilon(t)$ is small. Lemma 3.2 b) implies that:

$$\lim_{\epsilon\searrow 0}\mathbb{P}\{\sup_{t\le t_0}|f^\epsilon(t) - f(T_t^\epsilon,\theta)| \ge \alpha\} = 0$$

for every $\alpha > 0$ and $t_0 < +\infty$.
It then remains to prove that for every finite t_0, the family $\{A^\epsilon f^\epsilon(t),\ 0<\epsilon\le 1\,, t \le t_0\}$ is uniformely integrable. This is done for each term appearing in $A^\epsilon f^\epsilon(t)$ in (13), in a similar way we proved lemma 3.2 a); the compact support property of θ enables us to take a supremum in x for $|x| \le M$ and by stationarity of V and its derivatives we remove the dependency in t/ϵ^2. ∎

3.2 Identification of the limit

Denoting by $(T_\cdot^\epsilon)$ a weakly convergent subsequence and by $(T_\cdot)$ its limit, according to [4], in order to identify the law of the limit, it is enough to identify the finite dimensional distributions of $\{(T_\cdot,\theta_1),...,(T_\cdot,\theta_n)\}$ for every $\theta_1,...,\theta_n$ in $\mathcal{D}(\mathbb{R}^d)$.

The martingale property of $f^\epsilon(t) - \int_0^t A^\epsilon f^\epsilon(s)ds$ implies that for every bounded continuous function h and for every sequence $0 < s_1 < \cdots < s_n \le s < t$ we have:

$$\mathbb{E}\left\{\left(f^\epsilon(t) - f^\epsilon(s) - \int_s^t A^\epsilon f^\epsilon(u)du\right) h\left((T_{s_1}^\epsilon,\theta),\cdots,(T_{s_n}^\epsilon,\theta)\right)\right\} = 0. \tag{17}$$

Denoting by $A_i^\epsilon(t), i = 1,2,3,4$ the four terms in $A^\epsilon f^\epsilon(t)$ given by (13) we have that:

- $A_1^\epsilon(t)$ is uniformely bounded in ϵ and converges weakly to $\frac{\nu}{2} f'(T_t,\theta)(T_t,\triangle\theta)$.
- $\mathbb{E}\{\int_s^t |A_4^\epsilon f^\epsilon(u)|du\}$ converges to 0 for the same reason as in the proof of lemma 3.1 a).

The last two terms are given by:

$$
\begin{aligned}
A_2^\epsilon(t) &= \frac{1}{a} f''(T_t^\epsilon,\theta)(T_t^\epsilon,\Psi_t^\epsilon(\theta))^2 \\
A_3^\epsilon(t) &= \frac{1}{a} f'(T_t^\epsilon,\theta)\,(T_t^\epsilon,\Psi_t^\epsilon(\Psi_t^\epsilon(\theta)))
\end{aligned}
$$

One needs a *second perturbation* in order to center these two terms and get convergence in (17).

We introduce special notations for the quantities which appear in the limit:

Definition 3.5 *For θ,θ_1,θ_2 in $\mathcal{D}(\mathbb{R}^d)$ and ϕ in $L^2(\mathbb{R}^d)$:*

$$
\begin{aligned}
&H_{\theta_1,\theta_2}(x,y) = \sum_{i=1}^d \sum_{j=1}^d \partial_{x_i}\partial_{y_j}\left[\theta_1(x)\theta_2(y)\Gamma_{i,j}(x-y)\right] \\
&A_2(\phi) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \phi(x)\phi(y)H_{\theta,\theta}(x,y)dxdy \\
&B = \sum_{i=1}^d \sum_{j=1}^d \Gamma_{i,j}(0)\partial_{x_i}\partial_{x_j} \\
&b = \{b_j = \sum_{i=1}^d (\partial_{x_i}\Gamma_{i,j})(0), j = 1,\cdots,d\} \\
&A_3(\phi) = (\phi, b\cdot\nabla\theta + B\theta)
\end{aligned}
$$

With these notations, an easy computation shows:

Lemma 3.6 *$\mathbb{E}(\phi,\Psi_t^\epsilon(\theta))^2 = A_2(\phi)$ and $\mathbb{E}(\phi,\Psi_t^\epsilon(\Psi_t^\epsilon(\theta))) = A_3(\phi)$.*

The *second perturbation* involves the following two terms:

Definition 3.7

$$
\begin{aligned}
f_2^\epsilon(t) &= \frac{1}{a} f''(T_t^\epsilon,\theta)\int_t^{+\infty} \mathbb{E}_t^\epsilon\left[(T_t^\epsilon,\Psi_s^\epsilon(\theta))^2 - A_2(T_t^\epsilon)\right] ds \\
f_3^\epsilon(t) &= \frac{1}{a} f'(T_t^\epsilon,\theta)\int_t^{+\infty} \mathbb{E}_t^\epsilon\left[(T_t^\epsilon,\Psi_s^\epsilon(\Psi_s^\epsilon(\theta))) - A_3(T_t^\epsilon)\right] ds
\end{aligned}
$$

Using the conditional covariance:

$$\mathbb{E}_t^\epsilon \left\{ V_i(x, \frac{s}{\epsilon^2}) V_j(y, \frac{s}{\epsilon^2}) \right\} =$$
$$\Gamma_{i,j}(x-y) \left(1 - \exp(-2a\frac{s-t}{\epsilon^2}) \right) + V_i(x, \frac{t}{\epsilon^2}) V_j(y, \frac{t}{\epsilon^2}) \exp(-2a\frac{s-t}{\epsilon^2})$$

one easily gets:

$$f_2^\epsilon(t) = \frac{\epsilon^2}{2a^2} f''(T_t^\epsilon, \theta) \left((T_t^\epsilon, \Psi_t^\epsilon(\theta))^2 - A_2(T_t^\epsilon) \right) \tag{18}$$

and

$$f_3^\epsilon(t) = \frac{\epsilon^2}{2a^2} f'(T_t^\epsilon, \theta) \left((T_t^\epsilon, \Psi_t^\epsilon(\Psi_t^\epsilon(\theta))) - A_3(T_t^\epsilon) \right). \tag{19}$$

Using the explicit forms (18) and (19) for $f_2^\epsilon(t)$ and $f_3^\epsilon(t)$ and the argument given in lemma 3.2 a) for $f_1^\epsilon(t)$, we get:

Lemma 3.8 .

$$\sup_t \mathbb{E}|f_2^\epsilon(t)| = \epsilon^2 \cdot O(1)$$

and:

$$\sup_t \mathbb{E}|f_3^\epsilon(t)| = \epsilon^2 \cdot O(1)$$

If we follow the steps of the computation of $A^\epsilon f_1^\epsilon(t)$ in the proof of Proposition 3.3, after a long but straightforward computation, we get:

Proposition 3.9

$$\begin{aligned} A^\epsilon f_2^\epsilon(t) &= \frac{1}{a} f''(T_t^\epsilon, \theta) \left(A_2(T_t^\epsilon) - (T_t^\epsilon, \Psi_t^\epsilon(\theta))^2 \right) + R_2^\epsilon(t) \\ A^\epsilon f_3^\epsilon(t) &= \frac{1}{a} f'(T_t^\epsilon, \theta) \left(A_3(T_t^\epsilon) - (T_t^\epsilon, \Psi_t^\epsilon(\Psi_t^\epsilon(\theta))) \right) + R_3^\epsilon(t) \end{aligned}$$

with $\sup_t \mathbb{E}|R_2^\epsilon(t)| = O(\epsilon)$ *and* $\sup_t \mathbb{E}|R_3^\epsilon(t)| = O(\epsilon)$.

Let us set $R^\epsilon(t) = A_4^\epsilon(t) + R_2^\epsilon(t) + R_3^\epsilon(t)$. Recall that $A_4^\epsilon(t)$ is the fourth term of the decomposition of $A^\epsilon f^\epsilon(t)$ given in (13). Combining Proposition 3.3 and Proposition 3.9, we obtain that the quantity $M_t^{f,\epsilon}(\theta)$ defined by:

$$M_t^{f,\epsilon}(\theta) = f(T_t^\epsilon,\theta) - f_1^\epsilon(t) + f_2^\epsilon(t) + f_3^\epsilon(t) \tag{20}$$
$$-\int_0^t \left[f'(T_s^\epsilon,\theta)\left(\frac{\nu}{2}(T_s^\epsilon,\triangle\theta) + \frac{1}{a}A_3(T_s^\epsilon)\right) + \frac{1}{a}f''(T_s^\epsilon,\theta)A_2(T_s^\epsilon) + R^\epsilon(s)\right] ds$$

is a martingale with respect to $(\mathcal{F}_t^\epsilon)$. Introducing the nuclear operator K_ϕ, from $\mathcal{D}$ to $\mathcal{D}'$, defined for ϕ in L^2 by:

$$(\theta_1, K_\phi\theta_2) = \frac{2}{a}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\phi(x)\phi(y)H_{\theta_1,\theta_2}(x,y)dxdy \tag{21}$$

we can use the results of lemmas 3.2 and 3.8 and Propositions 3.3 and 3.9 to take a limit in the martingale problem (20). We then conclude that the limiting law of the converging subsequence $(T_\cdot^\epsilon)$ is a solution to the martingale problem:

$$\begin{aligned} M_t^f(\theta) &= f(T_t,\theta) \\ &- \int_0^t \left[f'(T_s,\theta)\left(\frac{\nu}{2}(T_s,\triangle\theta) + \frac{1}{a}A_3(T_s)\right) + \frac{1}{2}f''(T_s,\theta)(\theta, K_{T_s}\theta)\right] ds \end{aligned} \tag{22}$$

By the representation formula (7) and the boundedness hypothesis on T_0 we get:

$$|(T_t^\epsilon,\theta)| \le C(M)|T_0|_\infty \cdot |\theta|_2$$

and therefore $\mathbb{E}\{(T_t^\epsilon,\theta)^2\}$ converges to $\mathbb{E}\{(T_t,\theta)^2\}$. The same argument shows that the martingale property in (22) holds for $f(r) = r$ and $f(r) = r^2$ which enables us to conclude that:

$$M_t(\theta) = (T_t,\theta) - \int_0^t \left(\frac{\nu}{2}(T_s,\triangle\theta) + \frac{1}{a}A_3(T_s)\right) ds \tag{23}$$

is a martingale with quadratic variation given by:

$$< M(\theta) >_t = \int_0^t (\theta, K_{T_s}\theta)ds. \tag{24}$$

A similar computation with two test functions θ_1 and θ_2 gives the following quadratic covariation:

$$< M(\theta_1), M(\theta_2) >_t = \int_0^t (\theta_1, K_{T_s}\theta_2)ds. \tag{25}$$

The martingale problem (23),(25) has been studied by several authors. For instance it is shown in [8] that $M_t(\theta) = (M_t, \theta)$ defines an $L^2(\mathbb{R}^d)$-valued continuous square integrable martingale; this martingale can be represented as $M_t = \int_0^t K_{T_s}^{1/2} dW_s$, W being a cylindrical Wiener process. The law of (T_t) is then the same as the law of the unique solution of the following stochastic partial differential equation:

$$dT_t = \left(\frac{\nu}{2}\triangle + \frac{1}{a}B - \frac{1}{a}b\cdot\nabla\right)T_t dt + K_{T_t}^{1/2}dW_t. \tag{26}$$

Applying an infinite dimensional Ito's formula (see for instance [11]) to the square of the $L^2(\mathbb{R}^d)$-norm, one easily gets the following stochastic *a priori* estimates for every $t_0 < +\infty$:

$$\sup_{t\le t_0} \mathbb{E}|T_t|_2^2 < \infty\ , \qquad \mathbb{E}\int_0^{t_0} ||T_t||_{H^1}^2 dt < \infty$$

$$\lim_{\alpha\nearrow+\infty} \mathbb{P}\{\sup_{t\le t_0} |T_t|_2 \ge \alpha\} = 0$$

We summarize the results obtained in this section by:

Theorem 3.10 *The distributions of the solutions (T^ϵ) converge weakly on the space $D([0,+\infty);\mathcal{D}'(\mathbb{R}^d))$ toward the $L^2(\mathbb{R}^d)$-valued unique solution of the stochastic partial differential equation (26).*

Remarks:

1. In the equation (26) the term $\frac{1}{a}(B - b\cdot\nabla)T_t$ is the Stratonovich correction to the stochastic integral.

2. The first moment $m(t,x) = \mathbb{E}\{T_t(x)\}$ satisfies:

$$\frac{\partial m}{\partial t} = \left(\frac{\nu}{2}\triangle + \frac{1}{a}(B - b\cdot\nabla)\right)m, \qquad m_0 = T_0$$

3. The equation (26) can also be written as follows:

$$dT_t = \frac{\nu}{2}\triangle T_t dt + (2/a)^{1/2}\nabla T_t \circ dW_t^\Gamma$$

where W^Γ is an infinite dimensional brownian motion with spatial covariance Γ.

4. If the velocity field V is spatially symmetric (i.e. if the spatial covariance satisfies $\Gamma(-x) = \Gamma(x)$), then $b = 0$ in (26). In particular this is the case for *isotropic* velocity fields.

5. If the velocity field V is *divergence free* we also have $b = 0$ and much more as shown in the next section.

4 The *divergence free* case

In this section we assume that:

$$\sum_{i=1}^{d}\sum_{j=1}^{d}\partial_{x_i}\partial_{x_j}\Gamma_{i,j}=0 \tag{27}$$

which ensures the fact that our velocity field V has a null divergence.
Going back to equation (5), we deduce the following uniform estimates giving deterministic bounds:

$$|T_t^\epsilon|\leq|T_0|,\qquad \int_0^t ||T_s^\epsilon||_{H^1}^2ds\leq(t+\frac{1}{\nu})|T_0|^2 \tag{28}$$

The laws of $(T_\cdot^\epsilon)$ are naturally supported by the space of continuous $L^2(\mathbb{R}^d)$-valued trajectories which are $H^1(\mathbb{R}^d)$-locally integable. Considering the weak topologies on these L^2 spaces, we have:

Proposition 4.1 *The laws of $(T_\cdot^\epsilon)$ for $0<\epsilon\leq 1$ are tight on:*

$$\Omega=D([0,+\infty);L_w^2(\mathbb{R}^d))\cap L_{w,loc}^2([0,+\infty);H^1(\mathbb{R}^d))$$

equipped with its Borel σ-Field.

Proof:
For the tightness on $D([0,+\infty);L_w^2(\mathbb{R}^d))$ we have to prove the tightness of $\{(T_\cdot^\epsilon,\theta)\}_{0<\epsilon\leq 1}$ on $D([0,+\infty);\mathbb{R})$ for every θ in $L^2(\mathbb{R}^d)$. According to the first part of (28) it is enough to prove it for a dense subset in $L^2(\mathbb{R}^d)$. This is what we have done in section 3.1 for θ in $\mathcal{D}(\mathbb{R}^d)$. The second part of (28) implies the boundedness in $L_{loc}^2([0,+\infty);H^1(\mathbb{R}^d))$ and therefore the relative compactness in $L_{w,loc}^2([0,+\infty);H^1(\mathbb{R}^d))$. To conclude, we notice, as in [3], that tightness and relative compactness are equivalent on Ω. ∎

The identification of the limit is done as in section 3.2 and we obtain:

Theorem 4.2 $\{(T_\cdot^\epsilon),0<\epsilon\leq 1\}$ *converges in law on Ω to the continuous process (T_t), unique solution of the following stochastic partial differential equation:*

$$T_t=T_0+\int_0^t\left(\frac{\nu}{2}\triangle+\frac{1}{a}B\right)T_sds+(2/a)^{1/2}\int_0^t\nabla T_s\cdot dW_s^\Gamma \tag{29}$$

where $B=\sum_{i=1}^d\sum_{j=1}^d\Gamma_{i,j}(0)\partial_{x_i}\partial_{x_j}$ and $(W_t^\Gamma(x))$ has a covariance given by: $\Gamma(x-y)\delta(t-s)$.

Our final remark is that, in the *divergence free* case, the first moment of the solution satisfies a diffusion equation with an *effective diffusivity* equalled to $\frac{\nu}{2}\triangle+\frac{1}{a}B$.

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