Forward and Markov Approximation: The Strong Intensity Fluctuations Regime Revisited

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Abstract

The forward and Markov approximation for high frequency waves propagating in weakly fluctuating random media is the solution of a stochastic Schrödinger equation. In this context, the strong intensity fluctuations regime corresponds to long propagation distances. This regime has been studied by several different methods, such as expansion of the moment equations and path integral representations. It is a well-accepted fact that, in this regime, the field becomes Gaussian and completely decorrelated which implies, in particular, that the intensity has an exponential probability distribution. The aim of this paper is to give additional evidence for this by analyzing the stationary moment equations. Under the natural hypothesis of asymptotic spatial decorrelation of the field, we construct boundary conditions for these stationary equations which can then be solved explicitly. We note that the limiting probability distribution does not depend on the spectral content of the randomnes, which plays an essential role at finite propagation distances in the regime of saturation of the intensity fluctuations.

Keywords: stochastic Schrödinger equation, moment equations, strong intensity fluctuations, stationary moment equations.

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1 Introduction

During the last thirty years a great deal of work has been devoted to the study of high frequency wave propagation through weakly fluctuating random media. This problem arises in optics for light passing through a turbulent atmosphere or in acoustics for sound waves propagating in the ocean and elsewhere. A fairly recent and complete review can be found in the volume edited in 1993 by V.I. Tatarskii, A. Ishimaru and V.U. Zavorotny and entitled "Wave Propagation in Random Media (Scintillation)" [1]. Even though the full problem, the reduced wave equation with random index of refraction, is linear, it cannot be solved explicitly in the sense that it is impossible to compute the probability distribution of the wave field as a (highly nonlinear) function of the probability distribution of the medium. In most approaches, the different scales associated with the problem are first identified and then approximations are introduced that are valid in various regimes defined by the relative size of the scales.

In the high frequency regime, the parabolic or forward scattering approximation amounts to neglecting the back-scattered part of the wave. It transforms the original elliptic problem (the reduced wave equation) into an evolution equation along the direction of propagation. This evolution equation is a Schrödinger equation with a random potential and it is impossible to get from it closed evolution equations for the moments of the wave field. These moments are important quantities in the applications: the first moment, or mean field, corresponds to the coherent, or unscattered field; the second moment (the field and its conjugate) corresponds to the mutual coherence function and to the mean intensity of the field; the fourth moment gives the fluctuations of the intensity. In general, the moments describe the probability distribution of the wave field.

Another approximation that is valid for long propagation distances, compared to the correlation length of the inhomogeneities, and when the random coefficients are weakly fluctuating, is the so-called Markov approximation. It amounts to letting the inhomogeneities be δ-correlated in the direction of propagation. The corresponding equation for the wave field becomes a stochastic Schrödinger equation which has been studied in detail [2,3]. Their moment equations are closed at each order, as we recall in section 3. In a recent mathematical study, [4] and [5], it has been shown that these two approximations, parabolic and white noise, can be performed simultaneously in the regime of high frequency and weak noise, the wavelength being of the order of the $2/3$ power of the size of the noise. This stochastic Schrödinger equation is our starting point:

$$u(z,x) = u_0(x) + i\int_0^z \Delta u(z',x)dz' + i\int_0^z u(z',x)\circ W(dz',x) \tag{1}$$

where $z \geq 0$ is distance along the direction of propagation, $u(0,x) = u_0(x)$ is the initial complex field amplitude, x denotes the two-dimensional transverse space variables, Δ denotes the Laplacian with respect to x, $W(z,x)$ is a real, infinite dimensional Brownian Motion with covariance $\langle W(z,x)W(z',y)\rangle = Q(x-y)(z\wedge z')$, and $\circ W$ denotes a Stratonovich stochastic integral. This equation can be rewritten in the Itô sense as follows:

$$u(z,x) = u_0(x) + \int_0^z (i\Delta - \frac{1}{2}Q(0))u(z',x)dz' + i\int_0^z u(z',x)\cdot W(dz',x) \tag{2}$$

In atmospheric turbulence applications, we assume that Q has the spectral representation

$$Q(x) = \int_{R^2} e^{i\lambda\cdot x}\phi(|\lambda|)d\lambda \tag{3}$$

which may include the Kolmogorov case, $\phi(|\lambda|) = |\lambda|^{-11/3}$, with appropriate cut-offs at $|\lambda| = \frac{2\pi}{l_0}$ and $|\lambda| = \frac{2\pi}{L_0}$ corresponding to the inner and outer scales.

We note that the frequency does not appear in our equation since we have already gone to a high frequency limit and the variables are scaled. We could have left the wavenumber k in the equation (1) by dividing the Laplacian by k and multiplying the Brownian motion by k, and multiplying $Q(0)$ by k^2 in (2). Since we are dealing here with a single frequency we may set $k = 1$. In fact, it has been shown in [5] that the correct multi-frequency parabolic and white noise approximation is obtained by writing an equation like (1) for each frequency with a single Brownian Motion W.

The solution of (1) has been characterized as the unique solution of an infinite dimensional martingale problem in [2], in the case of an initial condition u_0 in L^2 for which the energy, $\|u\|^2 = \int_{\mathbf{R}^2} u(z,x)\overline{u(z,x)}dx$, is conserved (i.e. remains constant in z). We are interested here in the plane wave case, namely the case $u_0(x) = 1$ identically. This situation requires some extra work, but roughly speaking, the translation invariance property of the noise $W(z,x)$ with respect to x, enables us to consider the solution $u(z,x)$ as a stationary field in the transverse direction.

Using an infinite dimensional version of Itô's calculus we obtain equations for the moments of any order of the field. These equations are well-known (see for instance [6,7,8,9]). Up to second order they can be solved explicitly. The higher order moment equations, and in particular the fourth order moment equation, have been the subject of a lot of work, theoretical as well as numerical [10,11,12] and experimental. In this paper we revisit the so-called regime of strong intensity fluctuations which corresponds, in our notations, to the distance of propagation, z, becoming large. This large distance limit has been studied by various methods and it is a well-accepted fact that the field tends to an uncorrelated Gaussian. In particular, the probability distribution of the intensity is asymptotically exponential. This can be seen by a formal high frequency expansion of the moment equations, as was done in [13]. The limit exponential distribution cannot be completely proved by this approach because of the non-uniformity of the expansion with respect to the order of the moments. This is quite similar to what happens when path integral representations are used (see [14,15] or [16]).

The aim of this paper is to give another argument showing that the large z distribution of the intensity is exponential. The main step in our analysis is a change of variables in the equation for the higher order moments. We have used this to study only the equilibrium behavior of the moments. We expect that this tool will also be useful in the large z, or even not so large z, analysis of the higher moment equations.

2 Outline of the Argument

Our derivation of the exponential law for the field intensity relies on two hypotheses which are quite natural. We assume first that, as z goes to $+\infty$, the moments of any given order converge to a solution of the stationary equation corresponding to the evolution equation (section 3). Second, we assume that these limiting moments "decorrelate" at infinity, meaning that they become products of lower order moments as transverse points separate (section 4). This is usually assumed to be true at finte z (but has not been proved) and is used in numerical computations [11]. These hypotheses enable us in section 4 to construct, by induction, boundary conditions for the stationary equations, the explicit solution of the first and second order moment equations being the starting point of the induction.

The main technical part of our argument is a change of coordinates for the moment equations of arbitrary order, given in section 5, that generalizes the widely used change of variables for the fourth moment equations. The point of this transformation is that it allows us to obtain explicit solutions of the stationary moment equations that are compatible with the boundary conditions of section 4. We must also use here the special form of the potential term in the moment equations that makes it vanish when coordinates coalesce (property (18) section 5).

Finally we shall show in section 5 that the moments corresponding to the uncorrelated Gaussian field are solutions of the stationary moment equations with the prescribed boundary conditions.

We would like to point out that the field itself converges only in the sense of the finite dimensional distributions: for any given set of distinct points $\{x_1, ..., x_k\}$, the law of $\{u(z, x_1), ..., u(z, x_k)\}$ converges, as z goes to $+\infty$, to the law of k independent complex random variables having the same centered Gaussian law with independent real and imaginary part with variance 1/2. This is what we could call "a perfect speckle", which implies in particular that the probability distribution of the intensity at a given point, $I(z, x) = u(z, x)\overline{u(z, x)}$, converges to an exponential law with parameter 1, its density being given by $\exp(-I)$.

In order to illustrate this type of convergence one can think of the simplified model where only the phase is affected:

$$u(z, x) = 1 + i \int_0^z u(z', x) \circ W(dz', x) \tag{4}$$

Its solution is given by $u(z, x) = \exp(iW(z, x))$ and an elementary computation with Gaussians shows that for given distinct points $\{x_1, ..., x_k\}$, the moments

$$\langle u(z, x_1)^{m_1} \overline{u(z, x_1)}^{n_1} \cdots u(z, x_k)^{m_k} \overline{u(z, x_k)}^{n_k} \rangle$$

converge, as z goes to $+\infty$, to 0 unless $m_j = n_j$ for every $j = 1, ..., k$, in which case it converges to 1. These limiting moments are those of $\exp(i\phi(x))$ where $\{\phi(x), x \in I\!R^2\}$ is a collection of independent phases uniformly distributed over $(0, 2\pi)$.

In the other extreme case, where there is no randomness, $Q(0) = 0$, the field is simply constant and equal to 1. In the degenerate case where W depends only on z, the field is simply the free field, equal to 1, with a phase $\exp(iW(z))$ that does not depend on x and converges to $\exp(\phi)$ where ϕ is a uniformly distributed phase.

Clearly, our case is a nontrivial interaction between diffraction and random scattering. Nevertheless, as we shall see, the large z limit is independent of the spectral content of W in the transverse variables.

3 Moment Equations

For points $(x_1, ..., x_k, y_1, ..., y_l)$ in $I\!R^2$ we denote by $m_{k,l}$ the moment defined by:

$$m_{k,l}(x_1, ..., x_k; y_1, ..., y_l; z) = \langle u(z, x_1)...u(z, x_k)\overline{u(z, y_1)}...\overline{u(z, y_l)}\rangle \tag{5}$$

where it is understood that if $k = 0$ only conjugate fields appear, and if $l = 0$ only fields appear in the product. The points need not be distinct.

Equation (2) can be rewritten in differential form for the field and its conjugate:

$$\begin{cases} du(z, x) & = i\Delta u dz - \frac{1}{2}Q(0)u dz + iu \cdot W(dz, x) \\ d\overline{u(z, y)} & = -i\Delta \overline{u} dz - \frac{1}{2}Q(0)\overline{u} dz - i\overline{u} \cdot W(dz, y) \end{cases} \tag{6}$$

A straightforward application of the stochastic calculus enables us to write a stochastic differential equation for $u(z, x_1)...u(z, x_k)\overline{u(z, y_1)}...\overline{u(z, y_l)}$. Taking a mathematical expectation in this equation, the martingale terms disappear and we are left with the following linear differential equation for $m_{k,l}$:

$$\begin{aligned} \frac{dm_{k,l}}{dz} & = i(\textstyle\sum_{r=1}^k \Delta_{x_r} - \sum_{j=1}^l \Delta_{y_j})m_{k,l} - \frac{1}{2}(k + l)Q(0)m_{k,l} \\ & -\{\textstyle\sum_{r=1}^k \sum_{j=1}^l Q(x_r - y_j) - \sum_{1 \le r < j \le k} Q(x_r - x_j) - \sum_{1 \le r < j \le l} Q(y_r - y_j)\}m_{k,l} \end{aligned} \tag{7}$$

Introducing the structure function V given by $V(x) = Q(0) - Q(x)$ and the potentials $F_{k,l}$ defined by:

$$F_{k,l}(x_1, ..., x_k; y_1, ..., y_l) = \sum_{r=1}^{k}\sum_{j=1}^{l} V(x_r - y_j) - \sum_{1\le r<j\le k} V(x_r - x_j) - \sum_{1\le r<j\le l} V(y_r - y_j) \tag{8}$$

the moment equations become:

$$\frac{dm_{k,l}}{dz} = i(\sum_{r=1}^{k}\Delta_{x_r} - \sum_{j=1}^{l}\Delta_{y_j})m_{k,l} - \{F_{k,l} + \frac{1}{2}(k-l)^2 Q(0)\}m_{k,l} \tag{9}$$

These equations are well-known and have been derived before in many different ways [7,8,9].

In the plane wave case, the initial conditions are given by $m_{k,l}|_{z=0} = 1$. Up to second order these equations are explicitly solvable and we get:

$$m_{1,0}(x;z) = \langle u(z,x)\rangle = m_{0,1}(y;z) = \langle\overline{u(z,y)}\rangle = \exp\{-\frac{1}{2}Q(0)z\} \tag{10}$$

$$m_{1,1}(x;y;z) = \langle u(z,x)\overline{u(z,y)}\rangle = \exp\{-V(x-y)z\} \tag{11}$$

Our **first hypothesis** is that the moments $m_{k,l}$ converge, as z goes to $+\infty$, to limiting quantities $M_{k,l}$ which are solutions of the stationary equations associated to the evolution equations (9):

$$\lim_{z\nearrow+\infty} m_{k,l}(x_1, ..., x_k; y_1, ..., y_l; z) = M_{k,l}(x_1, ..., x_k; y_1, ..., y_l) \tag{12}$$

and

$$i(\sum_{r=1}^{k}\Delta_{x_r} - \sum_{j=1}^{l}\Delta_{y_j})M_{k,l} = (F_{k,l} + \frac{1}{2}(k-l)^2 Q(0))M_{k,l} \tag{13}$$

From (10) and (11) one can see that $M_{1,0} = M_{0,1} = 0$ and $M_{1,1}(x;y) = 1$ if $x = y$ and 0 if $x \ne y$ since $Q(0) > 0$, $V(0) = 0$ and $V(x-y) > 0$ if $x \ne y$. In particular, for $(k,l) = (1,0), (0,1)$ or $(1,1)$ the convergence part of the hypothesis is satisfied.

Since we are considering the plane wave case, $u_0 = 1$, and W stationary in x, $u(z,x)$ is also stationary in x; in particular $m_{k,l}$, and therefore $M_{k,l}$, do not depend on the center of mass $R = x_1 + ... + x_k + y_1 + ... + y_l$. When studying $M_{k,l}$ one can always assume that $R = 0$ although (13) has to be solved with respect to all the variables.

4 Boundary Conditions

In this section we formulate the decorrelation property of the $M_{k,l}$'s that we assume. We then show that $M_{k,l} = 0$ identically for $k \ne l$ and we derive boundary conditions for (13) in the case $k = l$.

4.1 Second Hypothesis

For a set $A_{k,l} = (x_1, ..., x_k; y_1, ..., y_l)$ of points in $I\!R^2$ we denote by $M_A = M_{A_{k,l}}$ the corresponding quantity defined in (12). For two such sets A and B, let $A + B$ be the combined set of points, the

x's with the x's and the y's with the y's. Denoting by $d(A,B)$ the distance between the two sets of points, our second hypothesis is

$$\lim_{d(A,B)\nearrow+\infty} |M_{A+B} - M_A M_B| = 0 \tag{14}$$

Note that this property is usually assumed to be valid for the m's at $z>0$ fixed: for instance, with (10) and (11), one can check that $|m_{1,1}(x;y;z) - m_{1,0}(x;z)m_{0,1}(y;z)|$ goes to 0 as $|x-y|$ goes to $+\infty$ since $V(+\infty) = Q(0)$. This, however, has not been shown to hold in general.

4.2 The case $k \neq l$

We shall show that $M_{k,l} = 0$ identically for $k \neq l$. This will be done by induction on $k+l$. It has already been observed that $M_{1,0} = M_{0,1} = 0$. Let us suppose that $M_{k',l'} = 0$ for every (k',l') such that $k' \neq l'$ and $k'+l' < l+k$ for (k,l) given such that $k \neq l$.

The boundary conditions for (13), satisfied by $M_{k,l}$, are obtained as follows. As was observed at the end of the previous section, one can assume that the center of mass of the set $A_{k.l}$ is at the origin without any influence on $M_{k,l}$. If the diameter of $A_{k,l}$, denoted by $\mathrm{diam}(A_{k,l})$, goes to $+\infty$, there exists a non trivial partition (B,C) of $A_{k,l}$ such that the distance between B and C goes to $+\infty$; moreover at least one of the sets B or C is such that $k' \neq l'$ where k' (resp. l') is the number of x's (resp. y's) in that set. Since the partition is non trivial we have $k'+l' < k+l$. From our second hypothesis and the induction hypothesis we deduce that $M_{k,l} \to 0$ as $\mathrm{diam}(A_{k,l}) \nearrow +\infty$.

Therefore $M_{k,l}$ is a solution of the linear equation (13) with boundary conditions 0 at $+\infty$ (in the sense that $\mathrm{diam}(A_{k,l}) \nearrow +\infty$), which implies that $M_{k,l} = 0$ identically, assuming a suitable uniqueness property of solutions of the linear stationary moment equations.

4.3 Boundary Conditions in the case $k = l$

The set $A_{n,n} = (x_1,...,x_n;y_1,...,y_n)$ will be denoted by A_n; we shall say that A_n is **coupled** if there exists a permutation σ of $\{1,...,n\}$ such that $x_j = y_{\sigma(j)}$ for every $j = 1,...,n$; if not, A_n is said to be **uncoupled**.

If A_n is coupled, $(x_1,...,x_n)$ is a set $\{x'_1,...,x'_k\}$ where all the x'_j's are distinct with $k \leq n$. We denote by n_j the multiplicity of x'_j in A_n so that $n_1 + ... + n_k = n$.

With these notations we can state the **main result** of this paper:

$$M_n = M_{n,n}(A_n) = \begin{cases} 0 & \text{if } A_n \text{ is uncoupled} \\ n_1!...n_k! & \text{if } A_n \text{ is coupled} \end{cases} \tag{15}$$

From previous explicit computations we have seen that this result is true for $n=1$ since $M_1 = M_{1,1}(x;y) = 0$ if $x \neq y$ and $M_1 = 1$ if $x = y$ in which case $k=1$ and $n_1 = 1$.

For a given $n > 1$, our **induction hypothesis** is that (15) holds for every $n' < n$. We are now going to derive the appropriate boundary conditions for M_n.

Since M_n does not depend on the center of mass R of the set A_n, we assume in the following that $R = 0$.

If A_n is uncoupled and $\mathrm{diam}(A_n) \nearrow +\infty$, there exists a non trivial partition (B,C) of A_n such that $d(B,C) \nearrow +\infty$ and B or C is either of the form $A_{k,l}$ with $k \neq l$ or of the form $A_{k,k}$ uncoupled with $k < n$. Then by the second hypothesis (14) and either the case $k \neq l$ or our induction hypothesis, M_n goes to 0 as $\mathrm{diam}(A_n)$ goes to $+\infty$.

If A_n is coupled with multiplicities $(n_1,...,n_k)$ being fixed, and if $\mathrm{diam}(A_n) \nearrow +\infty$, there exists a non trivial partition (B,C) of A_n such that B and C are coupled and $d(B,C) \nearrow +\infty$. The multiplicities $n_1, ..., n_k$ being obviously split between B and C, by the second hypothesis (14) and our induction hypothesis we deduce that M_n converges to

$$(\prod_{i\in B} n_i!)(\prod_{j\in C} n_j!) = n_1! \cdots n_k!$$

In order to express more compactly the boundary conditions for (13) in the case $k = l$, we denote by D_σ the linear subspace of $(I\!R^2)^{2n}$ defined by $\{x_j = y_{\sigma(j)}, j = 1, ..., n\}$, σ being any element of Σ_n, the set of permutations of $\{1, ..., n\}$. For a set $A_n = (x_1, ..., x_n; y_1, ..., y_n)$ with its center of mass R at the origin, we denote by $\Sigma_n(A_n)$ the set of permutations which couple A_n ($\sigma \in \Sigma_n(A_n) \Longleftrightarrow A_n \in D_\sigma$). If the diameter of A_n goes to $+\infty$, A_n remaining on the same D_σ's, we have shown that $M_n(A_n)$ converges to $|\Sigma_n(A_n)|$, the cardinality of the set $\Sigma_n(A_n)$, which is $n_1!...n_k!$ if A_n is coupled, and 0 if A_n is uncoupled.

We end this section with the following remark: a **fully coupled** A_n (i.e. $x_1 = ... = x_n = y_1 = ... = y_n$) cannot be sent to ∞, when $R = 0$, since its diameter is always 0. This case is not part of the boundary conditions for (13).

5 The Stationary Equations

The stationary equations (13) have been solved in the case $k \neq l$ and $M_{k,l} = 0$ identically. We are now interested in the case $k = l$. For a set of points $A_n = (x_1, ..., x_n; y_1, ..., y_n)$, the stationary equation (13) can be rewritten:

$$i \sum_{j=1}^{n} (\Delta_{x_j} - \Delta_{y_j}) M_n = F_n M_n \tag{16}$$

where the potential F_n is given by:

$$F_n(A_n) = \sum_{r=1}^{n}\sum_{j=1}^{n} V(x_r - y_j) - \sum_{1\leq r<j\leq n} \{V(x_r - x_j) + V(y_r - y_j)\} \tag{17}$$

It has already been observed that $F_n(A_n)$ does not depend on the center of mass $R = \sum_{j=1}^{n}(x_j + y_j)$. The potential F_n has another **essential property**: it vanishes on coupled sets of points:

$$\forall \sigma \in \Sigma_n \quad , \quad A_n \in D_\sigma \Longrightarrow F_n(A_n) = 0 \tag{18}$$

which is obtained by noting that, for $A_n \in D_\sigma$:

$$F_n(A_n) = \sum_{r=1}^{n}\sum_{j=1}^{n} V(y_{\sigma(r)} - y_j) - \sum_{1\leq r<j\leq n} \{V(y_{\sigma(r)} - y_{\sigma(j)}) + V(y_r - y_j)\} = 0$$

since

$$\sum_{1\leq r<j\leq n} V(y_{\sigma(r)} - y_{\sigma(j)}) = \sum_{1\leq r<j\leq n} V(y_{\sigma(j)} - y_{\sigma(r)}) = \frac{1}{2}\sum_{r=1}^{n}\sum_{j=1}^{n} V(y_{\sigma(r)} - y_{\sigma(j)})$$

where we have used the fact that V is even and $V(0) = 0$. Similarly

$$\sum_{1\leq r<j\leq n} V(y_r - y_j) = \frac{1}{2}\sum_{r=1}^{n}\sum_{j=1}^{n} V(y_r - y_j)$$

and obviously:

$$\sum_{r=1}^{n}\sum_{j=1}^{n} V(y_{\sigma(r)} - y_j) = \sum_{r=1}^{n}\sum_{j=1}^{n} V(y_r - y_j) = \sum_{r=1}^{n}\sum_{j=1}^{n} V(y_{\sigma(r)} - y_{\sigma(j)})$$

Since M_n is independent of the center of mass R, we want to perform a change of variables such that R appears as one of the new variables. Also, since $Q = x_1 + ... + x_n - (y_1 + ... + y_n)$ is 0 on any of the D_σ, we also choose Q as a new variable. The main property of equation (16), along with property (18) of the potential, is the following **change of variables** property, which is valid for finite z as well.

For every $\sigma \in \Sigma_n$, there exists a basis in $(\mathbb{R}^2)^{2n}$ such that in the new system of variables

$$\{R = \frac{1}{n}\sum_{j=1}^{n}(x_j + y_j), Q = \frac{1}{n}\sum_{j=1}^{n}(x_j - y_j), (v_j; j = 1, ..., n-1), (w_j; j = 1, ..., n-1)\}$$

equation (16) becomes:

$$i(\frac{4}{n}\nabla_R \cdot \nabla_Q + \frac{8}{n^2}\sum_{j=1}^{n-1}\nabla_{v_j} \cdot \nabla_{w_j})M_n = F_n(Q, v_1, ..., v_{n-1}, w_1, ..., w_{n-1})M_n$$

and D_σ is simply $\{w_1 = \cdots = w_{n-1} = Q = 0\}$

Proof of the change of variables property. We first perform a change of variables $(x_j, y_j) \longrightarrow (r_j, \rho_j)$ defined by:

$$r_j = x_{\sigma(j)} + y_j \ , \ \rho_j = x_{\sigma(j)} - y_j \ , \ j = 1, ..., n$$

Computing the operator $\sum_{j=1}^{n}(\Delta_{x_j} - \Delta_{y_j}) = \sum_{j=1}^{n}(\Delta_{x_{\sigma(j)}} - \Delta_{y_j})$ in the new variables gives:

$$4i(\sum_{j=1}^{n}\nabla_{r_j} \cdot \nabla_{\rho_j})M_n = F_n(r_1, ..., r_n, \rho_1, ..., \rho_n)M_n \tag{19}$$

where D_σ is simply $\{\rho_1 = \cdots = \rho_n = 0\}$ and F_n does not depend on $(\sum_{j=1}^{n} r_j)$.

Our next change of variables consists in choosing a new basis such that the first vector is orthogonal to $\{R = 0\}$, the second vector is orthogonal to $\{Q = 0\}$, completed by a particular basis of D_σ and a particular basis of a $D_{\sigma'}$ chosen such that $\sigma'(j) \neq \sigma(j)$ for every $j = 1, ..., n$. These vectors are those appearing as columns of the following $2n \times 2n$ matrix $P =$

$$\begin{pmatrix}
1 & 0 & 1-n & 1 & . & . & . & 1 & 1-\frac{n}{2} & 1 & . & . & . & 1 \\
1 & 0 & 1 & 1-n & 1 & . & . & 1 & 1-\frac{n}{2} & 1-\frac{n}{2} & 1 & . & . & 1 \\
1 & 0 & 1 & 1 & 1-n & 1 & . & 1 & 1 & 1-\frac{n}{2} & 1-\frac{n}{2} & 1 & . & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
1 & 0 & 1 & 1 & . & . & 1 & 1-n & 1 & . & . & . & 1-\frac{n}{2} & 1-\frac{n}{2} \\
1 & 0 & 1 & 1 & . & . & 1 & 1 & 1 & . & . & . & 1 & 1-\frac{n}{2} \\
0 & 1 & 0 & . & . & . & . & 0 & -\frac{n}{2} & 0 & . & . & . & 0 \\
0 & 1 & 0 & . & . & . & . & 0 & \frac{n}{2} & -\frac{n}{2} & 0 & . & . & 0 \\
0 & 1 & 0 & . & . & . & . & 0 & 0 & \frac{n}{2} & -\frac{n}{2} & 0 & . & 0 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . \\
0 & 1 & 0 & . & . & . & . & 0 & 0 & . & . & 0 & \frac{n}{2} & -\frac{n}{2} \\
0 & 1 & 0 & 0 & . & . & 0 & 0 & 0 & 0 & . & 0 & 0 & \frac{n}{2}
\end{pmatrix}$$

In the case $n = 2$, the matrix P is simply:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

associated to the usual change of variables $(r_1 + r_2, \rho_1 + \rho_2, r_2 - r_1, \rho_2 - \rho_1)$ in the study of the fourth moments.

The inverse of the matrix P, $P^{-1} =$

$$\begin{pmatrix} \frac{1}{n} & \frac{1}{n} & . & . & \frac{1}{n} & 0 & 0 & . & . & 0 \\ 0 & 0 & . & . & 0 & \frac{1}{n} & \frac{1}{n} & . & . & \frac{1}{n} \\ & & B_{11} & & & & & B_{12} & & \\ & & B_{21} & & & & & B_{22} & & \end{pmatrix}$$

where B_{11}, B_{12}, B_{21}, B_{22} are four $n-1 \times n$ matrices:

$$B_{11} = \frac{1}{n} \begin{pmatrix} -1 & 0 & . & . & 0 & 0 & 1 \\ 0 & -1 & . & . & 0 & 0 & 1 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & . & . & -1 & 0 & 1 \\ 0 & 0 & . & . & 0 & -1 & 1 \end{pmatrix}, \quad B_{21} = 0$$

$$B_{12} = \frac{1}{n^2} \begin{pmatrix} n-2 & -2 & -2 & . & . & . & -2 & -2 & n-2 \\ 2n-4 & n-4 & -4 & . & . & . & -4 & -4 & n-4 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 2n-2k & . & . & 2n-2k & n-2k & -2k & . & -2k & n-2k \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 2 & 2 & 2 & . & . & . & 2 & -(n-2) & -(n-2) \end{pmatrix}$$

and

$$B_{22} = \frac{1}{n^2} \begin{pmatrix} -2(n-1) & 2 & 2 & . & . & 2 & 2 \\ -2(n-2) & -2(n-2) & 4 & . & . & 4 & 4 \\ & . & . & . & . & . & . \\ . & . & . & . & . & . & \\ -2(n-k) & . & . & -2(n-k) & 2k & . & 2k \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ -2 & -2 & -2 & . & . & -2 & 2(n-1) \end{pmatrix}$$

The new variables are given by:

$$\begin{pmatrix} R \\ Q \\ v_1 \\ . \\ . \\ . \\ v_{n-1} \\ w'_1 \\ . \\ . \\ . \\ w'_{n-1} \end{pmatrix} = P^{-1} \begin{pmatrix} r_1 \\ . \\ . \\ . \\ . \\ r_n \\ \rho_1 \\ . \\ . \\ . \\ . \\ \rho_n \end{pmatrix} \tag{20}$$

Therefore, the first new variable is $R = \frac{1}{n}\sum_{j=1}^{n} r_j$ and the second is $Q = \frac{1}{n}\sum_{j=1}^{n} \rho_j$, $(v_j \; ; \; j = 1, ..., n-1)$ and $(w'_j \; ; \; j = 1, ..., n-1)$ being the $2(n-1)$ remaining new variables. In the old basis, the $2n \times 2n$ matrix of the coefficients of the second order operator $\sum_{j=1}^{n} \nabla_{r_j} \cdot \nabla_{\rho_j}$ is given by

$$A = \frac{1}{2}\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

In the new basis, the matrix of the coefficients of this same operator is given by

$$B = P^{-1}A(P^{-1})^t \tag{21}$$

This leads to the following expression for the operator in the new basis

$$\frac{1}{n}\nabla_R \cdot \nabla_Q + \frac{2}{n^2}\sum_{1 \le k \le l \le n-1} \nabla_{v_k} \cdot \nabla_{w'_l} \tag{22}$$

and D_σ becomes $\{Q = w'_1 = ... = w'_{n-1} = 0\}$.

A last change of variables given by

$$(w_j = \sum_{l=j}^{n-1} w'_l \; ; \; j = 1, ..., n-1)$$

and $(R, Q, v_j \; ; \; j = 1, ..., n-1)$ unchanged, gives:

$$4i(\frac{1}{n}\nabla_R \cdot \nabla_Q + \frac{2}{n^2}\sum_{j=1}^{n-1} \nabla_{v_j} \cdot \nabla_{w_j})M_n = F_n(Q, v_1, ..., v_{n-1}, w_1, ..., w_{n-1})M_n \tag{23}$$

where $D_\sigma = \{Q = w_1 = ... = w_{n-1} = 0\}$ which ends the proof of the change of variables property.

Since M_n is independent of R, equation (23) becomes:

$$\frac{8}{n^2}i(\sum_{j=1}^{n-1} \nabla_{v_j} \cdot \nabla_{w_j})M_n = F_n(Q, v_1, ..., v_{n-1}, w_1, ..., w_{n-1})M_n \tag{24}$$

This linear equation, with boundary conditions 1 at ∞ along D_σ and 0 at ∞ elsewhere, admits the solution $M_n^{(\sigma)} = 1$ on D_σ and $M_n^{(\sigma)} = 0$ on D_σ^c. This can be seen by observing that $\nabla_{v_j} M_n^{(\sigma)} = 0$ for $j = 1, ..., n-1$ and $F_n = 0$ on D_σ. Since $M_n^{(\sigma)}$ is 0 outside of D_σ, (24) is satisfied everywhere with the prescribed boundary conditions obviously satisfied by $M_n^{(\sigma)}$.

Since equation (13) is linear, we deduce that $M_n = \sum_{\sigma \in \Sigma_n} M_n^{(\sigma)}$ is a solution and by construction it satisfies the boundary conditions established in Section 3.3 and its value on the set A_n is simply the cardinality of $\Sigma_n(A_n)$, the number of permutations σ in Σ_n such that $A_n \in D_\sigma$ (or A_n is coupled by σ). This gives the value of M_n which completes our induction started in Section 3.3 (15).

Finally we note that the limiting moments correspond to those of an uncorrelated field with one-point moments $\langle u^m \overline{u}^n \rangle$ equal to $n! \delta_{mn}$. The real and imaginary parts of u, denoted respectively by X and Y, are easily shown to be independent and Gaussian as follows. For every real numbers a and b we denote by Z the real random variable $aX + bY$. Setting $\lambda = (a - ib)$, we have $Z = \frac{1}{2}(\lambda u + \overline{\lambda}\overline{u})$ with moments

$$\langle Z^n \rangle = \frac{1}{2^n} \sum_{k=0}^{n} C_n^k \lambda^k \overline{\lambda}^{n-k} \langle u^k \overline{u}^{n-k} \rangle = \begin{cases} 0 & \text{if n is odd} \\ (\lambda\overline{\lambda})^{\frac{n}{2}} \frac{n!}{2^n (\frac{n}{2})!} & \text{if n is even} \end{cases}$$

Therefore the real random variable $Z = aX + bY$ is $\mathcal{N}(0, \frac{a^2+b^2}{2})$ distributed, for every a and b. This implies that the pair (X, Y) is Gaussian, that X and Y are $\mathcal{N}(0, \frac{1}{2})$ distributed, and that they are independent since $\langle XY \rangle = \frac{1}{4i}(\langle u^2 \rangle - \langle \overline{u}^2 \rangle) = 0$.

6 Conclusions

We have studied the stationary moment equations (13), associated with the moment evolution equations (9), with boundary conditions obtained in Section 4. We have shown that, under the two hypothesis (13) and (14), the moments corresponding to those of an uncorrelated Gaussian field are natural solutions to these equations. They correspond to the regime of strong intensity fluctuations in the plane wave case, and are as follows:
for every set $\{x_1, ..., x_k\}$ of distinct points and every set of integers $m_1, n_1, ..., m_k, n_k$:

$$\lim_{z \nearrow +\infty} \langle \prod_{j=1}^{k} (u(z, x_j)^{m_j} \overline{u(z, x_j)}^{n_j}) \rangle = \prod_{j=1}^{k} (\lim_{z \nearrow +\infty} \langle u(z, x_j)^{m_j} \overline{u(z, x_j)}^{n_j} \rangle)$$

and

$$\lim_{z \nearrow +\infty} \langle u(z, x)^m \overline{u(z, x)}^n \rangle = n! \, \delta_{mn}$$

In particular, the intensity $I(z, x) = |u(z, x)|^2$ is asymptotically exponentially distributed since

$$\lim_{z \nearrow +\infty} \langle I(z, x)^n \rangle = n!$$

and the probability distribution of $u(z, x)$ converges, as z goes to $+\infty$, to a centered complex Gaussian random variable with independent real and imaginary parts with the same variance $1/2$.

Finally we note that the limiting probability distribution does not depend on the covariance Q provided that $Q(0) < +\infty$ and $Q(+\infty) = 0$.

We also note that the limiting Gaussian distribution has been derived by identification of its moments, without any use of the Central Limit Theorem.

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