Pressure fields generated by acoustical pulses propagating in randomly layered media

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Abstract

This paper investigates the pressure field generated at the bottom of a high-contrast randomly layered slab by an acoustical pulse emitted at the surface of the slab. This analysis takes place in the framework introduced by Asch, Kohler, Papanicolaou, Postel and White [1] where the incident pulse wave length is long compared to the correlation length of the random inhomogeneities, but short compared to the size of the slab. This problem has been studied in the one-dimensional case simultaneously by Clouet and Fouque [4] and Lewicki, Burridge and Papanicolaou [6] or for multimode plane wave pulses in Lewicki, Burridge and De Hoop [7]. These situations require only the use of classical diffusion-approximation results whereas the point-source problem studied in this paper requires a non-trivial combination of diffusion-approximation results with stationary phase methods. The stationary phase method has been used by De Hoop, Chang and Burridge [5] for weakly fluctuating media and in [1] for the study of the reflected pressure. The main statement of this paper is that, in order to apply simultaneously diffusion-approximation and stationary phase results, it is correct to apply them consecutively. We believe that this situation will be encountered again and again in this field and the goal of this paper is to present a clear statement in the most simple case of an acoustical pulse propagating in a randomly layered medium. In that case, the main result gives a formula describing the spreading of the pulse around its arrival time at the bottom of the slab which is obviously not contained in the classical geometrical acoustic approximation.

Key words : random media, diffusion-approximation, stationary phase, geometrical acoustics.
AMS subject classifications : 73D70, 60B10, 60H10

1 Introduction

An acoustical pulse emitted at the surface of a slab will create a pressure field at the bottom of the slab. We are interested in the pressure field at a given depth, the width of the slab, at different points and around their corresponding arrival times. The analysis takes place in the framework introduced by Papanicolaou and his coauthors (see [1] and references therein); the slab is randomly layered which means that coefficients such as Bulk modulus vary only in the orthogonal direction to the slab. Our limit theorem is based on a separation of scales : the typical wave lengths contained in the pulse are small compared to the deterministic macroscopic variation of the medium and large compared to the correlation length of the random inhomogeneities. In the limit of separation of these three scales, it is possible to obtain an explicit formula for the probability distribution of the pressure field created at the bottom of the slab, around the deterministic arrival times of the pulse at different points of the bottom. For simplicity we shall assume a condition of matched medium at the surface, where the pulse is emitted, and at the bottom, where the wave exits from the slab. It is noticeable that we do not assume that the random fluctuations are small as in [5], in the now classical O'Doherty and Anstey theory. In [4] and [6], the problem has been studied in the one-dimensional case : it has been proved that the exiting pulse is obtain by the convolution of the initial pulse with a gaussian density whose variance depends on the width of the slab and a correlation coefficient of the medium and by a random shift of the position by a gaussian random variable whose variance also depends on the same quantities. The methods in [4] and [6] are different: in [4], the transmitted pulse is observed at determistic arrival times and its limiting probability law is entirely characterized; in [6] , the transmitted pulse is observed at random arrival times so that its asymptotic shape is proved to be deterministic by computing first and second moments.

Our work is a generalization of the method in [4] to the three-dimensional randomly layered case; the main difference is that we have to combine a diffusion- approximation result with a stationary phase theorem due to the "horizontal" extra variables and we give the first rigorous treatment of such a combination. At the level of the result itself we show that the transmitted pressure field is obtained from the initial pulse by a convolution with the **derivative** of a gaussian density in constrast to the one-dimensional case.

In section 2 we formulate the problem following [1] and we give the Fourier representation of the transmitted pressure field by using the classical invariant imbedding technique.

In section 3 we restrict ourself to the uniform background case with no macroscopic variation : we derive an explicit formula for the limiting probability distribution of the transmitted pressure field : a classical diffusion-approximation result and an application of the stationary phase method are needed in this case, the difficulty being to show that these two limits can be taken simultaneously ; the idea of the proof will be presented while we refer to [3] for details, the main goal of the present work being to show that it is mathematically correct to first use a diffusion-approximation theorem and then a stationary phase result. This is done in the simplest case of a transmitted acoustical pulse but certainly applies to the study of the reflected pressure (see the remark at the end of paragraph 3.3).

In section 4, we come back to the general case with a slowly varying background. In that case an improved version of the diffusion-approximation result is needed in order to obtain a formula for the transmitted pressure field. In a recent paper, [7], Lewicki, Burridge and De Hoop studied this problem for a plane wave pulse. In that case only the diffusion-approximation results (without stationary phase) are needed. They deal with multimode waves and we believe that our result can be extended to that situation to obtain multimode transmitted pulses. Such a result would be a justification of the application of a high-contrast version of [5] to the result in [7].

In the conclusion we explain how these results constitute a useful complement to classical geometrical acoustics approximations.

2 Formulation of the problem

2.1 The medium

A point in a three-dimensional space will be denoted by $\vec{r} = (x, y, z)$. We assume that our model is stratified in the z-direction which means that the bulk modulus $K(\vec{r})$ and the density $\rho(\vec{r})$ depend only on z. The macroscopic variations of these quantities will be given by the deterministic functions $K_0(z)$ and $\rho_0(z)$: we assume that these functions are smooth (differentiable is enough) and that they are constant for $z > 0$ (above the surface $z = 0$) and for $z < -L$ (below the bottom of the slab $[-L, 0]$). The macroscopic sound speed $c_0(\vec{r})$ is given by $c_0(z) = \sqrt{\frac{K_0(z)}{\rho_0(z)}}$ which is a constant c_{upper} for $z > 0$ and c_{lower} for $z < -L$. In order to model the random fluctuations we introduce a Markov process $(q_s)_{s\in \mathbb{R}_-}$ valued in an auxiliary compact space $\mathcal{S}$, whose infinitesimal generator is denoted by Q. We denote by $\mathbb{P}$ its unique invariant probability distribution and we assume that Q satisfies the Fredholm alternative. The mathematical expectation will be denoted by $\mathbb{E}$ while the mathematical expectation with respect to $\mathbb{P}$ will be denoted by $\overline{\mathbb{E}}$. The following correlation coefficient α will be needed :

$$\alpha = \int_{-\infty}^{0} \overline{\mathbb{E}}\left\{\nu(q_0)\,\nu(q_s)\right\}\, ds \tag{1}$$

where ν is a real function defined on $\mathcal{S}$, bounded by a constant strictly less than one and such that the following centering condition holds :

$$\overline{\mathbb{E}}\{\nu(q_s)\} = 0,\ \forall s \in \mathbb{R}_-$$

Finally we set :

$$\rho(\vec{r}) \;=\; \rho_0(z)\ \text{, constant outside of } [-L,0] \tag{2}$$

$$\frac{1}{K(\vec{r})} \;=\; \frac{1}{K(z)} = \begin{cases} \frac{1}{K_0(z)}\left(1+\nu(q_{\frac{z}{\varepsilon^2}})\right) & ,\ -L<z<0 \\ \frac{1}{K_{upper}} & ,\ z>0 \\ \frac{1}{K_{lower}} & ,\ z<-L \end{cases} \tag{3}$$

where ε is a small positive parameter, ε^2 representing the size of the random inhomogeneities in the slab $-L<z<0$. Only the bulk modulus contains random fluctuations.

2.2 The acoustic equations

In nondimensional variables the acoustic equations for the velocity $\vec{u}$ and the pressure p are the linearized Euler equations :

$$\begin{cases} \rho\frac{\partial \vec{u}}{\partial t} + \overrightarrow{grad}\, p &= \vec{F}^{\varepsilon}(\vec{r},t) \\ \frac{1}{K}\frac{\partial p}{\partial t} + div\, \vec{u} &= 0 \end{cases} \tag{4}$$

where the ε-dependent acceleration $\vec{F}^{\varepsilon}$ is assumed to be only in the z-direction and the coefficients $\rho(\vec{r})$ and $K(\vec{r})$ are defined in the previous paragraph (2.1).

Denoting by u the vertical (third) component of $\vec{u}$ and using the fact that our model is stratified, one can reduce this system to the following :

$$\begin{cases} \rho\frac{\partial u}{\partial t} + \frac{\partial p}{\partial z} &= F^{\varepsilon}(x,y,z,t) \\ \frac{1}{K}\frac{\partial^2 p}{\partial t^2} - \frac{1}{\rho}\frac{\partial^2 p}{\partial x^2} - \frac{1}{\rho}\frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 u}{\partial z \partial t} &= 0 \end{cases} \tag{5}$$

The initial pulse represented by F^{ε} will be defined in the Fourier domain as follow : in order to work with wave lengths of order ε, intermediate scale between ε^2, the size of random inhomogeneities, and 1, the macroscopic scale, we define a specific Fourier transform in time and transverse variables x and y:

$$\hat{u}^{\varepsilon}(\omega,h,k,z) = \iiint e^{i\frac{\omega}{\varepsilon}(t-hx-ky)}\, u(t,x,y,z)\, dt\, dx\, dy \tag{6}$$

and similar definitions for $\hat{p}^{\varepsilon}$ and $\hat{F}^{\varepsilon}$.

The system (5) becomes in the Fourier domain :

$$\begin{cases} -\frac{i\omega}{\varepsilon}\rho(z)\,\hat{u}^{\varepsilon} + \frac{\partial \hat{p}^{\varepsilon}}{\partial z} &= \hat{F}^{\varepsilon}(\omega,h,k,z) \\ \frac{i\omega}{\varepsilon}\left(\frac{1}{K(z)} - \frac{h^2+k^2}{\rho(z)}\right)\hat{p}^{\varepsilon} - \frac{\partial \hat{u}^{\varepsilon}}{\partial z} &= 0 \end{cases} \tag{7}$$

Let us suppose that F^ε is of the form :

$$F^\varepsilon(x,y,z,t) = \varepsilon\, f(\frac{t}{\varepsilon}) * \varphi(x,y,t)\,\delta(z) \tag{8}$$

where $*$ is the convolution in the time variable t and δ the Dirac measure at $z=0$.

Denoting by $\hat{f}$ and $\hat{\varphi}$ the Fourier transforms :

$$\left\{ \begin{array}{ccc} \hat{f}(\omega) & = & \int e^{i\omega t}\, f(t)\, dt \\ \hat{\varphi}(\omega,h,k) & = & \iiint e^{i\omega(t-hx-ky)}\, \varphi(t,x,y)\, dt\, dx\, dy \end{array} \right. \tag{9}$$

we get the specific Fourier transform of F^ε :

$$\hat{F}^\varepsilon(\omega,h,k,z) = \varepsilon^2 \hat{f}(\omega)\, \hat{\varphi}(\omega,h,k)\, \delta(z) \tag{10}$$

The shape of the emitted pulse is represented by f which we suppose C^∞ with compact support. In the Fourier domain, $\hat{\varphi}(\omega,h,k)$ is the amplitude of the signal emitted in the direction (h,k) : we assume that $\hat{\varphi}$ depends only on h^2+k^2, and that $\hat{\varphi}$ is a truncation function such that :

- $\hat{\varphi}(h^2+k^2) = 0$ on $\{\sqrt{h^2+k^2} > c_0^{-1}(1-\eta)\}$
- $\hat{\varphi}(h^2+k^2) = 1$ on $\{\sqrt{h^2+k^2} < c_0^{-1}(1-2\eta)\}$

for a constant η such that $0 < \eta < \frac{1}{2}$. This choice insures that $\frac{1}{{c_0}^2} - h^2 - k^2 > 0$ and therefore a propagation in the vertical direction staying above turning points.

Let us observe that the factor ε in (8) makes the total energy, released by the source, small of order ε^2 . The equations being linear, this has no other effect. In the following the slowness $\sqrt{h^2+k^2}$ will be denoted by κ.

2.3 Upgoing and downgoing waves

Let $\tau = \tau(z,\kappa)$ be the deterministic time relative to $z=0$ for the oblique plane wave :

$$\tau(z,\kappa) = \int_0^z \frac{\left(1 - c_0(s)^2\,\kappa^2\right)^{\frac{1}{2}}}{c_0(s)}\, ds \tag{11}$$

Note that with this definition τ is negative but increasing in z.

The acoustic impedance is defined by :

$$\zeta_0(z,\kappa) = \frac{\rho_0(z)\, c_0(z)}{\sqrt{1 - c_0(z)^2\,\kappa^2}} \tag{12}$$

These definitions are also valid above and below the random slab where the coefficients are constant. Let us observe that $\frac{\partial}{\partial z}\tau = \frac{\rho_0(z)}{\zeta_0(z,\kappa)}$. Since we have assumed a "matched medium" condition at the interfaces $z=0$ and $z=-L$, the corresponding reflexion coefficients are zero.

The upgoing wave A and the downgoing wave B are defined by :

$$\hat{p}^\varepsilon(\omega,h,k,z) = \sqrt{\zeta_0}\,(A\, e^{i\omega\frac{\tau}{\varepsilon}} - B\, e^{-i\omega\frac{\tau}{\varepsilon}}) \tag{13}$$

$$\hat{u}^\varepsilon(\omega,h,k,z) = \frac{1}{\sqrt{\zeta_0}}\,(A\, e^{i\omega\frac{\tau}{\varepsilon}} + B\, e^{-i\omega\frac{\tau}{\varepsilon}}) \tag{14}$$

They depend on w, h, k and z (τ given by (11) depends on z and κ as well as ζ_0 given by (12)).

Using equations (7) for $\hat{u}^\varepsilon$ and $\hat{p}^\varepsilon$ and the fact that $\rho(z) = \rho_0(z)$ in our model, one can deduce a system of equations for A and B :

$$\frac{\partial A}{\partial z} = \frac{i\omega\zeta_0(z)}{2\varepsilon K_0(z)}\nu(q_{\frac{z}{\varepsilon^2}})(A - Be^{-2i\omega\frac{\tau}{\varepsilon}}) + \frac{1}{2\zeta_0(z)}\frac{\partial\zeta_0(z)}{\partial z}Be^{-2i\omega\frac{\tau}{\varepsilon}} \tag{15}$$

$$\frac{\partial B}{\partial z} = \frac{i\omega\zeta_0(z)}{2\varepsilon K_0(z)}\nu(q_{\frac{z}{\varepsilon^2}})(Ae^{2i\omega\frac{\tau}{\varepsilon}} - B) + \frac{1}{2\zeta_0(z)}\frac{\partial\zeta_0(z)}{\partial z}Ae^{2i\omega\frac{\tau}{\varepsilon}} \tag{16}$$

to which we have to add the following boundary conditions :

$$A(\omega, h, k, -L) = 0 \tag{17}$$

since there is no upgoing wave below the slab, and :

$$B(\omega, h, k, 0) = \frac{1}{2}\,\varepsilon^2\,\hat{f}(\omega)\,\hat{\varphi}(h, k) \tag{18}$$

which is obtained in solving the wave equation in the homogeneous upper half space $\{z > 0\}$ and using the fact that the reflection coefficient is zero (matched medium assumption).

2.4 Representation of the pressure field at $z = -L$

Our quantity of interest is the pressure at the depth $z = -L$; in the Fourier domain this quantity is given by $\hat{p}^\varepsilon(\omega, h, k, -L) = -\sqrt{\zeta_0}\,B(\omega, h, k, -L)\,e^{-i\omega\frac{\tau(-L)}{\varepsilon}}$ by setting $z = -L$ in (13) and using the boundary condition (17).

The randomness is contained in $B(-L)$ where we omit the variables ω, h, k. This quantity is obtained by solving the system (15), (16) with the two-point boundary conditions (17) and (18). Introducing the notations :

$$Q_1(z) = \frac{i\omega\zeta_0(z)}{2K_0(z)}\nu(q_{\frac{z}{\varepsilon^2}})\begin{bmatrix} 1 & -e^{-2i\omega\frac{\tau}{\varepsilon}} \\ e^{2i\omega\frac{\tau}{\varepsilon}} & -1 \end{bmatrix} \tag{19}$$

$$Q_2(z) = \frac{1}{2\zeta_0(z)}\frac{\partial\zeta_0(z)}{\partial z}\begin{bmatrix} 0 & e^{-2i\omega\frac{\tau}{\varepsilon}} \\ e^{2i\omega\frac{\tau}{\varepsilon}} & 0 \end{bmatrix} \tag{20}$$

the system (15) , (16) can be rewritten as :

$$\frac{d}{dz}\begin{pmatrix} A \\ B \end{pmatrix} = \left(\frac{1}{\varepsilon}Q_1(z) + Q_2(z)\right)\begin{pmatrix} A \\ B \end{pmatrix} \tag{21}$$

Considering the solution (a, b) to this system with the initial conditions $(a, b) = (1, 0)$ at $z = 0$, we obtain the propagator P of (21) as the matrix solution defined by :

$$P = \begin{bmatrix} a & \overline{b} \\ b & \overline{a} \end{bmatrix}$$

and with the identity matrix as initial condition. The relation :

$$P(-L)\begin{pmatrix} A(0) \\ B(0) \end{pmatrix} = \begin{pmatrix} A(-L) \\ B(-L) \end{pmatrix}$$

combined with (17) and (18) gives :

$$B(-L) = \frac{a(-L)\overline{a}(-L) - b(-L)\overline{b}(-L)}{a(-L)}\,B(0)$$

On the other hand the trace of $(\frac{1}{\varepsilon}Q_1 + Q_2)$ being 0, the determinant of P is constant equal to $det\, I = 1$ so that $a\bar{a} - b\bar{b} = 1$, which leads to :

$$B(-L) = \frac{1}{a(-L)}\, B(0) \tag{22}$$

B(0) being given by (18), it is natural to study the quantity :

$$C^{\varepsilon}(z) = \frac{1}{a(z)} \tag{23}$$

where we have again omitted ω and h and k. The coefficient $C^{\varepsilon}(z)$ does not satisfy a closed equation. Introducing the quantity Γ^{ε} defined by :

$$\Gamma^{\varepsilon}(z) = \frac{b(z)}{a(z)} \tag{24}$$

one can easily show that $(C^{\varepsilon}, \Gamma^{\varepsilon})$ is the unique solution to the following system of equations :

$$\begin{aligned} \frac{\partial \Gamma^{\varepsilon}(z)}{\partial z} &= \frac{i\omega\zeta_0(z)}{2\varepsilon K_0(z)}\nu(q_{\frac{z}{\varepsilon^2}})\left[e^{2i\omega\frac{\tau}{\varepsilon}} + \Gamma^{\varepsilon}(z)^2 e^{-2i\omega\frac{\tau}{\varepsilon}} - 2\Gamma^{\varepsilon}(z)\right] \\ &\quad + \frac{1}{2\zeta_0(z)}\frac{\partial\zeta_0}{\partial z}\left[e^{2i\omega\frac{\tau}{\varepsilon}} - \Gamma^{\varepsilon}(z)^2 e^{-2i\omega\frac{\tau}{\varepsilon}}\right] \end{aligned} \tag{25}$$

$$\begin{aligned} \frac{\partial C^{\varepsilon}(z)}{\partial z} &= \frac{i\omega\zeta_0(z)}{2\varepsilon K_0(z)}\nu(q_{\frac{z}{\varepsilon^2}})\left(\Gamma^{\varepsilon}(z)e^{-2i\omega\frac{\tau}{\varepsilon}} - 1\right)C^{\varepsilon}(z) \\ &\quad - \frac{1}{2\zeta_0(z)}\frac{\partial\zeta_0}{\partial z}\Gamma^{\varepsilon}(z)C^{\varepsilon}(z)e^{-2i\omega\frac{\tau}{\varepsilon}} \end{aligned} \tag{26}$$

with the initial conditions $(\Gamma^{\varepsilon}(0), C^{\varepsilon}(0)) = (0, 1)$.

Remark that Γ^{ε} satisfies a Riccatti equation while C^{ε} satisfies a linear equation with Γ^{ε} entering in the coefficients. Γ^{ε} is not the usual reflection coefficient used to study the reflected field ; it differs from a phase factor $\frac{a}{\bar{a}}$; this will not affect the study of the transmitted field. Finally let us remark that the relation $|a|^2 - |b|^2 = 1$ implies the uniform boundedness for Γ^{ε} and C^{ε} :

$$\left\{ \begin{array}{lll} |\Gamma^{\varepsilon}| & \leq & 1 \\ |C^{\varepsilon}| & \leq & 1 \end{array} \right. \tag{27}$$

Replacing $B(-L)$ by $C^{\varepsilon}(-L)\, B(0)$ thanks to (22) and (23) in the quantity $\hat{p}^{\varepsilon}(-L)$, we obtain the representation :

$$\hat{p}^{\varepsilon}(\omega, h, k, -L) = -\sqrt{\zeta_0(-L)}C^{\varepsilon}(\omega, h, k, -L)\, B(0)\, e^{-i\omega\frac{\tau(-L)}{\varepsilon}}$$

$B(0)$ is now given by (18). Actually $B(0)$ should be $B(0^-)/\sqrt{\zeta_0(0)}$ but our matched medium condition gives $B(0) = B(0^-)$ and the values of the impedances at $z = 0$ and $z = -L$ being irrelevant in our study, we simply assume that their ratio is equal to 1 so that :

$$\hat{p}^{\varepsilon}(\omega, h, k, -L) = -\frac{1}{2}\varepsilon^2\, \hat{f}(\omega)\, \hat{\varphi}(\kappa)C^{\varepsilon}(\omega, h, k, -L)\, e^{-i\omega\frac{\tau(-L)}{\varepsilon}} \tag{28}$$

By Fourier inverse we get :

$$p^{\varepsilon}(t, x, y, -L) = \frac{1}{(2\pi\varepsilon)^3}\iiint e^{-i\frac{\omega}{\varepsilon}(t - hx - ky)}\, \hat{p}^{\varepsilon}(\omega, h, k, -L)\, \omega^2\, dh\, dk\, d\omega \tag{29}$$

the integral with respect to (h,k) being on the centered disc of radius $c_0^{-1}(1-\eta)$ containing the support of $\hat{\varphi}$.

The original time-scale of the pulse is ε, it is then natural to study the transmitted pulse on the same time-scale and to set :

$$\tilde{p}^\varepsilon(\sigma,x,y,-L) = p^\varepsilon(t_0+\varepsilon\sigma,x,y,-L) \tag{30}$$

where t_0 will be the deterministic time it takes to the wave to travel from the point source to $(x,y,-L)$.

The representation formula can be summarized in :

$$\tilde{p}^\varepsilon(\sigma,x,y,-L) = \frac{-1}{16\pi^3}\iiint e^{-i\omega\sigma}e^{-i\frac{\omega}{\varepsilon}(t_0-hx-ky+\tau(\kappa,-L))}\,C^\varepsilon(\omega,h,k,-L)\,\hat{f}(\omega)\,\hat{\varphi}(\kappa)\,\frac{\omega^2}{\varepsilon}\,dh\,dk\,d\omega \tag{31}$$

where C^ε is obtained from (26).

Our problem is now well posed and we are interested in the asymptotic probability distribution, as ε tends to 0, of the random field $\tilde{p}^\varepsilon(\sigma,x,y,-L)$ at $-L$ fixed.

The variable ω in the integral (31) makes this problem infinite-dimensional, even for σ, x and y fixed. In order to reduce the problem to finite-dimensional problems we shall characterize the law of $\tilde{p}^\varepsilon$ through its moments : let $\{(x_1,y_1),\ldots,(x_N,y_N)\}$ be N points in $\mathbb{R}^2$, $\{\sigma_1,\ldots,\sigma_N\}$ be N times in $\mathbb{R}$ and define :

$$E_N(\varepsilon) = \int \mathbb{E}\left\{\prod_{j=1}^N \tilde{p}^\varepsilon(\sigma_j,x_j,y_j,z)\right\}\Psi_{-L}(z)dz \tag{32}$$

where Ψ_{-L} is a smooth regularizing function around $z=-L$ which will be needed in the proof. Our goal is to derive the asymptotics of $E_N(\varepsilon)$, as ε tends to 0, and deduce the limiting law of $\tilde{p}^\varepsilon$.

To end this section, let us write E_N by pluging (31) in (29) and then in (32) :

$$\begin{aligned} E_N(\varepsilon) &= \left(-\frac{1}{16\pi^3\varepsilon}\right)^N \int_{\mathbb{R}^{2N}\times\mathbb{R}^N}\left\{\int_{\mathbb{R}^-}\exp\left(-i\sum_{j=1}^N\omega_j(\sigma_j+\frac{t_{0,j}(z)-h_jx_j-k_jy_j+\tau_j(z)}{\varepsilon})\right)\right. \\ &\quad \times \left.\mathbb{E}\left[\prod_{j=1}^N C^\varepsilon(\omega_j,h_j,k_j,z)\right]\Psi(z)\,dz\right\}\left(\prod_{j=1}^N \hat{f}(\omega_j)\omega_j^2\hat{\varphi}(\kappa_j)\right)\prod_{j=1}^N dh_jdk_jd\omega_j \end{aligned} \tag{33}$$

where the integral with respect to z is restricted to the support of the regularizing function Ψ_{-L} which can be chosen small around $-L$.

3 The uniform background case

In this simpler case, the sound speed c_0 is constant, equalled to $\sqrt{K_0/\rho_0}$. The travel time $\tau(\kappa,z)$ is now proportional to z :

$$\tau(z,k) = \frac{\sqrt{1-{c_0}^2\,\kappa^2}}{c_0}\,z \tag{34}$$

and the impedance ζ_0 is independent of z :

$$\zeta_0(\kappa) = \frac{\rho_0\,c_0}{\sqrt{1-{c_0}^2\,\kappa^2}}$$

The differential system (25) and (26) defining $(\Gamma^\varepsilon, C^\varepsilon)$ simplifies into :

$$\frac{\partial\Gamma^\varepsilon}{\partial z} = \frac{i\omega\zeta_0}{2\varepsilon K_0}\nu(q_{\frac{z}{\varepsilon^2}})\left[e^{2i\omega\frac{\tau}{\varepsilon}}+\Gamma^\varepsilon(z)^2e^{-2i\omega\frac{\tau}{\varepsilon}}-2\Gamma^\varepsilon(z)\right] \tag{35}$$

$$\frac{\partial C^\varepsilon}{\partial z} = \frac{i\omega\zeta_0}{2\varepsilon K_0}\nu(q_{\frac{z}{\varepsilon^2}})\left(\Gamma(z)e^{-2i\omega\frac{\tau}{\varepsilon}}-1\right)C^\varepsilon(z) \tag{36}$$

with the same initial conditions $(\Gamma^\varepsilon(0), C^\varepsilon(0)) = (0, 1)$. The main simplification comes from the fact that in the case of a constant speed c_0, the time t_0 is explicitely known :

$$t_0 = \frac{\sqrt{L^2 + x^2 + y^2}}{c_0} \tag{37}$$

Coming back to the moments $E_N(\varepsilon)$ given by (33) we see that we have two types of limits : one is of probabilistic nature – the mathematical expectations of the products of C^ε –, the other involving stationary phase results due to the exponentials.

3.1 Diffusion-approximation

Coupling systems (35) and (36) for N values of (ω, h, k) we obtain a complex valued system of $2N$ equations. In order to work with real systems we pass to real and imaginary parts and obtain a system of $4N$ equations which is of the form :

$$\frac{dY^\varepsilon}{dz} = \frac{1}{\varepsilon} F(Y^\varepsilon, \frac{z}{\varepsilon}, q_{\frac{z}{\varepsilon^2}})$$

with initial conditions at $z = 0$ and F periodical in the variable z/ε coming from terms such as $e^{2i\omega\frac{z}{\varepsilon}}$.

This system is quite lengthy to write but it is easy to see that it constitutes a classical diffusion-approximation problem treated in [8]. We also refer to [4] for a direct treatment of complex systems. From this system we are interested in the particular functional :

$$f(Y_z^\varepsilon) = \prod_{j=1}^{N} C^\varepsilon(\omega_j, h_j, k_j, z) \tag{38}$$

where (Y_z^ε) converges in law to a diffusion Y_z.

We refer to [3] where it is proved in details that $\mathbb{E}\{f(Y_z^\varepsilon)\}$ converges, as ε tends to 0, to $\mathbb{E}\{f(Y_z)\}$ which satisfies :

$$d\mathbb{E}\{f(Y_z)\} = \left(2\alpha \sum_{j=1}^{N} {\theta_j}^2 + 2\alpha \sum_{l \neq j} \theta_j \, \theta_l \right) \mathbb{E}\{f(Y_z)\}\, dz \tag{39}$$

where α is the correlation coefficient defined in (1), θ_j being given by :

$$\theta_j = \frac{\omega_j}{2\, c_0 \sqrt{1 - c_0^2 \kappa_j^2}} \tag{40}$$

and the condition $\mathbb{E}\{f(Y_0^\varepsilon)\} = \mathbb{E}\{f(Y_0)\} = 1$ at $z = 0$.

Obviously the linear equation (39) could be solved but, in order to identify the limiting law of $\tilde{p}^\varepsilon$, it is easier to remark that :

$$\mathbb{E}\{f(Y_z)\} = \mathbb{E}\left\{ \prod_{j=1}^{N} \tilde{C}_j(z) \right\}$$

where $\tilde{C}_j(z)$ is defined as the unique solution to the linear stochastic differential equation :

$$\begin{array}{rcl} d\tilde{C}_j(z) & = & \left(i\sqrt{2\alpha}\theta_j dW_{-z} + 2\alpha {\theta_j}^2 dz\right) \tilde{C}_j(z) \\ \tilde{C}_j(0) & = & 1 \end{array} \tag{41}$$

where W is a real standard brownian motion independant of N and j and the equation is a usual initial value stochastic differential equation after the change of variables $z' = -z$.

This means that the limiting law of $\tilde{p}^\varepsilon$ will be obtained by replacing C^ε by $\tilde{C}$ in (31) where $\tilde{C}$ is the explicit solution to (41) :

$$\tilde{C}(z) = \exp\left(i\sqrt{2\alpha}\theta W_{-z} + \alpha\theta^2 z\right) \tag{42}$$

with α given by (1) and $\theta = \frac{\omega}{2\, c_0\sqrt{1-c_0^2\kappa^2}}$. Note that the change of variables $z' = -z$ has to be performed before applying Itô's formula.

3.2 Stationary phases

In the representation (33) of E_N we have oscillatory phases given by :

$$\phi_z(\omega, h, k, x, y) = \omega\,(t_0 - hx - ky + \tau(\omega, \kappa, z))$$

which reduces, in the uniform background case, to :

$$\phi_z(\omega, h, k, x, y) = \omega\left(\frac{\sqrt{L^2+x^2+y^2}}{c_0} - hx - ky + \frac{\sqrt{1-c_0^2\,\kappa^2}}{c_0}\,z\right) \tag{43}$$

where we have used (34) and (37) at depth z.

Assuming that we can replace the expectation in (33) by its limit value given by (39), $E_N(\varepsilon)$ becomes :

$$\begin{aligned} E_N(\varepsilon) \quad = \quad & \left(-\frac{1}{16\pi^3\varepsilon}\right)^N \int_{\boldsymbol{R}^{2N}\times\boldsymbol{R}^N} \{\int_{\boldsymbol{R}} e^{-i\sum_{j=1}^N \omega_j\sigma_j} \exp\left(-\frac{i}{\varepsilon}\sum_{j=1}^N \phi_z(\omega_j, h_j, k_j, x_j, y_j)\right) \\ & I\!E\{f(Y_z)\}\,\Psi_{-L}(z)\,dz\} \prod_{j=1}^N \hat{f}(\omega_j)\,{\omega_j}^2\hat{\varphi}(\kappa_j) \prod_{j=1}^N dh_j\,dk_j\,d\omega_j \end{aligned} \tag{44}$$

where the phases ϕ_z are given by (43).

The limit, as $\varepsilon \to 0$, of the integral with respect to $(h_j, k_j)_{1\le j\le N}$ will be given by the application of the stationary phase theorem which can be found in [2], [5] or in [3] adapted to our situation : let us remark that $1/\varepsilon^N$ in front of the integral is the correct scaling for the application of this result which means that we have properly scaled the energy released by the pulse in (8). The stationary phase method tells us that the integral concentrates on values of (h_j, k_j) such that $\nabla_{(h,k)}\phi$ vanishes. These values are easily computable and give :

$$h_j = \frac{x_j}{c_0\sqrt{z^2+{x_j}^2+{y_j}^2}}\;,\; k_j = \frac{y_j}{c_0\sqrt{z^2+{x_j}^2+{y_j}^2}} \tag{45}$$

Since we have made the correct choice (37) for t_0, we also have that ϕ vanishes.

We shall not write the explicit value of the limit of $E_N(\varepsilon)$ which involves correct constants and the computation of the differential form of order two of $\sum_{j=1}^N \phi_z(\omega_j, h_j, k_j, x_j, y_j)$ taken for the value of (h_j, k_j) given by (45).

In order to indentify the limiting law of $\tilde{p}^\varepsilon$, it will be easier to apply the result directly to (31), after replacing C^ε by $\tilde{C}$ defined above. We delay this step to the end of this section.

3.3 Simultaneous limits

The main difficulty is to show that the diffusion-approximation limit and the stationary phase can be done simultaneously in $E_N(\varepsilon)$ given by (33). The diffusion-approximation result gives us only the limit of $\mathbb{E}\{f(Y_z^\varepsilon)\}$; what we need here is a full expansion in ε up to order $N+1$ with a control of the regularity in (h_j,k_j) of the coefficients of this expansion. This is very technical and quite long, and has be done in details in [3]. The linearity of the equation for the limit $f(Y_z)$ has been fully exploited in the proof. The weak form we have considered with the help of the regularizing function $\Psi_{-L}(z)$ enable us to do integration by parts as much as needed ; this is the reason for introducing this regularization.

Let us remark that this consecutive limits method has already been used, without a rigorous treatment of the simultaneous limits, in [1] (section 3.3) in the study of the coherent reflected pressure.

3.4 Identification of the limit

Another problem arising here is the tightness of the random field $\tilde{p}^\varepsilon(\sigma,x,y,-L)$ for L fixed and $(\sigma,x,y)\in\mathbb{R}^3$. This tightness is obtained by using classical criteria on moments of increments of the random field. These moments are controlled again by combining diffusion-approximation results and stationary phase methods. We refer again to [3] for details. Once this tightness is established, in order to identify the limiting law of $\tilde{p}^\varepsilon$, we come back to the representation (31), instead of computing explicitly all the limiting moments. Replacing C^ε by $\tilde{C}$ in (31) written in the simpler case of the uniform background, we obtain that the limiting law of $\tilde{p}^\varepsilon(\sigma,x,y,-L)$ is given by :

$$\lim_{\varepsilon\to 0}\frac{-1}{16\pi^3\varepsilon}\int\!\!\int\!\!\int e^{-i\omega\sigma}\,e^{-\frac{i}{\varepsilon}\phi_{-L}(h,k,x,y)}\,\tilde{C}(\omega,h,k,-L)\,\hat{f}(\omega)\,\hat{\varphi}(\kappa)\,\omega^2\,dh\,dk\,d\omega$$

where $\tilde{C}$ is given by (42) and ϕ_{-L} by (43).

The stationary phase theorem tells us that the integral with respect to (h,k) concentrates on the value of (h,k) given as in (45) (see also [5], section 4):

$$h=\frac{x}{c_0\sqrt{z^2+x^2+y^2}}\,,\;k=\frac{y}{c_0\sqrt{z^2+x^2+y^2}} \tag{46}$$

The coefficient $\theta(\kappa)$ appearing in $\tilde{C}$ is given in (40) and takes the value :

$$\theta(\kappa)=\frac{\omega\sqrt{L^2+x^2+y^2}}{2Lc_0}$$

for (h,k) given by (46). $\tilde{C}(\omega,h,k,x,y)$ will then concentrate on the value :

$$\tilde{C}(\omega,x,y,-L)=\exp\left(i\omega\frac{\sqrt{2\alpha(L^2+x^2+y^2)}}{2Lc_0}\,W_L-\omega^2\alpha\frac{L^2+x^2+y^2}{4L{c_0}^2}\right) \tag{47}$$

The computation of the second derivative of ϕ_{-L} at (h,k) given by (46) and a careful application of the stationary phase theorem gives :

$$\begin{aligned}\tilde{p}(\sigma,x,y,-L) \quad &= \quad \frac{-1}{8c_0\pi^2\sqrt{L^2+x^2+y^2}}\int\exp\left(i\omega\frac{\sqrt{2\alpha(L^2+x^2+y^2)}}{2Lc_0}W_L\right)\\ &\quad\times\exp\left(-\omega^2\alpha\frac{L^2+x^2+y^2}{4L{c_0}^2}\right)\,i\omega\,e^{-i\omega\sigma}\,\hat{f}(\omega)\,d\omega\end{aligned} \tag{48}$$

where we have used the fact that $\hat{\varphi}(\kappa)$ is equal to one as soon as $\kappa^2=\frac{x^2+y^2}{{c_0}^2(L^2+x^2+y^2)}<\frac{1}{{c_0}^2}(1-2\eta)$ (see section (2.2)) which is satisfied for $x^2+y^2<L^2(\frac{1-2\eta}{2\eta})$. If we restrict $\tilde{p}(\sigma,x,y,-L)$ to (x,y) in a given compact of $\mathbb{R}^2$, it is enough to choose η as small as needed for (48) to hold.

Let Z be a centered gaussian random variable with variance 1 ; denote by $Z_L(x,y)$ the gaussian random variable :

$$Z_L(x,y) = \frac{1}{c_0}\sqrt{\frac{\alpha(L^2+x^2+y^2)}{2L}}\, Z \tag{49}$$

Denote by $g_L(x,y)$ the centered gaussian density with variance $\frac{\alpha(L^2+x^2+y^2)}{2L^2{c_0}^2}$; we then have the main result of this section :

Theorem 1 *The pressure field* $p^\varepsilon(\frac{\sqrt{L^2+x^2+y^2}}{c_0}+\varepsilon\sigma, x, y, -L)$ *converges in law, as* $\varepsilon \to 0$ *and* L *fixed, to the random field* $\tilde p$ *defined by :*

$$\tilde p(\sigma,x,y,-L) = \frac{1}{4\pi c_0\sqrt{L^2+x^2+y^2}}\left(f * g_L'(x,y)\right)(\sigma - Z_L(x,y))$$

where the initial pulse is convoluted with the **derivative** *of the gaussian density* $g_L(x,y)$.

Proof

From (48), notation (49) and the definition of $g_L(x,y)$ we derive successively, our equalities being in law :

$$\begin{aligned}
\tilde p(\sigma,x,y,-L) &= \frac{-1}{8c_0\pi^2\sqrt{L^2+x^2+y^2}}\int e^{i\omega Z_L(x,y)} e^{-\omega^2\alpha\frac{L^2+x^2+y^2}{4L{c_0}^2}}\left(\int e^{i\omega t} f(t)\,dt\right) i\omega\, e^{-i\omega\sigma}\,d\omega \\
&= \frac{-1}{8c_0\pi^2\sqrt{L^2+x^2+y^2}}\int f(t)\left[\int e^{-\omega^2\alpha\frac{L^2+x^2+y^2}{4L{c_0}^2}}\, e^{i\omega(t-\sigma+Z_L(x,y))}\, i\,\omega\,d\omega\right] dt \\
&= \frac{-1}{8c_0\pi^2\sqrt{L^2+x^2+y^2}}\int f(t)\frac{\partial}{\partial t}\left[\int e^{-\omega^2\alpha\frac{L^2+x^2+y^2}{4L{c_0}^2}}\, e^{i\omega(t-\sigma+Z_L(x,y))} d\omega\right] dt
\end{aligned}$$

Using the fact that :

$$\int e^{i\omega s}\, e^{-a\omega^2}\, d\omega = \frac{\sqrt{\pi}}{\sqrt{a}}\, e^{-\frac{s^2}{4a}}$$

applied with $a = \alpha\frac{L^2+x^2+y^2}{4L{c_0}^2}$, we get :

$$\tilde p(\sigma,x,y,-L) = \frac{-1}{8c_0\pi^2\sqrt{L^2+x^2+y^2}}\int f(t)\frac{\partial}{\partial t}\left[\frac{\sqrt{\pi}}{\sqrt{a}}\, e^{-\frac{(t-\sigma+Z_L(x,y))^2}{4a}}\right] dt$$

Introducing the centered gaussian density $g_L(x,y)$ with variance $2a = \alpha\frac{L^2+x^2+y^2}{2L{c_0}^2}$, we now have :

$$\tilde p(\sigma,x,y,-L) = \frac{1}{4c_0\pi\sqrt{L^2+x^2+y^2}}\int f(t)\left(\frac{\partial}{\partial t} g_L(x,y)\right)(\sigma - t - Z_L(x,y))\, dt$$

which is nothing else than the formula announced in the theorem 1. □

It is noticeable that only one gaussian variable Z is needed to describe the asymptotical law of $\tilde p^\varepsilon(\sigma,x,y,-L)$ at depth $-L$ fixed. This comes from the diffusion-approximation result and the particular form of the functional $f(Y_z^\varepsilon)$ defined in (38) needed in the study of the moments $E_N(\varepsilon)$. Actually $(2N+1)$ independent real brownian motions are needed to describe the limit (Y_z) ; only one W is common to all frequencies, and each frequency ω_j requires two other independent brownian motions. On the other hand, the integral with respect to $\prod_{j=1}^N d\omega_j$ in (33) allows us to consider distinct frequencies $(\omega_j)_{1\leq j\leq N}$; the second order differential of $f(Y)$ contains only second order mixed partial derivatives so that only the brownian motion W will contribute to the Itô's correction in the equation (39) for $\mathbb{E}\{f(Y_z)\}$.

This is done in detail in [4] for the one-dimensional case and in [3] for our case.

The second important remark is the difference between the one-dimensional case and the three-dimensional layered case : in the first case the transmitted pressure is obtained by a convolution of the initial pulse f with a gaussian density while in the second case this convolution is made with the **derivative** of a gaussian density.

We delay the comparison with geometrical acoustics in the last section.

4 The general case

We start again with the system (25) and (26), the integral representation (31) and the moments $E_N(\varepsilon)$ given in (33). The difficulty comes essentially from the fact that $\tau(z,\kappa)$ given by (11) is not proportional to z as in the uniform background case. This will imply a complication in both the diffusion-approximation result and the stationary phases method.

4.1 Generalized diffusion-approximation result

As in 3.1, in order to study (25) and (26) for N values of (ω,h,k) we have to deal with an equation of the form :

$$\frac{\partial Y^\varepsilon}{\partial z}=\frac{1}{\varepsilon}F\left(Y^\varepsilon_z,q_{\frac{z}{\varepsilon^2}},z,\frac{\tau_1(z)}{\varepsilon},\dots,\frac{\tau_N(z)}{\varepsilon}\right)+G\left(Y^\varepsilon_z,z,\frac{\tau_1(z)}{\varepsilon},\dots,\frac{\tau_N(z)}{\varepsilon}\right)$$

where Y^ε is in $\mathbb{R}^{4N}$, $\tau_j(z)=\tau(z,\kappa_j)$ and G comes from the terms involving the derivative of the impedance $\zeta_0(z)$ in (25) and (26).

The generalization of classical diffusion-approximation results established in details in [3] tells us the following :

Define the infinitesimal generator :

$$\begin{aligned}L_{z,\tau_1,\dots,\tau_N}f &= \int_0^\infty \overline{\mathbb{E}}\left\{F(Y,q_0,z,\tau_1,\dots,\tau_N).\nabla_Y[F(Y,q_s,z,\tau_1,\dots,\tau_N).\nabla_Y f]\right\}ds\\ &\quad +G(Y,z,\tau_1,\dots,\tau_N).\nabla_Y f+\frac{\partial}{\partial z}f\end{aligned}\tag{50}$$

Suppose that :

$$L_z=\lim_{T\to+\infty}\frac{1}{T}\int_0^T L_{z,\tau_1+s\tau_1'(z),\dots,\tau_N+s\tau_N'(z)}ds\tag{51}$$

exists and is independent of $(\tau_1,\dots,\tau_N)$ for almost all z (where τ' is the derivative of τ). If the martingale problem associated to L_z is well-posed, the process (Y^ε_z) converges in law, as $\varepsilon\to 0$, to the diffusion (Y_z) with infinitesimal generator L_z.

In our case, the functions F and G are periodical in each of the variables $\tau_1,\dots,\tau_N$ and this ensures the convergence in (51). Note also that G will not affect the limit after the averaging over a period. The particular form of $\tau_j'(z,\kappa_j)=\frac{\sqrt{1-c_0(z)^2\kappa_j{}^2}}{c_0(z)}$ enables us to show that the limit above is independent of $(\tau_1,\dots,\tau_N)$ for almost all z due to the fact that $c_0(z)$ does not vanish.

Once the result is applied we get a similar equation to (41) with θ_j now defined by :

$$\theta_j=\frac{\omega_j}{2c_0(z)\sqrt{1-c_0(z)^2\kappa_j^2}}$$

Of course this linear equation is not anymore with constant coefficients.

4.2 Stationary phases

As in section 3, a full proof would require to apply stationary phase methods to the moments $E_N(\varepsilon)$ to justify the simultaneous limits (diffusion-approximation and stationary phase) and to obtain, as a by product, the tightness of the field $\tilde{p}^\varepsilon$. We will not repeat all these steps and simply apply the stationary phase method to (31) where C^ε has been replaced by the solution $\tilde{C}$ of our generalized equation (41).

Again in (31) we have the phases $\phi_z(\omega, h, k, x, y)$ given by (43). The difference here is that we do not have an a priori value for the correct t_0 at a given point (x, y, z) : this is due to the fact that the acoustic rays are not straight lines anymore. Nevertheless the gradient of ϕ (with respect to (h, k)) will vanish if :

$$\left\{ \begin{array}{rcl} x & = & h \int_z^0 \frac{c_0(s)}{\sqrt{1-c_0(s)^2\kappa^2}}\, ds \\ y & = & k \int_z^0 \frac{c_0(s)}{\sqrt{1-c_0(s)^2\kappa^2}}\, ds \end{array} \right. \tag{52}$$

which gives implicitely the unique value of (h_0, k_0) which makes $\nabla\phi(h_0, k_0)$ equalled to 0 (recall that $\kappa^2 = h^2 + k^2$).

It is a simple computation to show that the wronskian of ϕ is non-negative which is the required condition in the application of the stationary phase theorem.

(h_0, k_0) being obtained from (x, y, z) by solving (52), we now determine t_0 by solving the equation $\phi_z(\omega, h_0, k_0, x, y) = 0$ so that :

$$\begin{aligned} 0 &= t_0 - h_0\, x - k_0\, y + \tau(\omega, \kappa_0, z) \\ &= t_0 - h_0\, x - k_0\, y + \int_0^z \frac{\sqrt{1 - c_0(s)^2\kappa_0^2}}{c_0(s)}\, ds \end{aligned}$$

Using (52) we get :

$$\begin{aligned} t_0 &= {\kappa_0}^2 \int_z^0 \frac{c_0(s)}{\sqrt{1 - c_0(s)^2\kappa_0^2}}\, ds - \int_0^z \frac{\sqrt{1 - c_0(s)^2\kappa_0^2}}{c_0(s)}\, ds \\ &= \int_z^0 \frac{1}{c_0(s)\sqrt{1 - c_0(s)^2\kappa_0^2}}\, ds \end{aligned} \tag{53}$$

4.3 Identification of the limit

Rewriting (31) with C^ε replaced by $\tilde{C}$, we get that the limit law of $p^\varepsilon(t_0 + \varepsilon\sigma, x, y, -L)$ is given by :

$$\begin{aligned} p^\varepsilon(t_0 + \varepsilon\sigma, x, y, -L) &= \lim_{\varepsilon\to 0} \frac{-1}{16\pi^3\varepsilon} \iiint e^{-i\omega\sigma}\, e^{-\frac{i}{\varepsilon}\phi_{-L}(\omega,h,k,x,y)}\, e^{i\sqrt{2\alpha}\int_0^{-L}\theta(s,\kappa)\, dW_{-s} + \alpha\int_0^{-L}\theta^2(s,\kappa)ds} \\ & \quad \hat{f}(\omega)\, \hat{\varphi}(\kappa)\, \omega^2\, dh\, dk\, d\omega \end{aligned} \tag{54}$$

with t_0 given by (53) for ${\kappa_0}^2 = {h_0}^2 + {k_0}^2$.

Again by the stationary phase theorem, the integral with respect to (h, k) will concentrate on $(h, k) = (h_0, k_0)$ and a computation similar to the proof of theorem 1 gives that the limit can be written as :

$$\beta(x, y, -L)\, (f * g_L'(x, y))\, (\sigma - Z_L(x, y)) \tag{55}$$

where $g_L(x, y)$ is the centered gaussian density with variance v defined by :

$$v = \frac{\alpha}{2} \int_{-L}^0 \frac{1}{c_0(s)^2(1 - c_0(s)^2{\kappa_0}^2)}\, ds$$

$Z_L(x,y)$ is defined by :

$$Z_L = \left(\alpha \int_{-L}^{0} \frac{1}{2c_0(s)^2(1 - c_0(s)^2 \kappa_0{}^2)} \, ds \right)^{\frac{1}{2}} Z$$

(Z being a centered gaussian random variable with variance 1), and $\beta(x,y,-L)$ is the coefficient corresponding to the amplitude attenuation obtained through the wronskian of the stationary phase as in the classical geometrical acoustics approximation.

Remark : a simple computation shows that, at depth $-L$ fixed, $\nabla_{(x,y)} t_0(x,y) = (h_0, k_0)$. This gives a practical method to estimate $\kappa_0{}^2 = h_0{}^2 + k_0{}^2$ by observing the variation of the delay of the signal arriving at depth $-L$ in function of the position (x,y).

5 Conclusion

The classical geometrical acoustics approximation gives t_0, the delay of the signal arriving at depth $-L$, it gives also the amplitude attenuation $\beta(x,y,-L)$ in (55). In the uniform background case this coefficient is $(4\pi c_0 \sqrt{L^2 + x^2 + y^2})^{-1}$, which means that the energy decreases as $\frac{1}{r^2}$ (this is the spherical wave front case). Our study brings a complete description of how the initial pulse spreads after travelling through the random slab and the separation of scales limit. Moreover the law of this spreading has been obtained jointly for all horizontal positions (x,y). To summarize, it is a convolution by the derivative of a gaussian density and a random gaussian time delay whose variance has been computed. Our final comment is that we have presented the problem in the context of a markovian model (the ergodic Markov process q). All these results can be generalized to a model driven by a random process satisfying nice mixing conditions.

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