Introduction to Pricing and Hedging in Continuous Time

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Chapter 1
The Black-Scholes Theory of Derivative Pricing

1.1 Market Model

One riskless asset (savings account):

$$d\beta_t = r\beta_t dt, \tag{1}$$

where $r \geq 0$ is the instantaneous interest rate.
Setting $\beta_0 = 1$, we have $\beta_t = e^{rt}$ for $t \geq 0$.

The price X_t of the other asset, the risky stock or stock index, evolves according to the
stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \tag{2}$$

where μ is a constant mean return rate, $\sigma > 0$ is a constant *volatility* and $(W_t)_{t\geq 0}$ is a standard *Brownian motion*.

1.1.1 Brownian Motion

A Brownian motion is a real-valued stochastic process with continuous trajectories that have independent and stationary increments. The trajectories are denoted by $t \to W_t$ and for the standard Brownian motion:

- $W_0 = 0$;
- for any $0 < t_1 < \cdots < t_n$, the random variables $(W_{t_1}, W_{t_2} - W_{t_1}, \cdots, W_{t_n} - W_{t_{n-1}})$ are independent;
- for any $0 \leq s < t$, the increment $W_t - W_s$ is a centered (mean-zero) normal random variable with variance $\mathbb{E}\{(W_t - W_s)^2\} = t - s$. In particular W_t is $\mathcal{N}(0, t)$-distributed.

$\mathcal{F}_t$ denotes the σ-algebra generated by $(W_s)_{s \leq t}$, the information on W up to time t.

Conditional characteristic functions

For $0 \le s < t$ and $u \in \mathbb{R}$

$$\mathbf{E}\left\{\mathbf{e}^{\mathbf{iu(W_t - W_s)}} \mid \mathcal{F}_\mathbf{s}\right\} \quad = \quad \mathbf{e}^{-\frac{\mathbf{u^2(t-s)}}{\mathbf{2}}}. \tag{3}$$

If W is a Brownian motion, by independence of the increment $W_t - W_s$ from the past $\mathcal{F}_s$, the left-hand side of (3) is simply

$$\mathbb{E}\left\{e^{iu(W_t - W_s)}\right\},$$

which is the characteristic function of a centered normal random variable with variance $t-s$, and is equal to the right-hand side.

Conversely, if (3) holds, then the continuous process (W_t) is a standard Brownian motion.

Gaussian white noise

This independence of increments makes the Brownian motion an ideal candidate to define a complete family of **independent infinitesimal increments $\mathbf{dW_t}$**, which are $\mathcal{N}(\mathbf{0}, \mathbf{dt})$-distributed (centered, normally distributed with variance dt) and which will serve as a model of (Gaussian white) noise.

The drawback is that the trajectories of (W_t) are **not of bounded variation**.

Let $t_0 = 0 < t_1 < \cdots < t_n = t$ be an evenly spaced subdivision of $[0, t]$ so that $t_i - t_{i-1} = t/n$. The quantity

$$\mathbb{E}\left\{\sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}|\right\} = n\mathbb{E}\{|W_{\frac{t}{n}}|\} = n\sqrt{\frac{t}{n}}\mathbb{E}\{|W_1|\},$$

goes to $+\infty$ as $n \nearrow +\infty$.

The integral with respect to dW_t cannot be defined in the usual way "trajectory by trajectory".

1.1.2 Stochastic Integrals

For T fixed, let $(X_t)_{0\leq t\leq T}$ be a stochastic process adapted to $(\mathcal{F}_t)_{0\leq t\leq T}$, and such that

$$\mathbb{E}\left\{\int_0^T (X_t)^2 dt\right\} \quad < \quad +\infty. \tag{4}$$

Then for $t_0 = 0 < t_1 < \cdots < t_n = t \leq T$ we have:

$$\mathbb{E}\left\{\left(\sum_{i=1}^{n} X_{t_{i-1}}\left(W_{t_i} - W_{t_{i-1}}\right)\right)^2\right\} \quad = \quad \mathbb{E}\left\{\sum_{i=1}^{n}\left(X_{t_{i-1}}\right)^2 (t_i - t_{i-1})\right\}$$

The Brownian increments on the left are forward in time and the sum on the right converges to $\mathbb{E}\left\{\int_0^t (X_s)^2 ds\right\}$ which is finite by (4).

The *stochastic integral* of (X_t) with respect to (W_t) is defined as a limit in the mean-square sense $(L^2(\Omega))$

$$\int_0^t X_s dW_s = \lim_{n \nearrow +\infty} \sum_{i=1}^{n} X_{t_{i-1}} \left(W_{t_i} - W_{t_{i-1}}\right), \tag{5}$$

as the mesh size of the subdivision goes to zero.

As a function of time t, this stochastic integral defines a **continuous square integrable process** such that

$$\mathbf{E}\left\{\left(\int_0^t \mathbf{X_s dW_s}\right)^2\right\} = \mathbf{E}\left\{\int_0^t \mathbf{X_s^2 ds}\right\}, \tag{6}$$

and has the *martingale property*

$$\mathbf{E}\left\{\int_0^t \mathbf{X_u dW_u} \mid \mathcal{F}_\mathbf{s}\right\} = \int_0^\mathbf{s} \mathbf{X_u dW_u} \quad \mathbf{P}\text{-a.s., for } \mathbf{s} \leq \mathbf{t}, \tag{7}$$

as can be easily deduced from the definition (5).

The *quadratic variation* $\langle Y \rangle_t$ of the stochastic integral $Y_t = \int_0^t X_u dW_u$ is given by

$$\langle Y \rangle_t = \lim_{n \nearrow +\infty} \sum_{i=1}^{n} (Y_{t_i} - Y_{t_{i-1}})^2 = \int_0^t X_s^2 ds \tag{8}$$

in the mean-square sense.

Stochastic integrals are zero mean, continuous and square integrable martingales.

It is interesting to note that the converse **representation result** is also true: every zero mean, continuous and square integrable $(\mathcal{F}_t)$-martingale is a Brownian stochastic integral.

1.1.3 Risky Asset Price Model

Differential form:

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t \tag{9}$$

Integral form:

$$X_t = X_0 + \mu \int_0^t X_s ds + \sigma \int_0^t X_s dW_s \tag{10}$$

General class of *stochastic differential equations* driven by a Brownian motion:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \tag{11}$$

or in integral form

$$X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s. \tag{12}$$

Usual calculus does not apply!

The solution X_t of (9) is **NOT** given by

$$X_0 \exp(\mu t + \sigma W_t)$$

This is not correct because the usual chain rule is not valid for stochastic differentials. For instance

$$\mathbf{W_t^2 \neq 2 \int_0^t W_s dW_s}$$

as might be expected since, by the martingale property (7), this last integral has an expectation equal to zero but $\mathbb{E}\left\{W_t^2\right\} = t$.

This discrepancy is corrected by Itô's formula .

1.1.4 Itô's Formula

The purpose of the chain rule is to compute the differential $\mathbf{d(g(W_t))}$ or equivalently its integral $\mathbf{g(W_t) - g(W_0)}$.
Using the subdivision $t_0 = 0 < t_1 \cdots < t_n = t$ and Taylor's formula:

$$
\begin{aligned}
g(W_t) - g(W_0) &= \sum_{i=1}^{n} (g(W_{t_i}) - g(W_{t_{i-1}})) \\
&= \sum_{i=1}^{n} g'(W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}}) \\
&\quad + \frac{1}{2} \sum_{i=1}^{n} g''(W_{t_{i-1}})(W_{t_i} - W_{t_{i-1}})^2 + R
\end{aligned}
$$

where R contains all the higher order terms.

If (W_t) were differentiable only the first sum would contribute to the limit as the mesh size of the subdivision goes to zero, leading to the chain rule $dg(W_t) = g'(W_t)W_t' dt$ of classical calculus.

In the Brownian case (W_t) is **not differentiable** and, by (5), the first sum converges to the stochastic integral

$$\int_0^{\mathbf{t}} \mathbf{g}'(\mathbf{W_s})\mathbf{dW_s}.$$

The second sum, like the quadratic variation (8), converges to

$$\frac{\mathbf{1}}{\mathbf{2}}\int_0^{\mathbf{t}} \mathbf{g}''(\mathbf{W_s})\mathbf{ds}.$$

This can be seen by comparing it in L^2 with $\frac{1}{2}\sum_{i=1}^{n} g''(W_{t_{i-1}})(t_i - t_{i-1})$. The higher order terms contained in R converge to zero and do not contribute to the limit, which gives the simplest version of **Itô's formula**:

$$\mathbf{g}(\mathbf{W_t}) - \mathbf{g}(\mathbf{W_0}) = \int_0^{\mathbf{t}} \mathbf{g}'(\mathbf{W_s})\mathbf{dW_s} + \frac{\mathbf{1}}{\mathbf{2}}\int_0^{\mathbf{t}} \mathbf{g}''(\mathbf{W_s})\mathbf{ds} \qquad (13)$$

Itô's formula in differential form:

$$\mathbf{dg(W_t) = g'(W_t)dW_t + \frac{1}{2}g''(W_t)dt.} \tag{14}$$

More generally, when X_t satisfies (11)

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t,$$

and g depends also on t, one has

$$dg(t, X_t) = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)d\langle X\rangle_t, \tag{15}$$

where $\langle \mathbf{X}\rangle_{\mathbf{t}} = \int_{\mathbf{0}}^{\mathbf{t}} \sigma^{\mathbf{2}}(\mathbf{s}, \mathbf{X_s})\mathbf{ds}$ is the quadratic variation of the martingale part of X_t.
In terms of dt and dW_t the formula is

$$dg(t, X_t) = \left(\frac{\partial g}{\partial t} + \mu(t, X_t)\frac{\partial g}{\partial x} + \frac{1}{2}\sigma^2(t, X_t)\frac{\partial^2 g}{\partial x^2}\right)dt + \sigma(t, X_t)\frac{\partial g}{\partial x}dW_t, \tag{16}$$

where all the partial derivatives of g are evaluated at (t, X_t).

Application to the discounted price $g(t, X_t) = e^{-rt}X_t$

$$d\left(e^{-rt}X_t\right) = -re^{-rt}X_t dt + e^{-rt}dX_t$$
$$= e^{-rt}\left(-rX_t + \mu(t, X_t)\right)dt + e^{-rt}\sigma(t, X_t)dW_t \quad (17)$$
$$= (\mu - r)\left(e^{-rt}X_t\right)dt + \sigma\left(e^{-rt}X_t\right)dW_t.$$

The discounted price $\mathbf{\widetilde{X}_t = e^{-rt}X_t}$ satisfies the same equation as X_t where the return μ has been replaced by $\mu - r$:

$$\mathbf{d\widetilde{X}_t} = (\mu - \mathbf{r})\mathbf{\widetilde{X}_t dt} + \sigma\mathbf{\widetilde{X}_t dW_t}. \quad (18)$$

Integration by parts formula

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + d\langle X, Y\rangle_t\,, \quad (19)$$

where the covariation (also called "bracket") of X and Y is given by

$$d\langle X, Y\rangle_t = \sigma_X(t, X_t)\sigma_Y(t, Y_t)dt.$$

1.1.5 Lognormal Risky Asset Price

$$\mathbf{dX_t = X_t(\mu dt + \sigma dW_t)}$$

gives by Itô's formula

$$d\log X_t = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t \longrightarrow$$

$$\log X_t = \log X_0 + \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t \longrightarrow$$

$$\mathbf{X_t = X_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right)}. \tag{20}$$

The return X_t/X_0 is *lognormal* and the process (X_t) is called a *geometric Brownian motion*. which can also be obtained as a diffusion limit of binomial tree models which arise when the Brownian motion is approximated by a random walk.

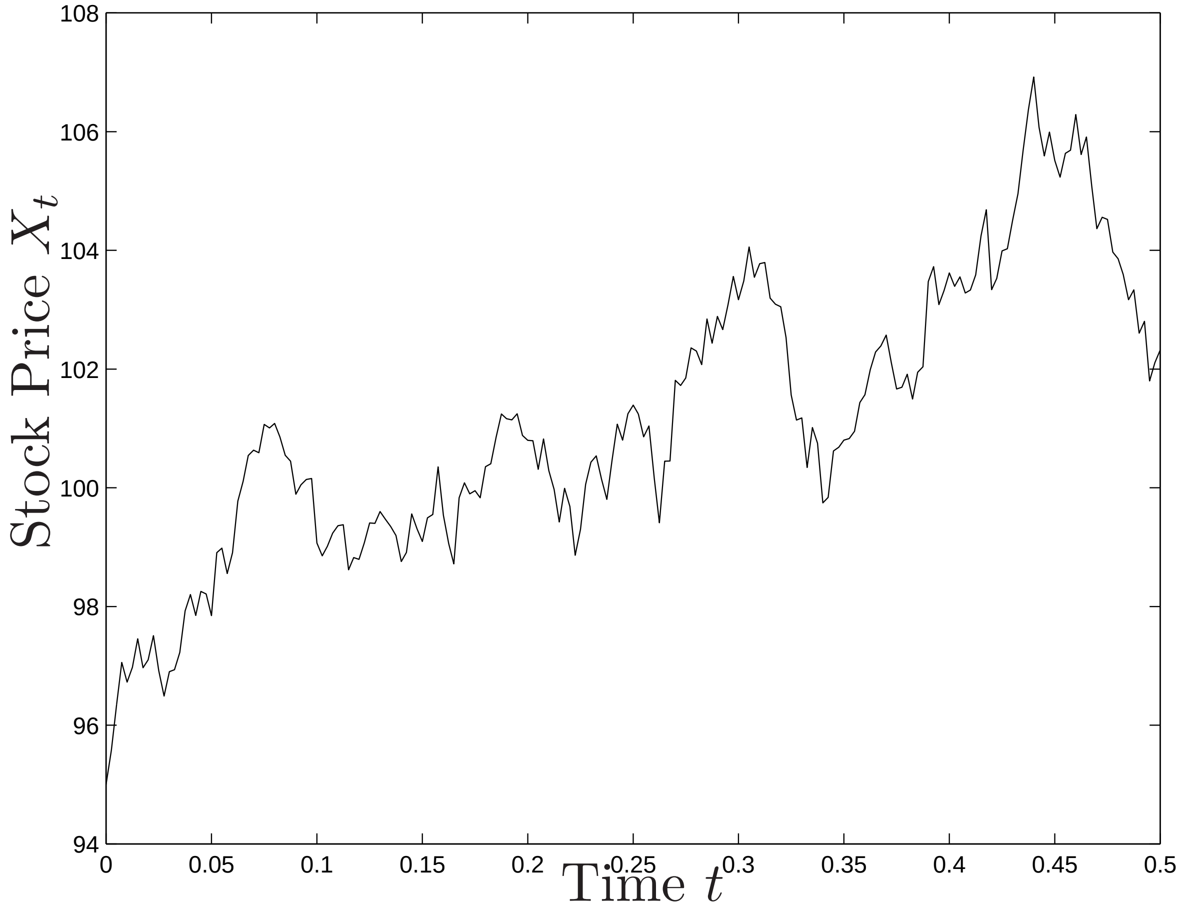

Figure 1: *A sample path of a geometric Brownian motion, with* $\mu = 0.15$, $\sigma = 0.1$ *and* $X_0 = 95$. *It exhibits the "average growth plus noise" behavior we expect from this model of asset prices.*

1.2 Derivative Contracts

Derivatives, also called *contingent claims*, are contracts based on the underlying asset X_t.

1.2.1 European Call and Put Options

A *European call option* is a contract that gives its holder the right, but not the obligation, to buy one unit of an underlying asset for a predetermined **strike price** K on the **maturity** date T. If X_T is the price of the underlying asset at maturity time T, then the value of this contract at maturity, its **payoff**, is

$$h(X_T) = (X_T - K)^+ = \begin{cases} X_T - K & \text{if } X_T > K \\ 0 & \text{if } X_T \leq K, \end{cases} \tag{21}$$

Similarly a *European put option* is a contract that gives its holder the right, but not the obligation, to sell a unit of the asset for a **strike price** K at the **maturity** date T. Its **payoff** is

$$h(X_T) = (K - X_T)^+ = \begin{cases} K - X_T & \text{if } X_T < K \\ 0 & \text{if } X_T \geq K\,, \end{cases} \tag{22}$$

More generally standard *path-independent* European derivatives are defined by their maturity time T and their payoff function $h(x)$.

At time $t < T$ this contract has a value, known as the **derivative price**, which will vary with t and the observed stock price X_t (by the *Markov property* which we will explain later). This option price at time t and for a stock price $X_t = x$ is denoted by $\mathbf{P(t, x)}$ and the problem of **derivative pricing** is to find this pricing function.

A naive approach

$$P(0,x) = \mathbb{E}\left\{e^{-rT}h(X_T)\right\} \tag{23}$$

$$= \mathbb{E}\left\{e^{-rT}h\left(xe^{(\mu-\frac{\sigma^2}{2})T+\sigma W_T}\right)\right\}, \tag{24}$$

The expectation reduces to a Gaussian integral since W_T is $\mathcal{N}(0,T)$-distributed.

In general (unless $\mu = r$) the option price given by formula (23) leads to an **arbitrage opportunity**, meaning that there will be a **risk-free** way to make a profit by managing a particular portfolio. This is one of the key ideas presented next that is used to determine the fair, or no-arbitrage, option price.

1.2.2 American Options

An *American option* is a contract in which the holder decides whether to exercise the option or not **at any time of his choice** before the option's expiration date T. The time τ at which the option is exercised is called the **exercise time**, it satisfies

$$\{\tau \leq t\} \in \mathcal{F}_t \quad \text{for any} \quad t \leq T$$

and is called a *stopping time* with respect to the filtration $(\mathcal{F}_t)$.

For an *American call option* the payoff is $h(X_\tau) = (X_\tau - K)^+$ for a given strike price K and a stopping time $\tau \leq T$ chosen by the holder of the option. Similarly, the payoff of an *American put option* is $h(X_\tau) = (K - X_\tau)^+$ and the option is exercised only if $K > X_\tau$.

Naive price:

$$P(0, x) = \sup_{\tau \leq T} I\!E\left\{e^{-r\tau} h(X_\tau)\right\}. \tag{25}$$

1.2.3 Other Exotic Options

The term *exotic option* refers here to any option contract which is not a standard European or American option described previously.

Barrier options are **path-dependent options** whose payoff depends on whether or not the underlying asset price hits a specified value during the option's lifetime. For instance a *down-and-out call option* becomes worthless, or *knocked out*, if, at any time t before the expiration date T, the stock price X_t falls below a predetermined level B. The payoff at expiration T is a function of the trajectory of the stock price

$$\mathbf{h}(\mathbf{X}) = (\mathbf{X_T} - \mathbf{K})^+ \mathbf{1}_{\{\inf_{\mathbf{t} \leq \mathbf{T}} \mathbf{X_t} > \mathbf{B}\}}. \tag{26}$$

This option is obviously less valuable than a standard European call option given by (21) with the same strike K and maturity T and it will lead to a *knock-out discount*.

Lookback options are **path-dependent options** whose payoff functions depend on the minimum or maximum price of the underlying asset during the lifetimes of the options. In particular, a *standard lookback call option* has a payoff at maturity given by

$$\mathbf{h}(\mathbf{X}) = \left(\mathbf{X_T} - \inf_{\mathbf{t}\le\mathbf{T}} \mathbf{X_t}\right)^+ = \mathbf{X_T} - \inf_{\mathbf{t}\le\mathbf{T}} \mathbf{X_t}, \tag{27}$$

where the lowest price plays the role of a *floating* strike price.

Similarly a *standard lookback put option* has a payoff given by

$$\mathbf{h}(\mathbf{X}) = \left(\sup_{\mathbf{t}\le\mathbf{T}} \mathbf{X_t} - \mathbf{X_T}\right)^+ = \sup_{\mathbf{t}\le\mathbf{T}} \mathbf{X_t} - \mathbf{X_T}, \tag{28}$$

where the highest price plays the role of the floating strike price. Note that these options are **not genuine option contracts** since they are almost always exercised, since $h > 0$ ($I\!\!P$ a.s.) as can be seen from (27) and (28).

Forward-start or **cliquet** options are like call options for instance where the strike price is set at a later time. If $\mathbf{t < T_1 < T}$, then the payoff at maturity T is given by

$$\mathbf{h(X) = (X_T - X_{T_1})^+}, \tag{29}$$

where the stock price at time T_1 becomes the strike price.

Compound options are *options on options*. For instance, a *call-on-call* is the right to buy a call option at a later time for a predetermined price.. If $\mathbf{t < T_1 < T}$, then the payoff at maturity T_1 is given by

$$\mathbf{h(X) = (C_{T_1}(K, T) - K_1)^+}, \tag{30}$$

where $C_{T_1}(K,T)$ is the price at time T_1 of a call option which pays $(X_T - K)^+$ at maturity time T.

Asian options: the payoff depends on the average stock price during a specified period of time before maturity. They can be European or American with typical payoffs like

$$\mathbf{h(X)} = \left(\mathbf{X_T} - \frac{\mathbf{1}}{\mathbf{T}} \int_{\mathbf{0}}^{\mathbf{T}} \mathbf{X_s ds} \right)^+ , \tag{31}$$

for an *arithmetic-average strike call option* (European style), where the strike price is the average stock price.

A *geometric-average strike call option* would be given by

$$\mathbf{h(X)} = \left(\mathbf{X_T} - \mathbf{e}^{\frac{\mathbf{1}}{\mathbf{T}} \int_{\mathbf{0}}^{\mathbf{T}} \log \mathbf{X_s ds}} \right)^+ .$$

1.3 Replicating Strategies

The Black-Scholes analysis of a European style derivative yields an *explicit trading strategy* in the underlying risky asset and riskless bond whose terminal payoff is equal to the payoff $h(X_T)$ of the derivative at maturity, no matter what path the stock price takes. This *replicating strategy* is a *dynamic hedging strategy* since it involves continuous trading, where *to hedge means to eliminate risk*. The essential step in the Black-Scholes methodology is the construction of this replicating strategy and arguing, based on *no-arbitrage*, that the value of the replicating portfolio at time t is the fair price of the derivative. We develop this idea now.

1.3.1 Replicating Self-Financing Portfolios

We consider a European style derivative with payoff $h(X_T)$. Assume that the stock price (X_t) follows the geometric Brownian motion model (20), solution of the SDE (2).

A *trading strategy* is a pair $(\mathbf{a_t}, \mathbf{b_t})$ of adapted processes specifying the number of units held at time t of the underlying asset and the riskless bond, respectively.

We suppose that $\mathbb{E}\left\{\int_0^T a_t^2 dt\right\}$ and $\int_0^T |b_t| dt$ are finite so that the stochastic integral involving (a_t) and the usual integral involving (b_t) are well-defined.

The value at time t of this portfolio is $\mathbf{a_t X_t + b_t e^{rt}}$. It will *replicate* the derivative at maturity if its value at time T is almost surely equal to the payoff:

$$\mathbf{a_T X_T + b_T e^{rT} = h(X_T)} \tag{32}$$

In addition, this portfolio is to be *self-financing*,

$$\mathbf{d}\left(\mathbf{a_t X_t} + \mathbf{b_t e^{rt}}\right) = \mathbf{a_t dX_t} + \mathbf{r b_t e^{rt} dt}, \tag{33}$$

which implies the **self-financing property**

$$\mathbf{X_t da_t} + \mathbf{e^{rt} db_t} + \mathbf{d\langle a, X\rangle_t} = \mathbf{0}. \tag{34}$$

In integral form:

$$\mathbf{a_t X_t} + \mathbf{b_t e^{rt}} = \mathbf{a_0 X_0} + \mathbf{b_0} + \int_0^{\mathbf{t}} \mathbf{a_s dX_s} + \int_0^{\mathbf{t}} \mathbf{r b_s e^{rs} ds}\,,\ \ \mathbf{0} \leq \mathbf{t} \leq \mathbf{T}.$$

In discrete time:

$$a_{t_n} X_{t_{n+1}} + b_{t_n} e^{r t_{n+1}} = a_{t_{n+1}} X_{t_{n+1}} + b_{t_{n+1}} e^{r t_{n+1}} \quad \longrightarrow$$

$$\begin{aligned}\left(a_{t_{n+1}} X_{t_{n+1}} + b_{t_{n+1}} e^{r t_{n+1}}\right) \quad - \quad & \left(a_{t_n} X_{t_n} + b_n e^{r t_n}\right) \\ = \quad & a_{t_n}\left(X_{t_{n+1}} - X_{t_n}\right) + b_{t_n}\left(e^{r t_{n+1}} - e^{r t_n}\right),\end{aligned}$$

which in continuous time becomes (33).

1.3.2 The Black-Scholes Partial Differential Equation

Assume that the price of a European-style contract with payoff $h(X_T)$ is given by $\mathbf{P(t, X_T)}$ where the *pricing function* $\mathbf{P(t, x)}$ is to be determined.

Cconstruct a **self-financing** portfolio (a_t, b_t) that will **replicate** the derivative at maturity (32).

The **no-arbitrage condition** requires that

$$\mathbf{a_t X_t + b_t e^{rt} = P(t, X_t)}\,, \quad \text{for any } 0 \leq t \leq T. \tag{35}$$

Differentiating (35) and using the self-financing property (33) on the left-hand side, **Itô's formula** (16) on the right-hand side and equation (2), we obtain

$$\begin{aligned} &\left(\mathbf{a_t} \mu \mathbf{X_t} + \mathbf{b_t r e^{rt}}\right) \mathbf{dt} + \mathbf{a_t} \sigma \mathbf{X_t dW_t} \\ &= \left(\frac{\partial \mathbf{P}}{\partial \mathbf{t}} + \mu \mathbf{X_t} \frac{\partial \mathbf{P}}{\partial \mathbf{x}} + \frac{\mathbf{1}}{\mathbf{2}} \sigma^{\mathbf{2}} \mathbf{X_t^2} \frac{\partial^{\mathbf{2}} \mathbf{P}}{\partial \mathbf{x^2}}\right) \mathbf{dt} + \sigma \mathbf{X_t} \frac{\partial \mathbf{P}}{\partial \mathbf{x}} \mathbf{dW_t} \end{aligned} \tag{36}$$

Eliminating risk (*or equating the* dW_t *terms*) gives

$$\mathbf{a_t} = \frac{\partial \mathbf{P}}{\partial \mathbf{x}}(\mathbf{t}, \mathbf{X_t}). \tag{37}$$

From (35) we get

$$\mathbf{b_t} = (\mathbf{P}(\mathbf{t}, \mathbf{X_t}) - \mathbf{a_t X_t})\, \mathbf{e^{-rt}}. \tag{38}$$

Equating the dt *terms* in (36) gives

$$\mathbf{r}\left(\mathbf{P} - \mathbf{X_t}\frac{\partial \mathbf{P}}{\partial \mathbf{x}}\right) = \frac{\partial \mathbf{P}}{\partial \mathbf{t}} + \frac{\mathbf{1}}{\mathbf{2}}\sigma^{\mathbf{2}}\mathbf{X_t^2}\frac{\partial^{\mathbf{2}} \mathbf{P}}{\partial \mathbf{x^2}}\,, \tag{39}$$

which needs to be satisfied for any stock price X_t.

Note that μ disappeared!

$P(t, x)$ **is the solution of the Black-Scholes PDE**

$$\mathcal{L}_{\mathbf{BS}}(\sigma)\mathbf{P} = \mathbf{0}, \tag{40}$$

where the *Black-Scholes operator* is defined by

$$\mathcal{L}_{\mathbf{BS}}(\sigma) = \frac{\partial}{\partial \mathbf{t}} + \frac{\mathbf{1}}{\mathbf{2}}\sigma^{\mathbf{2}}\mathbf{x}^{\mathbf{2}}\frac{\partial^{\mathbf{2}}}{\partial \mathbf{x}^{\mathbf{2}}} + \mathbf{r}\left(\mathbf{x}\frac{\partial}{\partial \mathbf{x}} - \cdot\right). \tag{41}$$

Equation (40) is to be solved *backward in time* with the *terminal condition* $P(T, x) = h(x)$, on the upper half-plane $x > 0$.

Knowing P**, the portfolio** (a_t, b_t) **is uniquely determined by (37) and (38).**

a_t **is the "Delta" of the portfolio.**

Only the **volatility** σ is needed.

1.3.3 Pricing to Hedge (*alternative derivation)*

If we sell N_t options and hold A_t units of the risky asset X_t, then the change in this self-financing portfolio should produce a return identical to a riskless asset:

$$\mathbf{A_t dX_t - N_t dP_t = r(A_t X_t - N_t P_t)dt} \quad \longrightarrow$$

$$\mathbf{A_t(\mu X_t dt + \sigma X_t dW_t) -}$$
$$\mathbf{N_t \left\{ \left(\frac{\partial P}{\partial t} + \mu X_t \frac{\partial P}{\partial x} + \frac{1}{2}\sigma^2 X_t^2 \frac{\partial^2 P}{\partial x^2} \right) dt - \sigma X_t \frac{\partial P}{\partial x} dW_t \right\}}$$
$$\mathbf{= r(A_t X_t - N_t P_t)dt.}$$

Eliminating the dW_t terms gives

$$\mathbf{A_t = N_t \frac{\partial P}{\partial x}(t, X_t),}$$

the terms involving μ cancel, $P(t,x)$ satisfies the Black-Scholes PDE (40), and the *hedge ratio* is given by A_t/N_t.

1.3.4 The Black-Scholes Formula

For **European call options** the Black-Scholes PDE (40) is solved with the final condition $h(x) = (x - K)^+$. There is a closed-form solution known as the *Black-Scholes formula*:

$$\mathbf{C_{BS}(t, x; K, T; \sigma) = xN(d_1) - Ke^{-r(T-t)}N(d_2)}, \tag{42}$$

$$d_1 = \frac{\log(x/K) + \left(r + \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}}, \tag{43}$$

$$d_2 = d_1 - \sigma\sqrt{T - t}, \tag{44}$$

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2} dy. \tag{45}$$

(By direct check or probabilistic derivation later)

The Delta hedging ratio a_t for a call is given by $\mathbf{\frac{\partial C_{BS}}{\partial x} = N(d_1)}$.

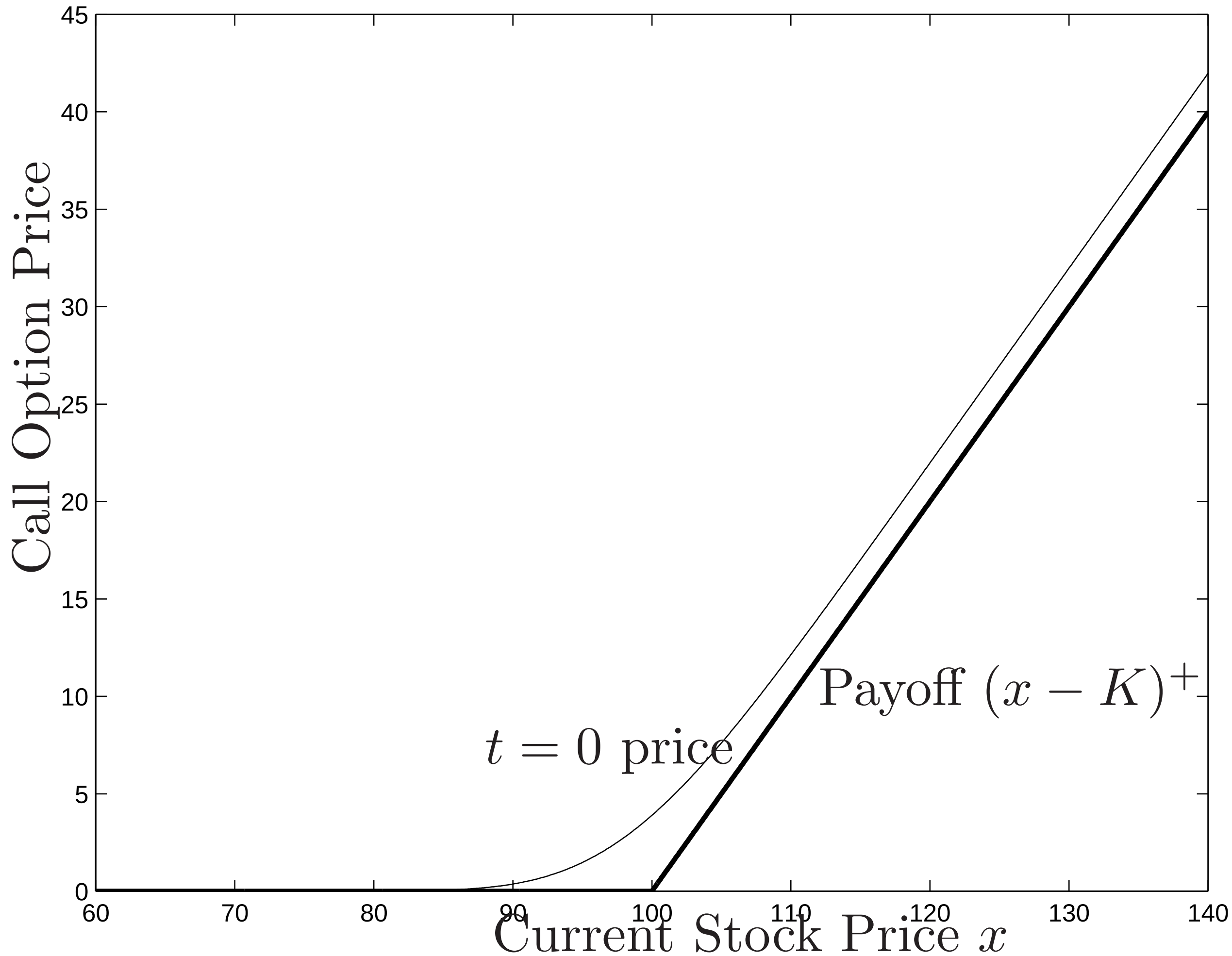

Figure 2: *Black-Scholes call option price* $C_{BS}(0, x; 100, 0.5; 10\%)$ *at time* $t = 0$*, with* $K = 100$*,* $T = 0.5$*,* $\sigma = 0.1$ *and* $r = 0.04$*.*

European put options

We have the *put-call parity* relation

$$\mathbf{C_{BS}(t, X_t) - P_{BS}(t, X_t) = X_t - Ke^{-r(T-t)}}\,, \tag{46}$$

between put and call options with the same maturity and strike price.

This is a *model-free* relationship that follows from simple *no-arbitrage* arguments. *If, for instance, the left side is smaller than the right side then buying a call and selling a put and one unit of the stock, and investing the difference in the bond, creates a profit no matter what the stock price does.*

Under the lognormal model, this relationship can be checked directly since the difference $C_{BS} - P_{BS}$ *satisfies the PDE (40) with the final condition* $h(x) = x - K$. *This problem has the unique simple solution* $x - Ke^{-r(T-t)}$.

Using the Black-Scholes formula (42) for C_{BS} and the put-call parity relation (46), we deduce the following explicit formula for the price of a **European put option**:

$$\mathbf{P_{BS}(t,x) = Ke^{-r(T-t)}N(-d_2) - xN(-d_1)}\,, \tag{47}$$

where d_1, d_2 and N are as in (43), (97) and (45) respectively.

Other types of options do not lead in general to such explicit formulas. Determining their prices requires *solving numerically* the Black-Scholes PDE (40) with *appropriate boundary conditions*. Nevertheless probabilistic representations can be obtained as explained in the following section. In particular American options lead to *free-boundary value problems* associated with equation (40).

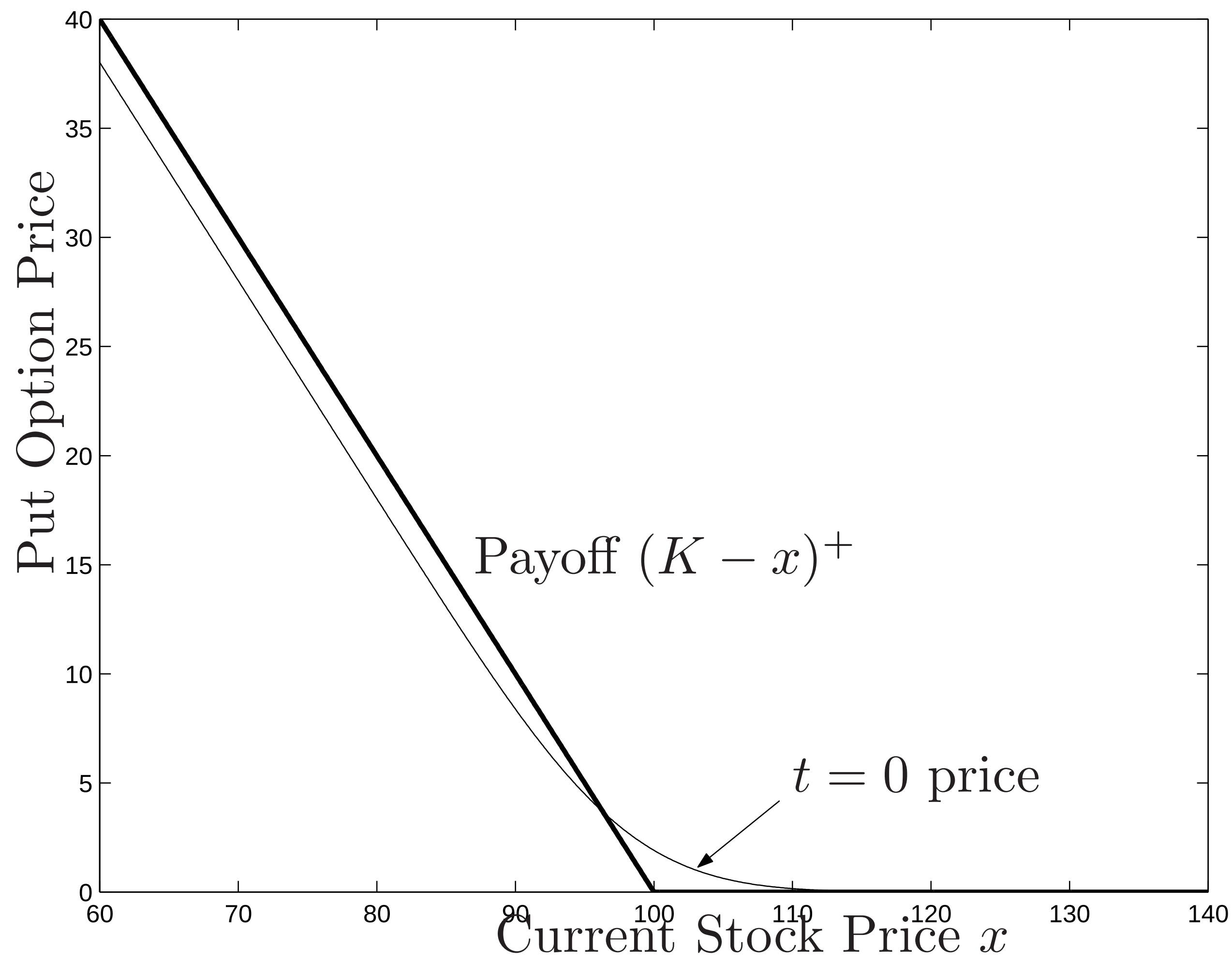

Figure 3: *Black-Scholes put option price* $P_{BS}(0, x; 100, 0.5; 10\%)$ *at time* $t = 0$, *with* $K = 100$, $T = 0.5$, $\sigma = 0.1$ *and* $r = 0.04$.

1.3.5 The Greeks:

$$\text{“Delta”} : \quad \mathbf{\Delta_{BS}} = \frac{\partial \mathbf{C_{BS}}}{\partial \mathbf{x}} = \mathbf{N(d_1)} \tag{48}$$

$$\text{“Gamma”} : \quad \mathbf{\Gamma_{BS}} = \frac{\partial^2 \mathbf{C_{BS}}}{\partial \mathbf{x}^2} = \frac{\partial \mathbf{\Delta_{BS}}}{\partial \mathbf{x}} = \frac{\mathbf{e}^{-\mathbf{d}_1^2/2}}{\mathbf{x}\sigma\sqrt{2\pi(\mathbf{T}-\mathbf{t})}} \tag{49}$$

$$\text{“Vega”} : \quad \mathcal{V}_{\mathbf{BS}} = \frac{\partial \mathbf{C_{BS}}}{\partial \sigma} = \frac{\mathbf{x}\mathbf{e}^{-\mathbf{d}_1^2/2}\sqrt{\mathbf{T}-\mathbf{t}}}{\sqrt{2\pi}} \tag{50}$$

The **sensitivities** with respect to *time to maturity* $T - t$ and *short rate* r are respectively named the **“Theta”** and the **“Rho”**.

In the **general case** of an European derivative whose price satisfies the Black-Scholes PDE (40) with a terminal condition $P(T, x) = h(x)$, there are simple and important relations between some of the Greeks $\longrightarrow$

For instance, differentiating with respect to σ leads to the following equation for the *Vega*:

$$\mathcal{L}_{\mathbf{BS}}(\sigma)\mathcal{V} + \sigma \mathbf{x^2} \frac{\partial^2 \mathbf{P}}{\partial \mathbf{x^2}} = \mathbf{0}, \tag{51}$$

with a zero terminal condition.

One can easily check that the Black-Scholes operator $\mathcal{L}_{BS}(\sigma)$ commutes with $x^2\partial^2/\partial x^2$, and therefore that $(T-t)\sigma x^2 \frac{\partial^2 P}{\partial x^2}$ satisfies equation (51). If the second derivative with respect to x remains bounded as $t \to T$, this solution satisfies the zero terminal condition, and we obtain the following relation between the **Vega** and the **Gamma**

$$\frac{\partial \mathbf{P}}{\partial \sigma} = (\mathbf{T} - \mathbf{t})\sigma \mathbf{x^2} \frac{\partial^2 \mathbf{P}}{\partial \mathbf{x^2}}. \tag{52}$$

In the case of a call option this relation can be directly obtained from (49) and (50).

Using the same argument, by differentiating the Black-Scholes equation **with respect to** r, one can obtain the relation between the **Rho** and the **Delta**:

$$\frac{\partial \mathbf{P}}{\partial \mathbf{r}} = (\mathbf{T} - \mathbf{t}) \left(\mathbf{x} \frac{\partial \mathbf{P}}{\partial \mathbf{x}} - \mathbf{P} \right). \tag{53}$$

Note that these relations may not be satisfied by more complex derivatives involving additional boundary conditions, such as barrier options for instance.

1.4 Risk-Neutral Pricing

Unless $\mu = r$, the expected value under the *objective* probability $I\!P$ of the discounted payoff of a derivative (23) would lead to an opportunity for arbitrage. This is closely related to the fact that the discounted price $\widetilde{\mathbf{X}_\mathbf{t}} = \mathbf{e}^{-\mathbf{rt}}\mathbf{X}_\mathbf{t}$ **is not a martingale** since, from (18),

$$\mathbf{d}\widetilde{\mathbf{X}_\mathbf{t}} = (\mu - \mathbf{r})\widetilde{\mathbf{X}_\mathbf{t}}\mathbf{dt} + \sigma\widetilde{\mathbf{X}_\mathbf{t}}\mathbf{dW}_\mathbf{t}\,, \tag{54}$$

which contains a non zero *drift* term if $\mu \neq r$.

The main result we want to build in this section is that there is a **unique probability measure** $I\!P^\star$ **equivalent to** $I\!P$ such that, under this probability, (i) the discounted price $\widetilde{X_t}$ **is a martingale** and (ii) the expected value under $I\!P^\star$ of the discounted payoff of a derivative gives its **no-arbitrage price**. Such a probability measure describing a **risk-neutral** world is called an *Equivalent Martingale Measure*.

1.4.1 Equivalent Martingale Measure

In order to find a probability measure under which the discounted price $\widetilde{X}_t$ is a martingale, we rewrite (54) in such a way that the drift term is "absorbed" in the martingale term:

$$\mathbf{d}\widetilde{\mathbf{X}_\mathbf{t}} = \sigma\widetilde{\mathbf{X}_\mathbf{t}}\left[\mathbf{dW_t} + \left(\frac{\mu - \mathbf{r}}{\sigma}\right)\mathbf{dt}\right].$$

$$\theta = \frac{\mu - \mathbf{r}}{\sigma} \tag{55}$$

is called the *market price of asset risk*, and we define

$$\mathbf{W}^\star_\mathbf{t} = \mathbf{W_t} + \int_\mathbf{0}^\mathbf{t} \theta\mathbf{ds} = \mathbf{W_t} + \theta\mathbf{t}, \tag{56}$$

so that

$$\mathbf{d}\widetilde{\mathbf{X}_\mathbf{t}} = \sigma\widetilde{\mathbf{X}_\mathbf{t}}\mathbf{dW}^\star_\mathbf{t}. \tag{57}$$

Using the characterization (3), it is easy to check that

$$\xi_{\mathbf{T}}^{\theta} = \exp\left(-\theta \mathbf{W_T} - \frac{\mathbf{1}}{\mathbf{2}}\theta^{\mathbf{2}}\mathbf{T}\right), \tag{58}$$

has an $I\!\!P$-expected value equal to 1 (Cameron-Martin formula).

It has a conditional expectation with respect to $\mathcal{F}_t$ given by

$$\mathbf{E}\{\xi_{\mathbf{T}}^{\theta} \mid \mathcal{F}_{\mathbf{t}}\} = \exp\left(-\theta \mathbf{W_t} - \frac{\mathbf{1}}{\mathbf{2}}\theta^{\mathbf{2}}\mathbf{t}\right) = \xi_{\mathbf{t}}^{\theta}, \text{ for } 0 \leq t \leq T,$$

which defines a **martingale** denoted by $(\xi_t^{\theta})_{0\leq t\leq T}$.

$I\!\!P^{\star}$ is the equivalent measure to $I\!\!P$ (they have the same null sets), which has the **density** ξ_T^{θ} with respect to $I\!\!P$:

$$\mathbf{dP}^{\star} = \xi_{\mathbf{T}}^{\theta}\mathbf{dP}, \tag{59}$$

or denoting by $I\!\!E^{\star}\{\cdot\}$ the expectation with respect to $I\!\!P^{\star}$, for any integrable random variable Z we have

$$\mathbf{E}^{\star}\{\mathbf{Z}\} = \mathbf{E}\{\xi_{\mathbf{T}}^{\theta}\mathbf{Z}\}.$$

For any adapted and integrable process (Z_t),

$$\mathbf{E}^\star\{\mathbf{Z_t} \mid \mathcal{F}_\mathbf{s}\} = \frac{\mathbf{1}}{\xi^\theta_\mathbf{s}}\mathbf{E}\{\xi^\theta_\mathbf{t}\mathbf{Z_t} \mid \mathcal{F}_\mathbf{s}\}, \tag{60}$$

for any $0 \le s \le t \le T$. The process $(\xi^\theta_t)_{0\le t\le T}$ is called the **Radon-Nikodym density** .

The main result of this section asserts that the process $(W^\star_t)$ given by (56) is a standard Brownian motion under the probability $\mathbb{P}^\star$.

This result in its full generality (when θ is an adapted stochastic process) is known as **Girsanov's Theorem**.

In our simple case (θ **constant**), it is easily derived by using the characterization (3) and formula (60) as follows:

$$
\begin{aligned}
\mathbb{E}^{\star}\left\{e^{iu(W_t^{\star}-W_s^{\star})} \mid \mathcal{F}_s\right\} &= \frac{1}{\xi_s^{\theta}}\mathbb{E}\left\{\xi_t^{\theta} e^{iu(W_t^{\star}-W_s^{\star})} \mid \mathcal{F}_s\right\} \\
&= e^{\theta W_s+\frac{1}{2}\theta^2 s}\mathbb{E}\left\{e^{-\theta W_t-\frac{1}{2}\theta^2 t} e^{iu(W_t-W_s+\theta(t-s))} \mid \mathcal{F}_s\right\} \\
&= e^{\left(-\frac{1}{2}\theta^2+iu\theta\right)(t-s)}\mathbb{E}\left\{e^{i(u+i\theta)(W_t-W_s)} \mid \mathcal{F}_s\right\} \\
&= e^{\left(-\frac{1}{2}\theta^2+iu\theta\right)(t-s)} e^{-\frac{(u+i\theta)^2(t-s)}{2}} \\
&= e^{-\frac{u^2(t-s)}{2}} .
\end{aligned}
$$

1.4.2 Self-Financing Portfolios

$$\mathbf{V_t = a_t X_t + b_t e^{rt}.}$$

The self-financing property (33), namely $dV_t = a_t dX_t + rb_t e^{rt} dt$, implies that the discounted value of the portfolio, $\widetilde{\mathbf{V_t}} = \mathbf{e^{-rt} V_t}$, is a **martingale** under the risk-neutral probability $I\!\!P^\star$. This **essential property** of self-financing portfolios is obtained as follows:

$$\begin{aligned}
d\widetilde{V_t} &= -re^{-rt} V_t dt + e^{-rt} dV_t \\
&= -re^{-rt}(a_t X_t + b_t e^{rt}) dt + e^{-rt}(a_t dX_t + rb_t e^{rt} dt) \\
&= -re^{-rt} a_t X_t dt + e^{-rt} a_t dX_t \\
&= a_t d(e^{-rt} X_t) \\
&= a_t d\widetilde{X_t} \\
&= \sigma a_t \widetilde{X_t} dW_t^\star \qquad \text{(by (57))}, \qquad (61)
\end{aligned}$$

Connection between martingales and *no-arbitrage*

Suppose that $(a_t, b_t)_{0\le t\le T}$ is a **self-financing arbitrage strategy** such that

$$\mathbf{V_T \geq e^{rT} V_0} \ \ (\mathbb{P}\text{-a.s.}), \tag{62}$$

$$\mathbb{P}\{\mathbf{V_T > e^{rT} V_0}\} > \mathbf{0}, \tag{63}$$

so that the strategy never makes less than money in the bank and there is some chance of making more. But

$$\mathbb{E}^{\star}\{\mathbf{e^{-rT} V_T}\} = \mathbf{V_0}$$

by the martingale property, so (62) and (63) **cannot hold**.

This is because $I\!\!P$ and $I\!\!P^{\star}$ are equivalent and so (62) and (63) also hold with $I\!\!P$ replaced by $I\!\!P^{\star}$.

1.4.3 Risk-Neutral Valuation

Let (a_t, b_t) be a self-financing portfolio replicating the European style derivative with nonnegative square integrable payoff H:

$$\mathbf{a_T X_T + b_T e^{rT} = H}. \tag{64}$$

This includes European calls and puts or more general standard European derivatives for which $H = h(X_T)$, as well as other European style exotic derivatives presented in Section 1.2.3.

On one hand, a **no-arbitrage argument** shows that the price at time t of this derivative should be the value V_t of this portfolio.

On the other hand the discounted values $(\widetilde{V_t})$ of this portfolio form a **martingale under the risk-neutral probability** $\mathbb{P}^\star$:

$$\widetilde{\mathbf{V_t}} = \mathbf{E}^\star \left\{ \widetilde{\mathbf{V_T}} \mid \mathcal{F}_\mathbf{t} \right\} \quad \longrightarrow$$

$$\mathbf{V_t} = \mathbb{E}^\star \left\{ \mathbf{e^{-r(T-t)} H} \mid \mathcal{F}_\mathbf{t} \right\}, \tag{65}$$

after reintroducing the discounting factor and using the replicating property (64).

Alternatively, given the **risk-neutral valuation formula (65)**, we can find a self-financing replicating portfolio for the payoff H. The existence of such a portfolio is guaranteed by an application of the **martingale representation theorem**: for $0 \leq t \leq T$

$$\mathbf{M_t} = \mathbb{E}^\star \left\{ \mathbf{e^{-rT} H} \mid \mathcal{F}_\mathbf{t} \right\},$$

defines a square integrable martingale under $\mathbb{P}^\star$ with respect to the filtration $(\mathcal{F}_t)$, which is also the natural filtration of the Brownian motion $W^\star$.

The **representation theorem** says that any such martingale is a stochastic integral with respect to $W^\star$, so that

$$\mathbf{E}^\star \left\{ \mathbf{e}^{-\mathbf{rT}}\mathbf{H} \mid \mathcal{F}_\mathbf{t} \right\} = \mathbf{M_0} + \int_\mathbf{0}^\mathbf{t} \eta_\mathbf{s} \mathbf{dW}^\star_\mathbf{s},$$

where (η_t) is some adapted process with $I\!E^\star \left\{ \int_0^T \eta_t^2 dt \right\}$ finite.

By defining $\mathbf{a_t} = \eta_\mathbf{t}/(\sigma\widetilde{\mathbf{X_t}})$ and $\mathbf{b_t} = \mathbf{M_t} - \mathbf{a_t}\widetilde{\mathbf{X_t}}$, we construct a portfolio $(\mathbf{a_t}, \mathbf{b_t})$, which is shown to be **self-financing** by checking that its discounted value is the martingale M_t and using the characterization (61) obtained in Section 1.4.2.

Its value at time T is $e^{rT} M_T = H$ and therefore it is a **replicating** portfolio.

1.4.4 Using the Markov Property

For a standard European derivative with payoff $H = h(X_T)$ the **Markov property** of (X_t) says that **conditioning with respect to the past $\mathcal{F}_t$ is the same as conditioning with respect to** X_t, so that the *risk-neutral pricing formula* becomes

$$\mathbf{V_t = E^\star \left\{ e^{-r(T-t)} h(X_T) \mid X_t \right\}}.$$

We will come back to this property in the next Section.

Denoting by $P(t,x)$ the price of this derivative at time t for an observed stock price $X_t = x$, we obtain the **pricing formula**

$$\mathbf{P(t,x) = E^\star \left\{ e^{-r(T-t)} h(X_T) \mid X_t = x \right\}}. \tag{66}$$

If we compare this formula (at time $t = 0$) with (23), the naive pricing a standard European derivative, we see that the essential step is to replace the "objective world" $\mathbb{P}$ by the "risk-neutral world" $\mathbb{P}^\star$ in order to obtain the fair no-arbitrage price.

Solving the SDE (2) from t to T starting from x gives

$$\mathbf{X_T} = \mathbf{x} \exp\left((\mu - \frac{\sigma^{\mathbf{2}}}{\mathbf{2}})(\mathbf{T} - \mathbf{t}) + \sigma(\mathbf{W_T} - \mathbf{W_t}) \right). \tag{67}$$

Using (56), this formula can be rewritten in terms of $(W_t^\star)$ as

$$\mathbf{X_T} = \mathbf{x} \exp\left((\mathbf{r} - \frac{\sigma^{\mathbf{2}}}{\mathbf{2}})(\mathbf{T} - \mathbf{t}) + \sigma(\mathbf{W_T^\star} - \mathbf{W_t^\star}) \right).$$

As $(W_t^\star)$ is a standard Brownian motion under the risk-neutral probability $\mathbb{P}^\star$, the increment $W_T^\star - W_t^\star$ is $\mathcal{N}(0, T-t)$-distributed, and (66) gives the **Gaussian integral**

$$P(t,x) = \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{+\infty} e^{-r(T-t)} h\left(x e^{(r-\frac{\sigma^2}{2})(T-t)+\sigma z} \right) e^{-\frac{z^2}{2(T-t)}} dz. \tag{68}$$

In the case of a **European call option**, $h(x) = (x - K)^+$, this integral reduces to the **Black-Scholes formula** (42) obtained in Section 1.3.4, as the following computation shows:

$$\mathbf{P(t,x)} = \frac{\mathbf{x}}{\sqrt{\mathbf{2}\pi\tau}} \int_{\mathbf{z}^\star}^{+\infty} \mathbf{e}^{-\frac{(\mathbf{z}-\sigma\tau)^{\mathbf{2}}}{\mathbf{2}\tau}} \mathbf{dz} - \frac{\mathbf{K}\mathbf{e}^{-\mathbf{r}\tau}}{\sqrt{\mathbf{2}\pi\tau}} \int_{\mathbf{z}^\star}^{+\infty} \mathbf{e}^{-\frac{\mathbf{z}^{\mathbf{2}}}{\mathbf{2}\tau}} \mathbf{dz},$$

where $\tau = T - t$ and $z^\star$ is defined by

$$\mathbf{x} \exp\left((\mathbf{r} - \frac{\mathbf{1}}{\mathbf{2}}\sigma^{\mathbf{2}})\tau + \sigma\mathbf{z}^\star \right) = \mathbf{K}.$$

We then set

$$\frac{z^\star - \sigma\tau}{\sqrt{\tau}} = -d_1, \quad \frac{z^\star}{\sqrt{\tau}} = -d_2,$$

which coincide with the definitions (43) and (97) of d_1 and d_2. The Black-Scholes formula (42) follows by introducing the normal cumulative distribution function N given by (45).

Binary or digital options

It pays at time T a fixed amount (say one), if $X_T \geq K$, and nothing otherwise. The corresponding discontinuous payoff function is simply $\mathbf{h(x) = 1_{\{x \geq K\}}}$. Its value at time t is given by (66), which, in this case, becomes

$$\mathbf{P_{digital}(t, x) = \frac{e^{-r\tau}}{\sqrt{2\pi\tau}} \int_{z^\star}^{+\infty} e^{-\frac{z^2}{2\tau}} dz = e^{-r\tau} N(d_2).} \qquad (69)$$

The two approaches developed in Sections 1.3 (PDE) and 1.4 (risk-neutral valuation) should give the same fair price to the same derivative. This is indeed the case, and is the content of the following section, where we explain that a formula like (66) is just a probabilistic representation of the solution of a partial differential equation like (40).

1.5 Risk-Neutral Expectations and PDEs

We denote by $(\mathbf{X_s^{t,x}})_{\mathbf{s \geq t}}$ the solution of the SDE (11) starting from x at time t:

$$\mathbf{X_s^{t,x} = x + \int_t^s \mu(u, X_u^{t,x}) du + \int_t^s \sigma(u, X_u^{t,x}) dW_u},$$

and we assume enough regularity in the coefficients μ and σ for $(X_s^{t,x})$ to be jointly continuous in the three variables (t, x, s). The *flow property* for deterministic differential equations can be extended to stochastic differential equations like (11); it says that, in order to compute the solution at time $s > t$ starting at time 0 from point x, one can use

$$\mathbf{x \longrightarrow X_t^{0,x} \longrightarrow X_s^{t, X_t^{0,x}} = X_s^{0,x}} \qquad (\mathbf{P}\text{-a.s.}). \tag{70}$$

In other words, one can solve the equation from 0 *to* t*, starting from* x*, to obtain* $X_t^{0,x}$*. Then we solve the equation from* t *to* s*, starting from* $X_t^{0,x}$*. This is the same as solving the equation from* 0 *to* s*, starting from* x*.*

The Markov property is a consequence and can be stated as follows:

$$\mathbf{E}\left\{\mathbf{h}(\mathbf{X_s}) \mid \mathcal{F}_\mathbf{t}\right\} = \mathbf{E}\left\{\mathbf{h}(\mathbf{X_s^{t,x}})\right\}|_{\mathbf{x}=\mathbf{X_t}}, \tag{71}$$

which is what we have used with $s = T$ to derive (66).

Observe that the discounting factor could be pulled out of the conditional expection since the interest rate is constant (not random).

In the *time homogeneous case* (μ and σ independent of time) we further have

$$\mathbf{E}\left\{\mathbf{h}(\mathbf{X_s^{t,x}})\right\} = \mathbf{E}\left\{\mathbf{h}(\mathbf{X_{s-t}^{0,x}})\right\},$$

which could have been used with $s = T$ to derive (68) since $W^\star_{T-t}$ is $\mathcal{N}(0, T-t)$-distributed.

1.5.1 Infinitesimal Generators and Associated Martingales

Consider first a time homogeneous diffusion process (X_t), solution of the SDE

$$\mathbf{dX_t} = \mu(\mathbf{X_t})\mathbf{dt} + \sigma(\mathbf{X_t})\mathbf{dW_t}. \tag{72}$$

Let g be a twice continuously differentiable function of the variable x with bounded derivatives, and define the **differential operator** $\mathcal{L}$ acting on g according to

$$\mathcal{L}\mathbf{g}(\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{2}}\sigma^{\mathbf{2}}(\mathbf{x})\mathbf{g}''(\mathbf{x}) + \mu(\mathbf{x})\mathbf{g}'(\mathbf{x}). \tag{73}$$

In terms of $\mathcal{L}$, **Itô's formula** (16) gives

$$\mathbf{dg}(\mathbf{X_t}) = \mathcal{L}\mathbf{g}(\mathbf{X_t})\mathbf{dt} + \mathbf{g}'(\mathbf{X_t})\sigma(\mathbf{X_t})\mathbf{dW_t} \quad \longrightarrow$$

$$\mathbf{M_t} = \mathbf{g}(\mathbf{X_t}) - \int_{\mathbf{0}}^{\mathbf{t}} \mathcal{L}\mathbf{g}(\mathbf{X_s})\mathbf{ds}, \tag{74}$$

defines a **martingale**.

Consequently, if $X_0 = x$, we obtain

$$\mathbf{E}\{\mathbf{g}(\mathbf{X_t})\} = \mathbf{g}(\mathbf{x}) + \mathbf{E}\left\{\int_0^{\mathbf{t}} \mathcal{L}\mathbf{g}(\mathbf{X_s})\mathbf{ds}\right\}.$$

Under the assumptions made on the coefficients μ and σ and on the function g, the *Lebesgue dominated convergence theorem* is applicable and gives

$$\begin{aligned}
\frac{\mathbf{d}}{\mathbf{dt}}\mathbf{E}\{\mathbf{g}(\mathbf{X_t})\}|_{\mathbf{t=0}} &= \lim_{t\downarrow 0} \frac{\mathbb{E}\{g(X_t)\} - g(x)}{t} \\
&= \lim_{t\downarrow 0} \mathbb{E}\left\{\frac{1}{t}\int_0^t \mathcal{L}g(X_s)ds\right\} = \mathcal{L}\mathbf{g}(\mathbf{x}).
\end{aligned}$$

The differential operator $\mathcal{L}$ given by (73) is called the **infinitesimal generator** of the Markov process (X_t).

For nonhomogeneous diffusions $(\sigma(t,x),\mu(t,x))$ and functions $g(t,x)$ which depend also on time, (74) can be generalized by using the full Itô formula (16) to yield the **martingale**

$$\mathbf{M_t = g(t, X_t) - \int_0^t \left(\frac{\partial g}{\partial t} + \mathcal{L}_s g\right)(s, X_s) ds,} \tag{75}$$

where the **infinitesimal generator** $\mathcal{L}_t$ is defined by

$$\mathcal{L}_\mathbf{t} = \frac{\mathbf{1}}{\mathbf{2}}\sigma^\mathbf{2}(\mathbf{t},\mathbf{x})\frac{\partial^\mathbf{2}}{\partial \mathbf{x}^\mathbf{2}} + \mu(\mathbf{t},\mathbf{x})\frac{\partial}{\partial \mathbf{x}}, \tag{76}$$

and g is any smooth and bounded function.

Finally we incorporate a **discounting factor** by computing the differential of $\mathbf{e^{-\int_0^t r(s,X_s)ds} g(t, X_t)}$ and obtaining the martingales

$$\mathbf{M_t = e^{-\int_0^t r(s,X_s)ds} g(t, X_t) - \int_0^t e^{-\int_0^s r(u,X_u)du}\left(\frac{\partial g}{\partial t} + \mathcal{L}_s g - rg\right) ds,} \tag{77}$$

which introduces the *potential term* $-\mathbf{rg}$.

1.5.2 Conditional Expectations and Parabolic PDEs

Suppose that $u(t,x)$ is a solution of the PDE

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} + \frac{\mathbf{1}}{\mathbf{2}}\sigma^{\mathbf{2}}(\mathbf{t},\mathbf{x})\frac{\partial^{\mathbf{2}}\mathbf{u}}{\partial \mathbf{x}^{\mathbf{2}}} + \mu(\mathbf{t},\mathbf{x})\frac{\partial \mathbf{u}}{\partial \mathbf{x}} - \mathbf{r}\mathbf{u} = \mathbf{0}, \tag{78}$$

with the final condition $\mathbf{u}(\mathbf{T},\mathbf{x}) = \mathbf{h}(\mathbf{x})$ and assume that it is regular enough to apply Itô's formula (16). Using (77) we deduce that $\mathbf{M_t} = \mathbf{e}^{-\mathbf{rt}}\mathbf{u}(\mathbf{t},\mathbf{X_t})$ is a **martingale** when $\mathcal{L}_t$, given by (76), is the infinitesimal generator of the process (X_t) - in other words, when μ and σ are the drift and diffusion coefficients of (X_t).

The martingale property for times t and T reads $\mathbb{E}\{\mathbf{M_T} \mid \mathcal{F}_\mathbf{t}\} = \mathbf{M_t}$ which can be rewritten as

$$\mathbf{u}(\mathbf{t},\mathbf{X_t}) = \mathbb{E}\left\{\mathbf{e}^{-\mathbf{r}(\mathbf{T}-\mathbf{t})}\mathbf{h}(\mathbf{X_T}) \mid \mathcal{F}_\mathbf{t}\right\},$$

since $\mathbf{u}(\mathbf{T},\mathbf{X_T}) = \mathbf{h}(\mathbf{X_T})$ according to the final condition.

Using the **Markov property (71)**, we deduce the following **probabilistic representation** of the solution u:

$$\mathbf{u}(\mathbf{t},\mathbf{x}) = \mathbf{E}\left\{\mathbf{e}^{-\mathbf{r}(\mathbf{T}-\mathbf{t})}\mathbf{h}(\mathbf{X}_{\mathbf{T}}^{\mathbf{t},\mathbf{x}})\right\}, \tag{79}$$

which may also be written as

$$\mathbf{u}(\mathbf{t},\mathbf{x}) = \mathbf{E}\left\{\mathbf{e}^{-\mathbf{r}(\mathbf{T}-\mathbf{t})}\mathbf{h}(\mathbf{X}_{\mathbf{T}}) \mid \mathbf{X}_{\mathbf{t}} = \mathbf{x}\right\} \text{ or } \mathbf{u}(\mathbf{t},\mathbf{x}) = \mathbf{E}_{\mathbf{t},\mathbf{x}}\left\{\mathbf{e}^{-\mathbf{r}(\mathbf{T}-\mathbf{t})}\mathbf{h}(\mathbf{X}_{\mathbf{T}})\right\}$$

If r depends on t and x, the discounting factor becomes $\exp\left(-\int_t^T r(s, X_s)ds\right)$.

The representation (79) is then called the *Feynman-Kac formula*.

1.5.3 Application to the Black-Scholes PDE

When $\mu(t,x) = rx$ and $\sigma(t,x) = \sigma x$ in the SDE (78), we have the **Black-Scholes PDE** (40) for the option price $P(t,x)$ on the domain $\{x > 0\}$, since

$$\mathcal{L}_{\mathbf{BS}} = \frac{\partial}{\partial \mathbf{t}} + \mathcal{L} - \mathbf{r},$$

where $\mathcal{L}$ is the infinitesimal generator of the geometric Brownian motion X. The **non-ellipticity**

$$\sigma^{\mathbf{2}}(\mathbf{t},\mathbf{x}) = \sigma^{\mathbf{2}}\mathbf{x}^{\mathbf{2}} \tag{80}$$

can be dealt with by the change of variable $P(t,x) = u(t, y = \log x)$ where equation (40) becomes

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} + \frac{\mathbf{1}}{\mathbf{2}}\sigma^{\mathbf{2}}\frac{\partial^{\mathbf{2}}\mathbf{u}}{\partial \mathbf{y}^{\mathbf{2}}} + \left(\mathbf{r} - \frac{\mathbf{1}}{\mathbf{2}}\sigma^{\mathbf{2}}\right)\frac{\partial \mathbf{u}}{\partial \mathbf{y}} - \mathbf{ru} = \mathbf{0}, \tag{81}$$

to be solved for $0 \le t \le T$, $y \in \mathbb{R}$ with $u(T,y) = h(e^y)$.

1.6 American Options and Free-boundary Problems

1.6.1 Optimal Stopping

$$\mathbf{P}(\mathbf{0},\mathbf{x}) = \sup_{\tau \le \mathbf{T}} \mathbf{E}^{\star}\left\{\mathbf{e}^{-\mathbf{r}\tau}\mathbf{h}(\mathbf{X}_{\tau})\right\},$$

is the price of the derivative at time $t = 0$, when $X_0 = x$ and where the supremum is taken over all the possible **stopping times** less that the expiration date T. This formula can be generalized to get the price of American derivatives at any time t before expiration T:

$$\mathbf{P}(\mathbf{t},\mathbf{x}) = \sup_{\mathbf{t} \le \tau \le \mathbf{T}} \mathbf{E}^{\star}\left\{\mathbf{e}^{-\mathbf{r}(\tau-\mathbf{t})}\mathbf{h}\left(\mathbf{X}_{\tau}^{\mathbf{t},\mathbf{x}}\right)\right\}, \tag{82}$$

where $(X_s^{t,x})_{s \ge t}$ is the stock price starting at time t from the observed price x.

$\tau = t \quad \longrightarrow \quad P(t,x) \ge h(x)$

$t = T \quad \longrightarrow \quad P(T,x) = h(x)$

Because an American derivative gives its holder more rights than the corresponding European derivative, the price of the American is always greater than or equal to the price of the European derivative which has the same payoff function and the same expiration date.

This can be seen by choosing $\tau = \mathbf{T}$ in (82).

The supremum in (82) is reached at the **optimal stopping time**,

$$\tau^\star = \tau^\star(\mathbf{t}) = \inf_{\mathbf{s}} \{\mathbf{t} \le \mathbf{s} \le \mathbf{T}\,,\, \mathbf{P}(\mathbf{s}, \mathbf{X_s}) = \mathbf{h}(\mathbf{X_s})\}\,, \tag{83}$$

the first time after t that the price of the derivative drops down to its payoff. In order to determine $\tau^\star$, one must first compute the price. In terms of PDEs, this leads to a so-called *free-boundary value problem*. To illustrate, we consider the case of an **American put option** defined in Section 1.2.2.

It can be shown by a no-arbitrage argument that, for nonnegative interest rates and no dividend paid, the price of an American call option is the same as its corresponding European option.

The price of an American put option

$$\mathbf{P^a(t,x) = \sup_{t \le \tau \le T} E^\star \left\{ e^{-r(\tau - t)} \left(K - X_\tau^{t,x} \right)^+ \right\}},$$

is in general **strictly higher** than the price of the corresponding European put option which has been obtained in closed-form (47). In fact, we saw in Figure 3 that the Black-Scholes European put option pricing function crosses below the payoff "ramp" function $(K - x)^+$ for small enough x. This violates $P(t,x) \ge h(x)$, so the European formula for a put cannot also give the price of the American contract, as is the case for call options.

1.6.2 Free-Boundary Value Problems

Pricing functions for American derivatives satisfy partial differential *inequalities*. For the nonnegative payoff function h, the price of the corresponding American derivative is the solution of the following **linear complementarity problem**:

$$
\begin{aligned}
\mathbf{P} &\geq \mathbf{h} \\
\mathcal{L}_{\mathbf{BS}}(\sigma)\mathbf{P} &\leq \mathbf{0} \qquad (84) \\
(\mathbf{h}-\mathbf{P})\mathcal{L}_{\mathbf{BS}}(\sigma)\mathbf{P} &= \mathbf{0}, \qquad (85)
\end{aligned}
$$

to be solved in $\{(\mathbf{t}, \mathbf{x}) : \mathbf{0} \leq \mathbf{t} \leq \mathbf{T}, \mathbf{x} > \mathbf{0}\}$ with the final condition $\mathbf{P}(\mathbf{T}, \mathbf{x}) = \mathbf{h}(\mathbf{x})$.

The second inequality is linked to the **supermartingale property of** $e^{-rt}P(t, X_t)$ through (77) applied to $\mathbf{g} = \mathbf{P}$.

To see that the price (82) is solution of the **differential inequalities**

$\longrightarrow$

For any stopping time $t \leq \tau \leq T$ we have

$$e^{-r\tau}P(\tau, X_\tau^{t,x}) = e^{-rt}P(t,x) + \int_t^\tau e^{-rs}\left(\frac{\partial}{\partial t} + \mathcal{L} - r\right)P(s, X_s^{t,x})ds + \int_t^\tau e^{-rs}\sigma X_s^{t,x}\frac{\partial P}{\partial t}(s, X_s^{t,x})dW_s^\star.$$

The integrand of the Riemann integral is nonpositive by (85) and, since τ is bounded, the expectation of the martingale term is zero by *Doob's optional stopping theorem*. This leads to

$$\mathbf{E}^\star\left\{\mathbf{e}^{-\mathbf{r}(\tau-\mathbf{t})}\mathbf{P}(\tau, \mathbf{X}_\tau^{\mathbf{t},\mathbf{x}})\right\} \leq \mathbf{P}(\mathbf{t}, \mathbf{x}),$$

and, using the first inequality in (85),

$$\mathbf{E}^\star\left\{\mathbf{e}^{-\mathbf{r}(\tau-\mathbf{t})}\mathbf{h}(\mathbf{X}_\tau^{\mathbf{t},\mathbf{x}})\right\} \leq \mathbf{P}(\mathbf{t}, \mathbf{x}).$$

It is easy to see now that if $\tau = \tau^\star$, then we have equalities throughout. This verifies that if (85) has a solution to which Itô's formula can be applied then it is the American derivative price (82).

In the case of the **American put option** there is an increasing function $x^\star(t)$ - the *free boundary* - such that, at time t,

$$\begin{aligned} \mathbf{P(t,x) = K - x} \qquad & \textbf{for } x < x^\star(t) \\ \mathcal{L}_{\mathbf{BS}}(\sigma)\mathbf{P = 0} \qquad & \textbf{for } x > x^\star(t), \end{aligned} \tag{86}$$

with

$$\mathbf{P(T,x) = (K - x)^+} \tag{87}$$

$$\mathbf{x^\star(T) = K.} \tag{88}$$

In addition, *P and $\frac{\partial P}{\partial x}$ are continuous across the boundary $x^\star(t)$*, so that

$$\mathbf{P(t, x^\star(t)) = K - x^\star(t),} \tag{89}$$

$$\mathbf{\frac{\partial P}{\partial x}(t, x^\star(t)) = -1.} \tag{90}$$

The exercise boundary $x^\star(t)$ separates the *hold* region, where the option is not exercised, from the *exercise* region, where it is:

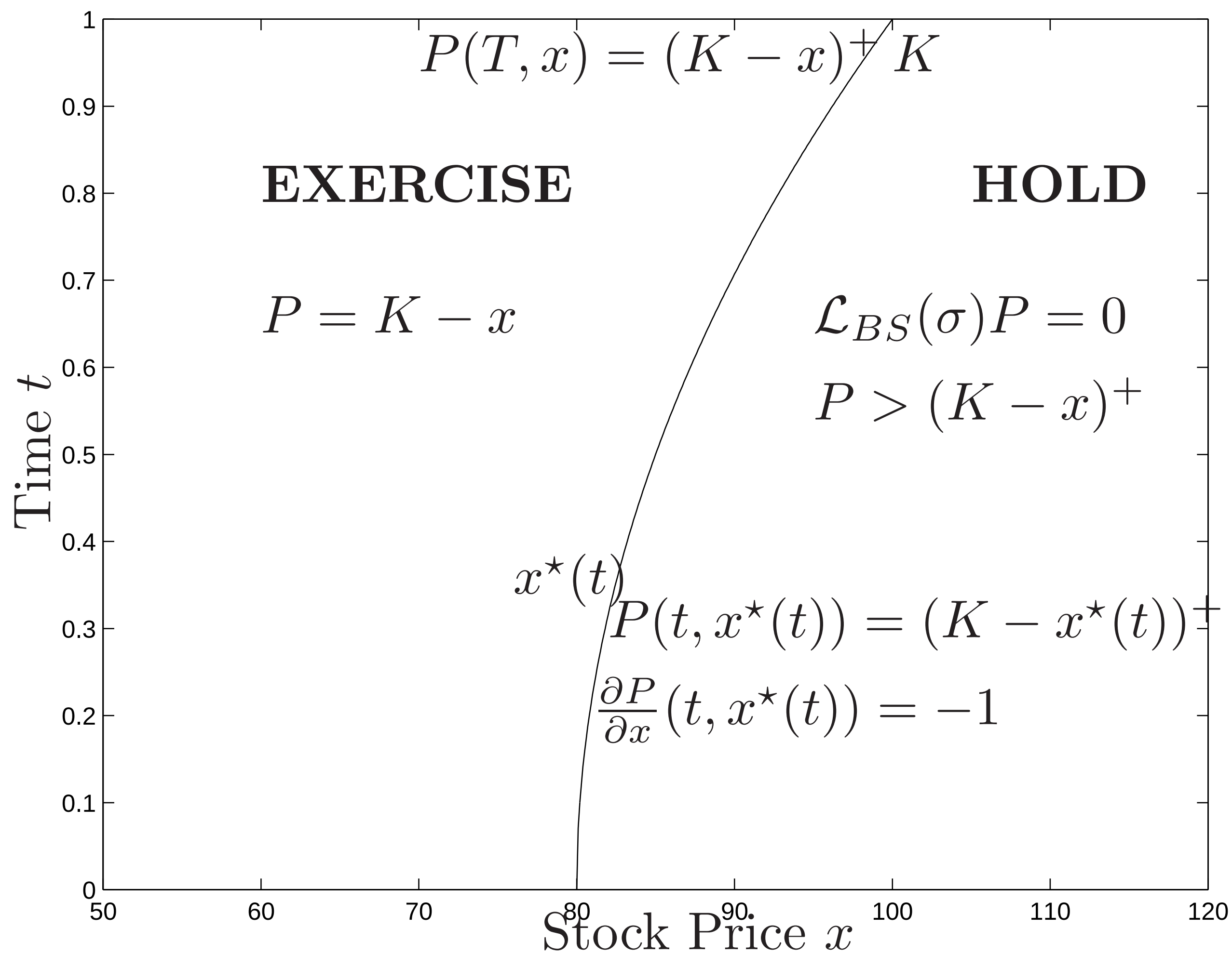

Figure 4: *The American put problem for $P(t,x)$ and $x^\star(t)$, with $\mathcal{L}_{BS}(\sigma)$ defined in (41).*

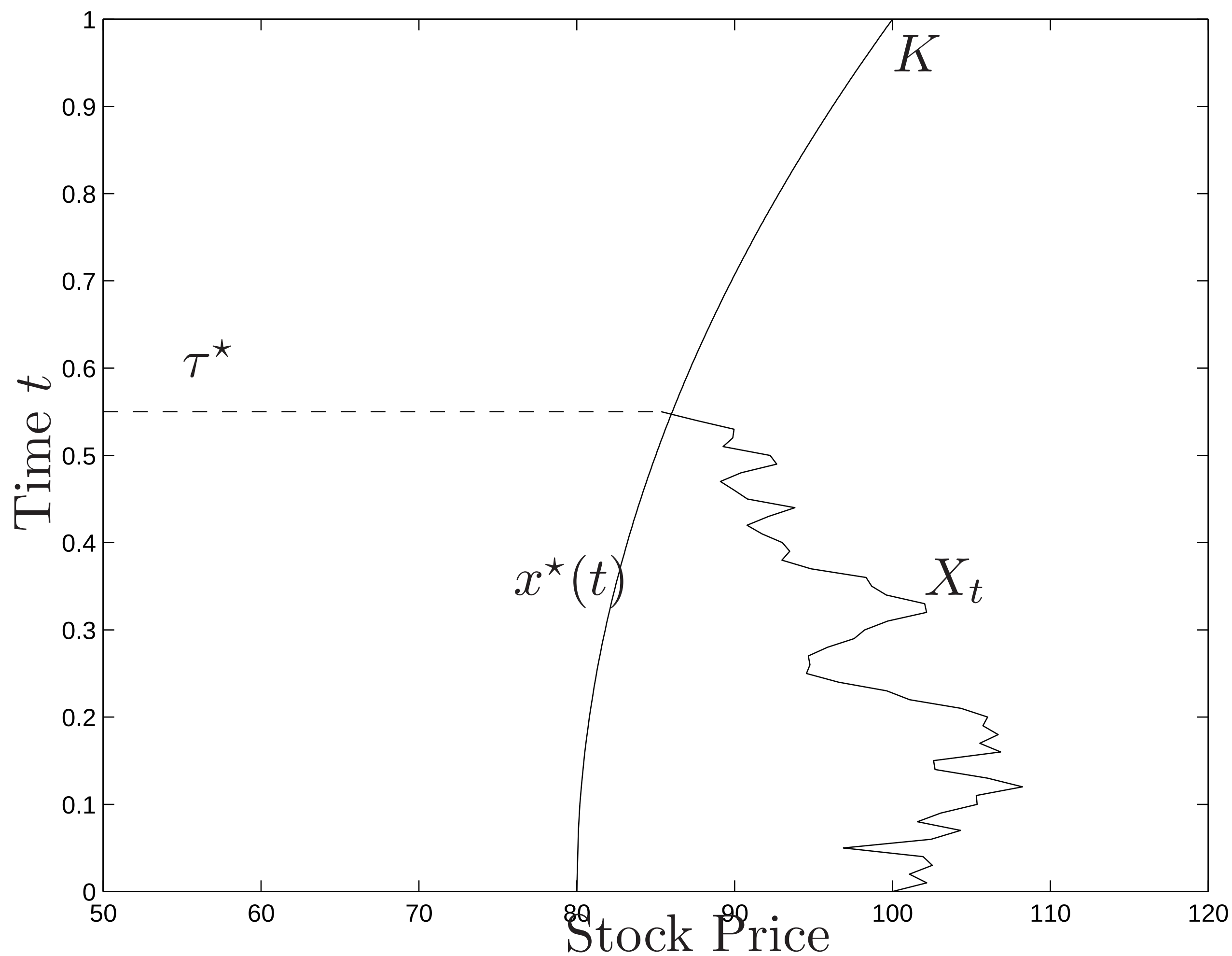

Figure 5: *Optimal exercise time $\tau^\star$ along a sample path for an American put option.*

1.7 Path-Dependent Derivatives

In order to price path-dependent derivatives, one has to compute *the expectations of their discounted payoffs with respect to the risk-neutral probability*. Here are some examples.

1.7.1 Barrier Options

A **down-and-out call option** (European style) is an example of a barrier option that has a payoff function given by (26).

$$\mathbf{P}(\mathbf{0}, \mathbf{x}) = \mathbb{E}^{\star}\left\{\mathbf{e}^{-\mathbf{r}\mathbf{T}}\left(\mathbf{X}_{\mathbf{T}} - \mathbf{K}\right)^{+}\mathbf{1}_{\{\inf_{\mathbf{0}\leq\mathbf{t}\leq\mathbf{T}}\mathbf{X}_{\mathbf{t}}>\mathbf{B}\}} \mid \mathbf{X}_{\mathbf{0}} = \mathbf{x}\right\}.$$

The price at time $t < T$ of this option is given by

$$\begin{aligned}\mathbf{P}_{\mathbf{t}} &= \mathbb{E}^{\star}\left\{\mathbf{e}^{-\mathbf{r}(\mathbf{T}-\mathbf{t})}\left(\mathbf{X}_{\mathbf{T}} - \mathbf{K}\right)^{+}\mathbf{1}_{\{\inf_{\mathbf{0}\leq\mathbf{s}\leq\mathbf{T}}\mathbf{X}_{\mathbf{s}}>\mathbf{B}\}} \mid \mathcal{F}_{\mathbf{t}}\right\} \\ &= \mathbf{1}_{\{\inf_{\mathbf{0}\leq\mathbf{s}\leq\mathbf{t}}\mathbf{X}_{\mathbf{s}}>\mathbf{B}\}}\mathbb{E}^{\star}\left\{\mathbf{e}^{-\mathbf{r}(\mathbf{T}-\mathbf{t})}\left(\mathbf{X}_{\mathbf{T}} - \mathbf{K}\right)^{+}\mathbf{1}_{\{\inf_{\mathbf{t}\leq\mathbf{s}\leq\mathbf{T}}\mathbf{X}_{\mathbf{s}}>\mathbf{B}\}} \mid \mathcal{F}_{\mathbf{t}}\right\} \\ &= \mathbf{1}_{\{\inf_{\mathbf{0}\leq\mathbf{s}\leq\mathbf{t}}\mathbf{X}_{\mathbf{s}}>\mathbf{B}\}}\mathbf{u}(\mathbf{t}, \mathbf{X}_{\mathbf{t}}), \quad \textit{by the Markov property.} \qquad (91)\end{aligned}$$

These expectations can be computed by using classical results on the *joint probability distribution of the Brownian motion and its minimum*, obtained by the *reflection principal*.

Alternatively, the function $u(t,x)$ given by (91) satisfies the following **boundary value problem** on $\{x > B\}$:

$$\begin{aligned} \mathcal{L}_{\mathbf{BS}}(\sigma)\mathbf{u} &= \mathbf{0} \\ \mathbf{u}(\mathbf{t},\mathbf{B}) &= \mathbf{0} \\ \mathbf{u}(\mathbf{T},\mathbf{x}) &= (\mathbf{x}-\mathbf{K})^{+}. \end{aligned}$$

The *method of images* leads to a formula for $\mathbf{u}(\mathbf{t},\mathbf{x})$:

$$\mathbf{u}(\mathbf{t},\mathbf{x}) = \mathbf{u}_{\mathbf{BS}}(\mathbf{t},\mathbf{x}) - \left(\frac{\mathbf{x}}{\mathbf{B}}\right)^{1-\frac{2\mathbf{r}}{\sigma^2}} \mathbf{u}_{\mathbf{BS}}\left(\mathbf{t}, \frac{\mathbf{B}^2}{\mathbf{x}}\right), \tag{92}$$

where $\mathbf{u}_{\mathbf{BS}}(\mathbf{t},\mathbf{x})$ is the Black-Scholes price of the European derivative with payoff function $h(x) = (x-K)^{+}\mathbf{1}_{\{x>B\}}$. In the case $B \leq K$, where the knock-out barrier is below the call strike, $\mathbf{u}_{\mathbf{BS}}(\mathbf{t},\mathbf{x})$ is simply the price $\mathbf{C}_{\mathbf{BS}}(\mathbf{t},\mathbf{x})$ of a call option given by the Black-Scholes formula (42).

1.7.2 Lookback Options

We consider for instance a *floating strike lookback put* which pays the difference $\mathbf{J_T} - \mathbf{X_T}$ where $\mathbf{J_T}$ is the *running maximum* $\mathbf{J_t} = \sup_{\mathbf{0 \le s \le t}} \mathbf{X_s}$.

$$\begin{aligned} \mathbf{P(0,x)} &= \mathbb{E}^\star \left\{ \mathbf{e^{-rT}} \left(\mathbf{J_T} - \mathbf{X_T} \right) \mid \mathbf{X_0 = x} \right\} \\ &= \mathbf{x e^{-rT}} \mathbb{E}^\star \left\{ \sup_{\mathbf{0 \le t \le T}} \left(\mathbf{e^{(r - \frac{1}{2}\sigma^2)t + \sigma W_t}} \right) \right\} - \mathbf{x}, \end{aligned}$$

by using the martingale property of the discounted stock price under the risk neutral probability $\mathbb{P}^\star$, and the explicit form of X_t.

Again, by using log-variables and a change of measure, these expectations can be reduced to integrals involving the joint distribution of a driftless Brownian motion and its running maximum.

The PDE approach:

The price $\mathbf{P(t, x, J)}$ of this option satisfies the problem

$$\begin{aligned} &\mathcal{L}_{\mathbf{BS}}(\sigma)\mathbf{P} = \mathbf{0} \quad \text{in } x < J \text{ and } t < T \\ &\frac{\partial \mathbf{P}}{\partial \mathbf{J}}(\mathbf{t}, \mathbf{J}, \mathbf{J}) = \mathbf{0} \\ &\mathbf{P}(\mathbf{T}, \mathbf{x}, \mathbf{J}) = \mathbf{J} - \mathbf{x}. \end{aligned}$$

The boundary condition at $J = x$ expresses the fact that the price of the lookback option for $X_t = J_t$ is insensitive to small changes in J_t because the realized maximum at time T is larger than the realized maximum at time t with probability one.

The problem of finding $\mathbf{P(t,x,J)}$ can be reduced to a one (space) dimensional boundary value problem with the following *similarity reduction*: $\xi = \mathbf{x/J}$, and $\mathbf{P(t,x,J) = JQ(t,\xi)}$, leading to

$$\mathbf{P(t,x,J)} = -\mathbf{x} + \mathbf{x}\left(\mathbf{1} + \frac{\sigma^{\mathbf{2}}}{\mathbf{2r}}\right)\mathbf{N(d_7)} + \mathbf{Je^{-r(T-t)}}\left(\mathbf{N(d_5)} - \frac{\sigma^{\mathbf{2}}}{\mathbf{2r}}\left(\frac{\mathbf{x}}{\mathbf{J}}\right)^{\mathbf{1}-\frac{\mathbf{2r}}{\sigma^{\mathbf{2}}}}\mathbf{N(d_6)}\right) \tag{93}$$

where

$$d_5 = \frac{\log(J/x) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_6 = \frac{\log(x/J) - \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_7 = \frac{\log(x/J) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}.$$

1.7.3 Forward-Start Options ("Cliquets")

A typical forward-start option is a *call option* maturing at time T such that the *strike price is set equal to X_{T_1} at time $T_1 < T$*.

Its payoff at maturity T is given by $\mathbf{h} = (\mathbf{X_T} - \mathbf{X_{T_1}})^+$.

If $T_1 \leq t \leq T$, the contract is simply a call option with $K = X_{T_1}$.

When $t < T_1 < T_2$, which is the case *when the contract is initiated*, its price at time t is given by $P(t, X_t)$ where

$$
\begin{aligned}
P(t,x) &= \mathbb{E}^\star \left\{ e^{-r(T-t)} (X_T - X_{T_1})^+ \mid X_t = x \right\} \\
&= \mathbb{E}^\star \left\{ e^{-r(T_1-t)} \mathbb{E}^\star \left\{ e^{-r(T-T_1)} (X_T - X_{T_1})^+ \mid \mathcal{F}_{T_1} \right\} \mid X_t = x \right\} \\
&= \mathbb{E}^\star \left\{ e^{-r(T_1-t)} C_{BS}(T_1, X_{T_1}; T, K = X_{T_1}) \mid X_t = x \right\} \\
&= \mathbb{E}^\star \left\{ e^{-r(T_1-t)} X_{T_1} \left(N(\bar{d}_1) - e^{-r(T-T_1)} N(\bar{d}_2) \right) \mid X_t = x \right\},
\end{aligned}
$$

where $\bar{d}_1$ and $\bar{d}_2$ are given here by

$$\bar{d}_1 = \left(r + \frac{1}{2}\sigma^2\right) \frac{\sqrt{T - T_1}}{\sigma} \quad , \quad \bar{d}_2 = \left(r - \frac{1}{2}\sigma^2\right) \frac{\sqrt{T - T_1}}{\sigma},$$

because **the underlying call option is computed at the money** $K = X_{T_1}$.

We then deduce

$$\begin{aligned} \mathbf{P}(\mathbf{t}, \mathbf{x}) &= \left(\mathbf{N}(\bar{\mathbf{d}}_1) - \mathbf{e}^{-\mathbf{r}(\mathbf{T}-\mathbf{T}_1)}\mathbf{N}(\bar{\mathbf{d}}_2)\right) \mathbb{E}^\star \left\{\mathbf{e}^{-\mathbf{r}(\mathbf{T}_1-\mathbf{t})}\mathbf{X}_{\mathbf{T}_1} \mid \mathbf{X}_\mathbf{t} = \mathbf{x}\right\} \\ &= \mathbf{x}\left(\mathbf{N}(\bar{\mathbf{d}}_1) - \mathbf{e}^{-\mathbf{r}(\mathbf{T}-\mathbf{T}_1)}\mathbf{N}(\bar{\mathbf{d}}_2)\right), \end{aligned} \tag{94}$$

by using the **martingale property of the discounted stock price** under the risk neutral probability $\mathbb{P}^\star$.

1.7.4 Compound Options

Example of a call-on-call option. For $t < T_1 < T$, at time T_1, the maturity time of the option, the payoff is given by

$$\mathbf{h}(\mathbf{C_{BS}}(\mathbf{T_1}, \mathbf{X_{T_1}}; \mathbf{K}, \mathbf{T})) = (\mathbf{C_{BS}}(\mathbf{T_1}, \mathbf{X_{T_1}}; \mathbf{K}, \mathbf{T}) - \mathbf{K_1})^+ .$$

The price at time t of this call-on-call is given by

$$\begin{aligned} P(t,x) &= \mathbb{E}^\star \left\{ e^{-r(T_1-t)} \left(C_{BS}(T_1, X_{T_1}; K, T) - K_1 \right)^+ \mid X_t = x \right\} \qquad (95) \\ &= \mathbb{E}^\star \left\{ e^{-r(T_1-t)} \left(C_{BS}(T_1, X_{T_1}; K, T) - K_1 \right) \mathbf{1}_{\{X_{T_1} \geq x_1\}} \mid X_t = x \right\}, \end{aligned}$$

where x_1 is defined by $\mathbf{C_{BS}}(\mathbf{T_1}, \mathbf{x_1}; \mathbf{K}, \mathbf{T}) = \mathbf{K_1}$.

Explicit formulas can be obtained by using the bivariate normal distribution (see notes).

1.7.5 Asian Options

As an example we consider an **Asian (European style) average-strike option** whose payoff is given by a function of the stock price at maturity and of the *arithmetically-averaged stock price* before maturity like in an average strike call option (31). One can introduce the integral process

$$\mathbf{I_t = \int_0^t X_s ds,}$$

and redo the replicating strategies analysis or the risk-neutral valuation argument for the pair of processes $(\mathbf{X_t}, \mathbf{I_t})$. Observe that $(\mathbf{I_t})$ does not introduce new risk or, in other words, there is no new Brownian motion in the equation $\mathbf{dI_t = X_t dt}$.

Using a **two-dimensional version of Itô's formula** presented in the following section, one can deduce the PDE

$$\frac{\partial \mathbf{P}}{\partial \mathbf{t}} + \frac{\mathbf{1}}{\mathbf{2}}\sigma^{\mathbf{2}}\mathbf{x}^{\mathbf{2}}\frac{\partial^{\mathbf{2}}\mathbf{P}}{\partial \mathbf{x}^{\mathbf{2}}} + \mathbf{r}\left(\mathbf{x}\frac{\partial \mathbf{P}}{\partial \mathbf{x}} - \mathbf{P}\right) + \mathbf{x}\frac{\partial \mathbf{P}}{\partial \mathbf{I}} = \mathbf{0}\,, \tag{96}$$

to be solved, for instance, with the final condition $\mathbf{P}(\mathbf{T},\mathbf{x},\mathbf{I}) = (\mathbf{x} - \frac{\mathbf{I}}{\mathbf{T}})^+$, in order to obtain the price $\mathbf{P}(\mathbf{t},\mathbf{X_t},\mathbf{I_t})$ of an *arithmetic-average* strike call option at time t. This is solved numerically in most examples (see the notes for a dimension reduction technique).

Note that *geometric-average Asian options* are much simpler since *log prices* are added leading to Gaussian random variables.

1.8 First Passage Structural Approach to Default

Credit risk. We consider here the problem of pricing a *defaultable zero-coupon bond* which pays a fixed amount (say \$1) at maturity T **unless default occurs**, in which case it is worth nothing. In other words we consider the simple case of *no recovery* in case of default.

1.8.1 Merton's Approach

In the **Merton's approach**, the underlying X_t follows a *geometric Brownian motion*, and **default occurs** if $\mathbf{X_T < B}$ for some threshold value $\mathbf{B}$. In this case the price at time t of the defaultable bond is simply the price of a *European digital option* which pays one if X_T exceeds the threshold and zero otherwise, as in (69). Assuming that the underlying is tradable and the risk free interest rate r is constant, by no-arbitrage argument, the price of this option is explicitly given by $\mathbf{u^d(t, X_t)}$ where $\longrightarrow$

$$
\begin{aligned}
u^d(t,x) &= I\!E^\star \left\{ e^{-r(T-t)} \mathbf{1}_{X_T > B} \mid X_t = x \right\} \\
&= e^{-r(T-t)} I\!P^\star \left\{ X_T > B \mid X_t = x \right\} \\
&= e^{-r(T-t)} I\!P^\star \left\{ \left(r - \frac{\sigma^2}{2} \right)(T-t) + \sigma (W_T^\star - W_t^\star) > \log\left(\frac{B}{x} \right) \right\} \\
&= e^{-r(T-t)} I\!P^\star \left\{ \frac{W_T^\star - W_t^\star}{\sqrt{T-t}} > -\frac{\log\left(\frac{x}{B}\right) + \left(r - \frac{\sigma^2}{2} \right)(T-t)}{\sigma\sqrt{T-t}} \right\} \\
&= e^{-r\tau} N(d_2(\tau)), \qquad (108)
\end{aligned}
$$

with the usual notation $\tau = T - t$ and the *distance to default*:

$$
d_2(\tau) = \frac{\log\left(\frac{x}{B}\right) + \left(r - \frac{\sigma^2}{2} \right)\tau}{\sigma\sqrt{\tau}}. \qquad (109)
$$

1.8.2 The First Passage Model

In the *first passage structural approach*, default occurs if X_t goes below B at **some time before maturity**. In this extended Merton, or **Black and Cox model**, the payoff is

$$\mathbf{h}(\mathbf{X}) = \mathbf{1}_{\{\inf_{\mathbf{0}\leq \mathbf{s}\leq \mathbf{T}} \mathbf{X_s} > \mathbf{B}\}}.$$

The defaultable bond can then be viewed as a *path-dependent derivative*. Its value at time $t \leq T$, denoted by $\mathbf{P^B(t,T)}$, is given by

$$\begin{aligned}\mathbf{P^B(t,T)} &= \mathbf{E}^\star \left\{ \mathbf{e}^{-\mathbf{r}(\mathbf{T}-\mathbf{t})} \mathbf{1}_{\{\inf_{\mathbf{0}\leq \mathbf{s}\leq \mathbf{T}} \mathbf{X_s} > \mathbf{B}\}} \mid \mathcal{F}_\mathbf{t} \right\} \quad (110)\\ &= \mathbf{1}_{\{\inf_{\mathbf{0}\leq \mathbf{s}\leq \mathbf{t}} \mathbf{X_s} > \mathbf{B}\}} \mathbf{e}^{-\mathbf{r}(\mathbf{T}-\mathbf{t})} \mathbf{E}^\star \left\{ \mathbf{1}_{\{\inf_{\mathbf{t}\leq \mathbf{s}\leq \mathbf{T}} \mathbf{X_s} > \mathbf{B}\}} \mid \mathcal{F}_\mathbf{t} \right\}.\end{aligned}$$

Indeed $\mathbf{P^B(t,T)} = \mathbf{0}$ if the asset price has reached B before time t, which is reflected by the factor $\mathbf{1}_{\{\inf_{\mathbf{0}\leq \mathbf{s}\leq \mathbf{t}} \mathbf{X_s} > \mathbf{B}\}}$.

Introducing the *default time* τ_t defined by

$$\tau_{\mathbf{t}} = \inf\{\mathbf{s} \geq \mathbf{t}, \mathbf{X_s} \leq \mathbf{B}\},$$

one has

$$\mathbf{E}^{\star}\left\{\mathbf{1}_{\{\inf_{\mathbf{t}\leq\mathbf{s}\leq\mathbf{T}}\mathbf{X_s}>\mathbf{B}\}} \mid \mathcal{F}_{\mathbf{t}}\right\} = \mathbf{P}^{\star}\{\tau_{\mathbf{t}} > \mathbf{T} \mid \mathcal{F}_{\mathbf{t}}\},$$

which shows that the problem reduces to the characterization of the *distribution of default times*. Observe that the default time τ_t is a *predictable* stopping time, in the sense that it can be *announced* by an increasing sequence of stopping times. For instance one can consider the sequence $(\tau_{\mathbf{t}}^{(\mathbf{n})})$ defined by

$$\tau_{\mathbf{t}}^{(\mathbf{n})} = \inf\{\mathbf{s} \geq \mathbf{t}, \mathbf{X_s} \leq \mathbf{B} + \mathbf{1}/\mathbf{n}\}.$$

These stopping times are illustrated in Figure 6:

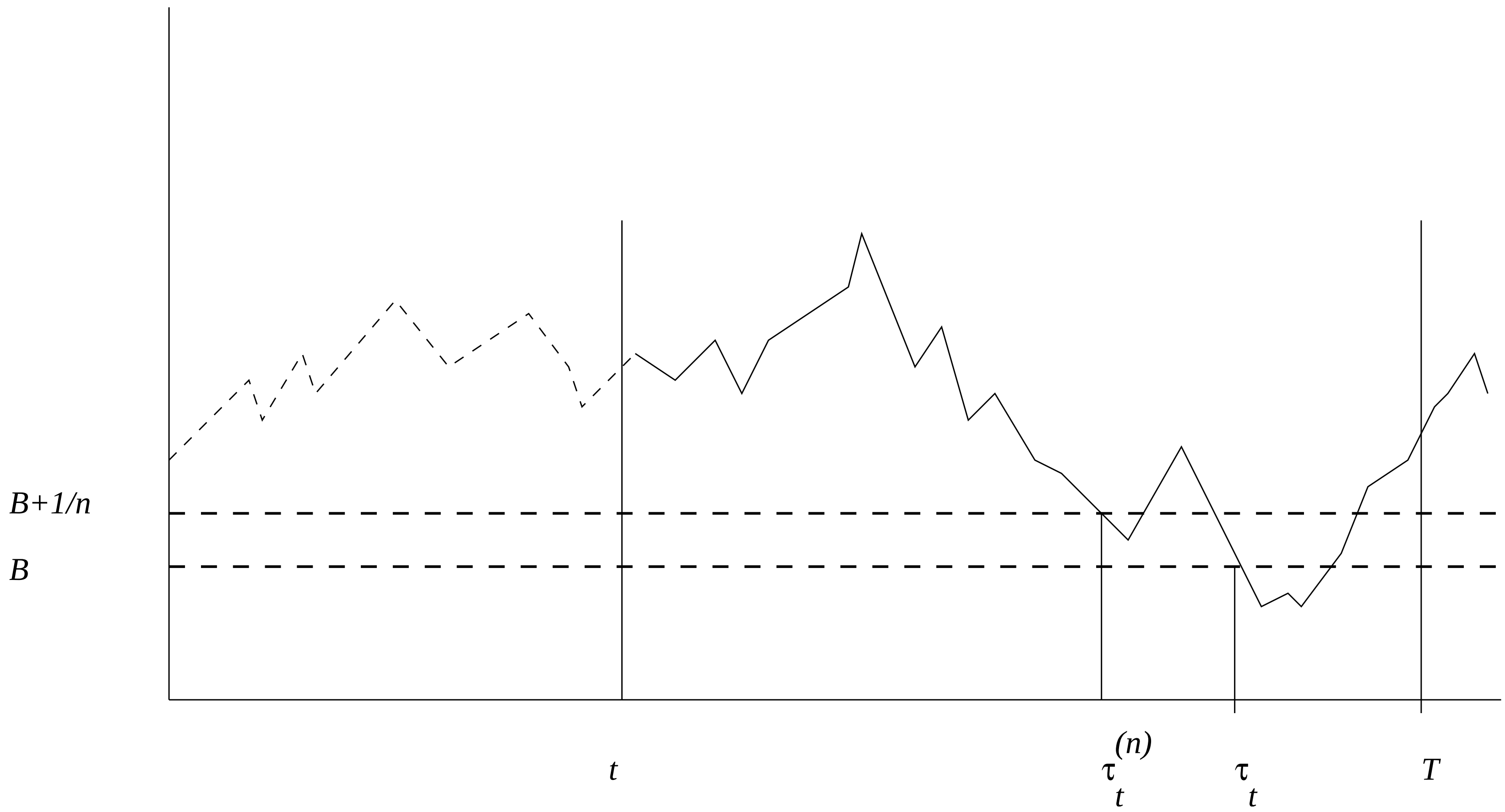

Figure 6: *The cartoon shows a sample trajectory of the geometric Brownian motion X_t, and the corresponding values of the first hitting times $\tau_t^{(n)}$ and τ_t after t of the levels $B + 1/n$ and B.*

An alternative *intensity based* approach to default consists in introducing default times which are *unpredictable*.

In the **first passage model**, a defaultable zero-coupon bond is in fact a **binary down-an-out barrier option** where the barrier level and the strike price coincide. As presented in Section 1.7.1, from a *probabilistic point of view*, we have

$$\begin{aligned}
&\mathbf{E}^\star\left\{\mathbf{1}_{\{\inf_{\mathbf{t}\leq\mathbf{s}\leq\mathbf{T}}\mathbf{X_s}>\mathbf{B}\}}\mid\mathcal{F}_\mathbf{t}\right\}\\
=\;&\mathbf{P}^\star\left\{\inf_{\mathbf{t}\leq\mathbf{s}\leq\mathbf{T}}\left((\mathbf{r}-\frac{\sigma^\mathbf{2}}{\mathbf{2}})(\mathbf{s}-\mathbf{t})+\sigma(\mathbf{W}^\star_\mathbf{s}-\mathbf{W}^\star_\mathbf{t})\right)>\log\left(\frac{\mathbf{B}}{\mathbf{x}}\right)\mid\mathbf{X_t}=\mathbf{x}\right\},
\end{aligned}$$

which can be computed by using the distribution of the minimum of a (non standard) Brownian motion. From the *point of view of PDEs*, we have

$$\mathbf{E}^\star\left\{\mathbf{e}^{-\mathbf{r}(\mathbf{T}-\mathbf{t})}\mathbf{1}_{\{\inf_{\mathbf{t}\leq\mathbf{s}\leq\mathbf{T}}\mathbf{X_s}>\mathbf{B}\}}\mid\mathcal{F}_\mathbf{t}\right\}=\mathbf{u}(\mathbf{t},\mathbf{X_t}),$$

where $\mathbf{u(t,x)}$ is the solution of the following problem

$$\mathcal{L}_{\mathbf{BS}}(\sigma)\mathbf{u} = \mathbf{0} \ \text{ on } \ \mathbf{x} > \mathbf{B},\ \mathbf{t} < \mathbf{T} \tag{111}$$
$$\mathbf{u(t,B)} = \mathbf{0} \ \ \text{for any } \ \mathbf{t} \leq \mathbf{T}$$
$$\mathbf{u(T,x)} = \mathbf{1} \ \text{ for } \ \mathbf{x} > \mathbf{B},$$

which is to be solved for $x > B$. This problem can be solved by introducing the corresponding *European digital option* which pays \$1 at maturity if $X_T > B$ and nothing otherwise. Its price at time $t < T$ is given by $\mathbf{u^d(t,X_t)}$ computed explicitly in (97).

The function $\mathbf{u^d(t,x)}$ is the solution to the PDE

$$\mathcal{L}_{\mathbf{BS}}(\sigma)\mathbf{u^d} = \mathbf{0} \ \text{ on } \ \mathbf{x} > \mathbf{0},\ \mathbf{t} < \mathbf{T} \tag{112}$$
$$\mathbf{u^d(T,x)} = \mathbf{1} \ \text{ for } \ \mathbf{x} > \mathbf{B}, \ \ \text{and 0 otherwise.}$$

By using the *method of images*, the solution $\mathbf{u(t,x)}$ can be written

$$\mathbf{u(t,x)} = \mathbf{u^d(t,x)} - \left(\frac{\mathbf{x}}{\mathbf{B}}\right)^{\mathbf{1}-\frac{\mathbf{2r}}{\sigma^{\mathbf{2}}}} \mathbf{u^d}\left(\mathbf{t}, \frac{\mathbf{B^2}}{\mathbf{x}}\right). \tag{113}$$

Combining with the formula for $\mathbf{u^d(t,x)}$, we get

$$\mathbf{u(t,x)} = \mathbf{e^{-r(T-t)}}\left(\mathbf{N(d_2^+(T-t))} - \left(\frac{\mathbf{x}}{\mathbf{B}}\right)^{\mathbf{1}-\frac{\mathbf{2r}}{\sigma^{\mathbf{2}}}} \mathbf{N(d_2^-(T-t))}\right), \tag{114}$$

where we denote

$$\mathbf{d_2^\pm}(\tau) = \frac{\pm\log\left(\frac{\mathbf{x}}{\mathbf{B}}\right) + \left(\mathbf{r} - \frac{\sigma^{\mathbf{2}}}{\mathbf{2}}\right)\tau}{\sigma\sqrt{\tau}}. \tag{115}$$

The yield spread $\mathbf{Y(0,T)}$ at time zero is defined by

$$\mathbf{e^{-Y(0,T)T} = \frac{P^B(0,T)}{P(0,T)}}, \tag{116}$$

where $\mathbf{P(0,T) = e^{-rT}}$ is the **default free zero-coupon bond** price. In other words, $\mathbf{r + Y(0,T)}$ is the **effective rate of return** over the period $(0,T)$, where the **spread** $Y(0,T)$ is due to the **default risk**.

The price of the defaultable bond is given by $\mathbf{P^B(0,T) = u(0,x)}$ given in (114),

leading to the **explicit formula for the yield spread**

$$\mathbf{Y(0,T) = -\frac{1}{T}\log\left(N\left(d_2(T)\right) - \left(\frac{x}{B}\right)^{1-\frac{2r}{\sigma^2}} N\left(d_2^-(T)\right)\right)}. \tag{117}$$

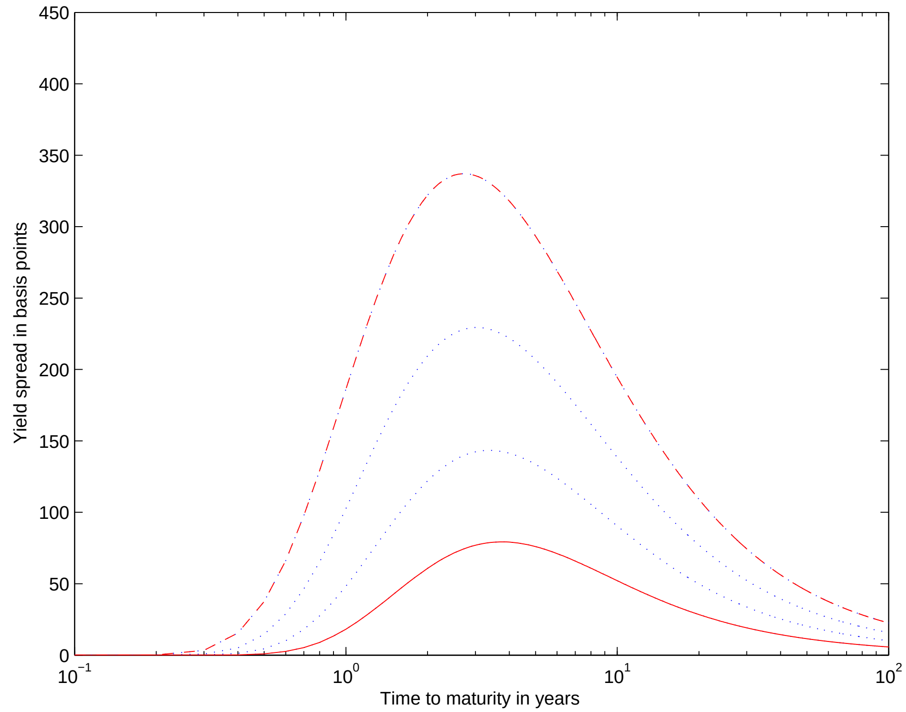

Figure 7: *The figure shows the sensitivity of the yield spread curve to the volatility level. The ratio of the initial value to the default level x/B is set to 1.3, the interest rate r is 6% and the curves increase with the values of σ: 10%, 11%, 12% and 13%. Time to maturity is in unit of years and plotted on the log scale and the yield spread is quoted in basis points.*

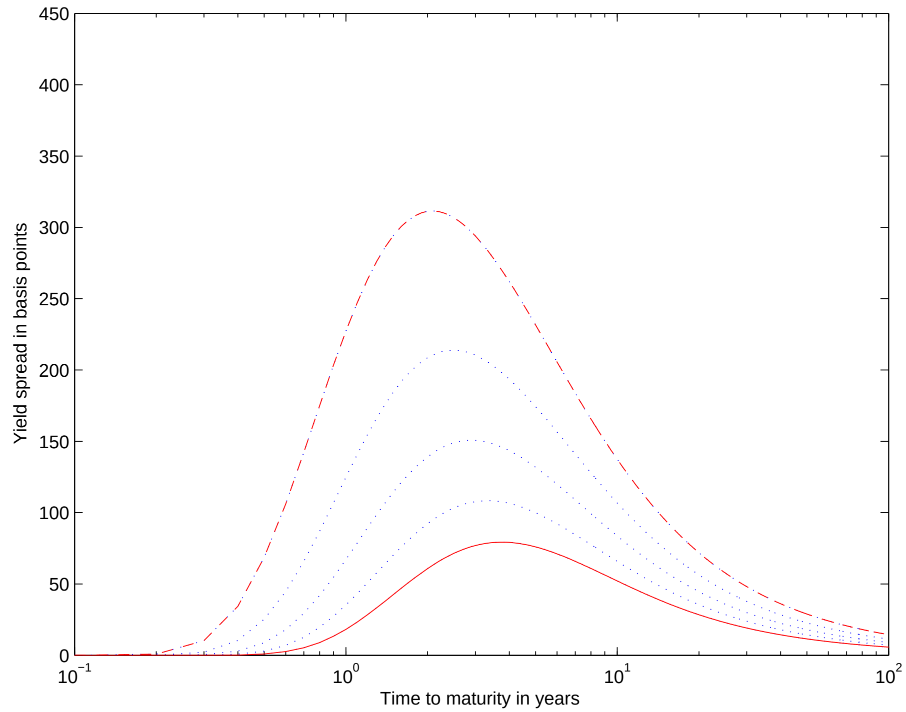

Figure 8: *This figure shows the sensitivity of the yield spread to the leverage level. The volatility level is set to 10%, the interest rate is 6%. The curves increases with the decreasing ratios x/B: $(1.3, 1.275, 1.25, 1.225, 1.2)$.*

1.9 Multidimensional Stochastic Calculus

1.9.1 Multidimensional Itô's Formula

We consider the generalization of the SDEs (11) to the case of **systems** of such equations:

$$\mathbf{dX_t^i} = \mu_{\mathbf{i}}(\mathbf{t}, \mathbf{X_t})\mathbf{dt} + \sum_{\mathbf{j=1}}^{\mathbf{d}} \sigma_{\mathbf{i,j}}(\mathbf{t}, \mathbf{X_t})\mathbf{dW_t^j}, \quad \mathbf{i} = \mathbf{1}, \cdots, \mathbf{d}, \tag{118}$$

where $\mathbf{W_t^j}$, $\mathbf{j} = \mathbf{1}, \cdots, \mathbf{d}$, are d **independent standard Brownian motions**, and

$$\mathbf{X_t} = (\mathbf{X_t^1}, \cdots, \mathbf{X_t^d}),$$

is a d-dimensional process.

We assume that the functions $\mu_i(t, x)$ and $\sigma_{i,j}(t, x)$ are smooth and at most linearly growing at infinity, so that this system has a unique solution adapted to the filtration $(\mathcal{F}_t)$ generated by the Brownian motions (W_t^j).

We now consider real processes of the form $\mathbf{f(t, X_t)}$ where the real function $f(t,x)$ is smooth on $I\!R_+ \times I\!R^d$ (*for instance continuously differentiable with respect to t, and twice continuously differentiable in the x-variable*). The **d-dimensional Itô's formula** can then be written:

$$\mathbf{df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t)dt + \sum_{i=1}^{d} \frac{\partial f}{\partial x^i}(t, X_t)dX_t^i}$$
$$\mathbf{+\frac{1}{2} \sum_{i,j=1}^{d} \frac{\partial^2 f}{\partial x^i \partial x^j}(t, X_t)d\langle X^i, X^j\rangle_t}, \qquad (119)$$

where

$$\begin{aligned} \mathbf{d\langle X^i, X^j\rangle_t} &= \mathbf{\sum_{k=1}^{d} \sigma_{ik}(t, X_t)\sigma_{jk}(t, X_t)dt} \\ &= \mathbf{(\sigma\sigma^T)_{i,j}(t, X_t)dt}, \qquad (120) \end{aligned}$$

Cross-variation rules

$$\mathbf{d\langle t, W_t^j\rangle = d\langle W_t^j, t\rangle = 0},$$

$$\mathbf{d\langle W_t^i, W_t^j\rangle = d\langle W_t^j, W_t^i\rangle = 0 \;\; for\; i \neq j},$$

$$\mathbf{d\langle W_t^i, W_t^i\rangle = dt}.$$

Formula (119) can then be rewritten:

$$\mathbf{df(t, X_t)} = \left(\frac{\partial \mathbf{f}}{\partial \mathbf{t}} + \sum_{\mathbf{i=1}}^{\mathbf{d}} \mu_{\mathbf{i}} \frac{\partial \mathbf{f}}{\partial \mathbf{x^i}} + \frac{\mathbf{1}}{\mathbf{2}} \sum_{\mathbf{i,j=1}}^{\mathbf{d}} (\sigma\sigma^{\mathbf{T}})_{\mathbf{i,j}} \frac{\partial^{\mathbf{2}} \mathbf{f}}{\partial \mathbf{x^i} \partial \mathbf{x^j}}\right) \mathbf{dt} + \sum_{\mathbf{i=1}}^{\mathbf{d}} \frac{\partial \mathbf{f}}{\partial \mathbf{x^i}} \left(\sum_{\mathbf{j=1}}^{\mathbf{d}} \sigma_{\mathbf{i,j}} \mathbf{dW_t^j}\right), \tag{121}$$

where the partial derivatives of f and the coefficients μ and σ are evaluated at (t, X_t).

1.9.2 Girsanov Theorem

In Section 1.4.1 we have used a *change of probability measure* so that the one-dimensional process $\mathbf{W_t^\star = W_t + \theta t}$ becomes a standard Brownian motion under the new probability $\mathbf{P}^\star$. We now give a multidimensional version of this result in the case where θ may also be a stochastic process. To simplify the presentation we assume that the d-dimensional process (θ_t) is of the form $(\theta_\mathbf{j}(\mathbf{X_t}), \mathbf{j = 1}, \cdots, \mathbf{d})$ where the functions $\theta_j(x)$ are bounded (see the notes for less restrictive conditions such as *Novikov condition*). Generalizing (58), we define the real process $(\xi_\mathbf{t}^\theta)_{\mathbf{0 \le t \le T}}$ by:

$$\xi_\mathbf{t}^\theta = \exp\left(-\sum_{\mathbf{j=i}}^{\mathbf{d}}\left(\int_\mathbf{0}^\mathbf{t} \theta_\mathbf{j}(\mathbf{X_s})\mathbf{dW_s^j} + \frac{\mathbf{1}}{\mathbf{2}}\int_\mathbf{0}^\mathbf{t} \theta_\mathbf{j}^\mathbf{2}(\mathbf{X_s})\mathbf{ds}\right)\right), \quad (122)$$

$$\mathbf{d}\xi_{\mathbf{t}}^{\theta} = -\xi_{\mathbf{t}}^{\theta} \sum_{\mathbf{j=1}}^{\mathbf{d}} \theta_{\mathbf{j}}(\mathbf{X_t})\mathbf{dW_t^j}.$$

and therefore $(\xi_{\mathbf{t}}^{\theta})$ is a martingale.

We then define, on $\mathcal{F}_{\mathbf{T}}$, the probability $\mathbf{P}^{\star}$ by $\mathbf{dP}^{\star} = \xi_{\mathbf{T}}^{\theta}\mathbf{dP}$.

Girsanov Theorem states that the processes $(\mathbf{W_t^{j\star}})_{\mathbf{0 \le t \le T}}, \mathbf{j = 1, \cdots, d}$, defined by

$$\mathbf{W_t^{j\star}} = \mathbf{W_t^j} + \int_{\mathbf{0}}^{\mathbf{t}} \theta_{\mathbf{j}}(\mathbf{X_s})\mathbf{ds}, \quad \mathbf{j = 1, \cdots, d}, \tag{123}$$

are **independent standard Brownian motions under $\mathbb{P}^{\star}$.**

See the notes for a justification using the martingale property of $(\xi_t^{\theta - iu})$, and the characterization of independent standard Brownian motions by conditional characteristic functions.

1.9.3 Feynman-Kac Formula

The **infinitesimal generator** of the (possibly non-homogeneous) Markovian process $X = (X^1, \cdots, X^d)$, introduced in (118), is given by

$$\mathcal{L}_t = \sum_{i=1}^{d} \mu_i(t,x) \frac{\partial}{\partial x^i} + \frac{1}{2} \sum_{i,j=1}^{d} (\sigma\sigma^T)_{i,j}(t,x) \frac{\partial^2}{\partial x^i \partial x^j}.$$

If $\mathbf{r(t,x)}$ is a function on $\mathbb{R}_+ \times \mathbb{R}^d$ (for instance bounded), then the function $\mathbf{u(t,x)}$ defined by

$$u(t,x) = \mathbb{E}\left\{ e^{-\int_t^T r(s,X_s)ds} h(X_T) \mid X_t = x \right\},$$

satisfies the **PDE**

$$\frac{\partial u}{\partial t} + \mathcal{L}_t u - ru = 0,$$

with the **terminal condition** $\mathbf{u(T,x) = h(x)}$ (*a call for instance).*

Such **parabolic PDEs** with an **additional source** are also important, If the function $\mathbf{g(t,x)}$ is, for instance, bounded, then the backward problem

$$\begin{aligned}&\frac{\partial \mathbf{v}}{\partial \mathbf{t}} + \mathcal{L}_{\mathbf{t}}\mathbf{v} - \mathbf{rv} + \mathbf{g} = \mathbf{0}\\ &\mathbf{v(T,x) = h(x)},\end{aligned}$$

admits the solution

$$\mathbf{v(t,x)} = \mathbb{E}\left\{\mathbf{e}^{-\int_{\mathbf{t}}^{\mathbf{T}} \mathbf{r(s,X_s)ds}}\mathbf{h(X_T)} + \int_{\mathbf{t}}^{\mathbf{T}} \mathbf{e}^{-\int_{\mathbf{t}}^{\mathbf{s}} \mathbf{r(u,X_u)du}}\mathbf{g(s,X_s)ds} \mid \mathbf{X_t = x}\right\}$$

Two Fundamental Questions:

1. Is the Geometric Brownian motion under $\mathbb{P}$ a good model for returns?

2. Under $\mathbb{P}^\star$ does it predict prices of traded options?

Possible generalizations: local volatility, stochastic volatility, jumps,...

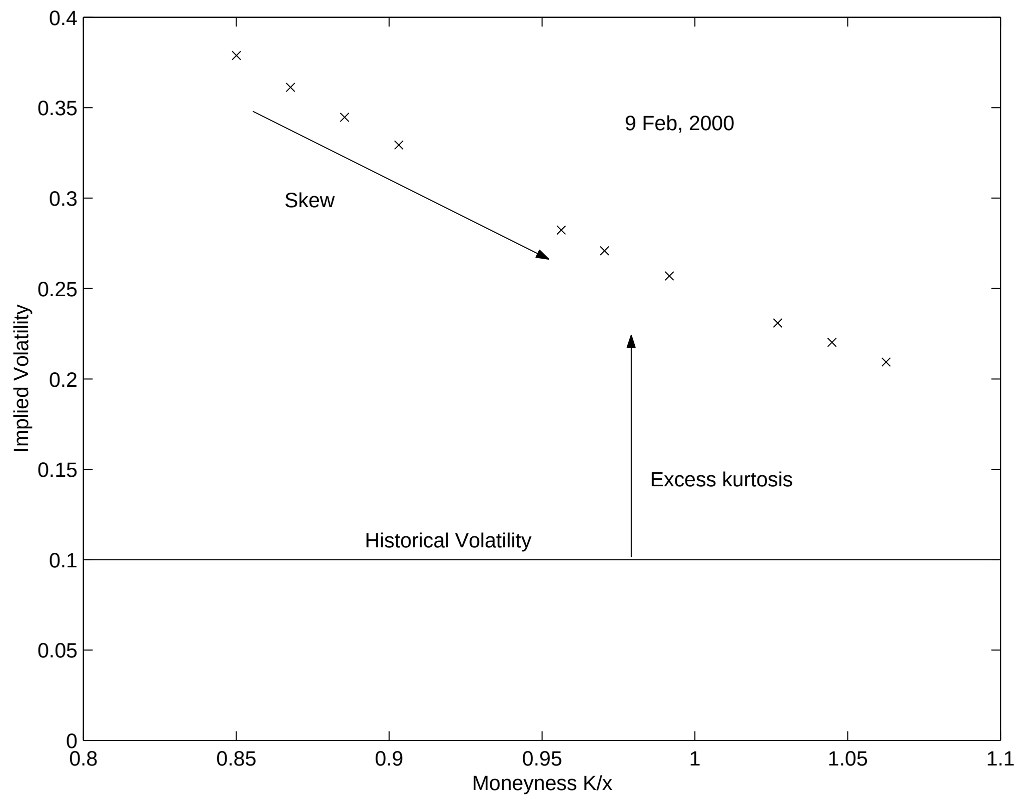

Figure 9: *S&P 500 Implied Volatility Curve as a function of moneyness from S&P 500 index options on February 9, 2000. The current index value is $x = 1411.71$ and the options have over two months to maturity. This is typically described as a downward sloping* ***skew****.*

"Parametrization" of the Implied Volatility Surface $I(t\,;T,K)$

REQUIRED QUALITIES

- **Universal Parsimonous Parameters**: *Model Independence*
- **Stability in Time**: *Predictive Power*
- **Easy Calibration**: *Practical Implementation*
- **Compatibility with Price Dynamics**: *Applicability to Pricing other Derivatives and Hedging*

Mean-Reverting Stochastic Volatility Models

$$dX_t = X_t \left(\mu dt + \sigma_t dW_t \right)$$

$$\sigma_t = f(Y_t)$$

For instance: $\quad 0 < \sigma_1 \leq f(y) \leq \sigma_2 \quad$ for every y

$$dY_t = \alpha(m - Y_t)dt + \beta(\cdots)d\hat{Z}_t$$

Brownian motion $\hat{Z}$ correlated to W:

$$\hat{Z}_t = \rho W_t + \sqrt{1-\rho^2} Z_t \,, \qquad |\rho| < 1$$

so that

$$d\langle W, \hat{Z} \rangle_t = \rho dt$$

Pricing under Stochastic Volatility

Risk-neutral probability chosen by the market: $I\!P^{\star(\gamma)}$

$$
\begin{aligned}
dX_t &= rX_t dt + f(Y_t)X_t dW_t^\star \\
dY_t &= \left[\alpha(m - Y_t) - \beta\left(\rho\frac{(\mu - r)}{f(Y_t)} + \gamma\sqrt{1-\rho^2}\right)\right]dt + \beta d\hat{Z}_t^\star \\
\hat{Z}_t^\star &= \rho W_t^\star + \sqrt{1-\rho^2}\, Z_t^\star
\end{aligned}
$$

Market price of volatility risk: $\gamma = \gamma(y)$

$$P_t = I\!E^{\star(\gamma)}\{e^{-r(T-t)}h(X_T)|\mathcal{F}_t\}$$

Markovian case:

$$P(t,x,y) = I\!E^{\star(\gamma)}\{e^{-r(T-t)}h(X_T)|X_t = x, Y_t = y\}$$

but y (or $f(y)$) is not directly observable!

Stochastic Volatility Pricing PDE

$$\begin{aligned}
\frac{\partial P}{\partial t} + \frac{1}{2} f(y)^2 x^2 \frac{\partial^2 P}{\partial x^2} + \rho\beta x f(y) \frac{\partial^2 P}{\partial x \partial y} + \frac{1}{2}\beta^2 \frac{\partial^2 P}{\partial y^2} & \\
+ r\left(x\frac{\partial P}{\partial x} - P \right) + \alpha(m-y)\frac{\partial P}{\partial y} - \beta\Lambda\frac{\partial P}{\partial y} &= 0
\end{aligned}$$

where

$$\Lambda = \rho\frac{(\mu - r)}{f(y)} + \gamma\sqrt{1-\rho^2}$$

Terminal condition: $P(T,x,y) = h(x)$

No perfect hedge!

Summary of the stochastic volatility approach

Positive aspects:

- More realistic returns distributions (fat tails and asymmetry)
- Smile effect with skew contolled by ρ

Difficulties:

- Volatility not directly observed, parameter estimation difficult
- No canonical model. Relevance of explicit formulas?
- Incomplete markets, no perfect hedge
- Volatility risk premium to be estimated from option prices
- Numerical difficulties due to higher dimension