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Option Pricing Under a Stressed-Beta Model

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Abstract Empirical studies have concluded that stochastic volatility is an important component of option prices. We introduce a regime-switching mechanism into a continuous-time Capital Asset Pricing Model (CAPM) which naturally induces stochastic volatility in the asset price. Under this *Stressed-Beta model*, the mechanism is relatively simple: the slope coefficient - which measures asset returns relative to market returns - switches between two values, depending on the market being above or below a given level. After specifying the model, we use it to price European options on the asset. Interestingly, these option prices are given explicitly as integrals with respect to known densities. We find that the model is able to produce a volatility skew, which is a prominent feature in option markets. This opens the possibility of forward-looking calibration of the slope coefficients, using option data, as illustrated in the paper.

1 Introduction

The concept of stock betas was developed in the context of the Capital Asset Pricing Model of Treynor [19], Sharpe [18], Lintner [15], and Mossin [17] and was based on previous portfolio theory in Markowitz [16]. The beta of a stock represents the scale of the risk of the asset relative to the systematic risk of the market and is critical in the development and performance of stock portfolios. It is now accepted that this linear model does not perform well empirically, and much research has been devoted to its improvement.

One popular approach to extending CAPM has been to retain linearity in the model, but to consider a beta which changes over time ([2], [3], [4], [5], and [12]). These

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papers argue that allowing a dynamic beta should improve the model. However, [10] finds that betas change very slowly over time, and linear factor models may overstate the time variation.

In [6], we proposed a "forward-looking" approach to the calibration of a constant beta-parameter in a continuous-time CAPM model which includes a fast mean-reverting stochastic volatility component in the dynamics of the market price.

In the present paper, we no longer assume beta constant but rather follow a second approach which consists in introducing nonlinearity through a state-switching mechanism. In [9], a two-state CAPM model is proposed in which the excess returns for the market and a particular security are bivariate normally distributed. The parameters of the distribution are determined by an unobservable Markov chain, where the two states represent business regimes of low and high volatility. The threshold CAPM model introduced in [1] expresses market risk as a function of an underlying economic variable termed a *threshold variable*. When the threshold variable is at or below a level λ, the beta takes value β_1, and when it is above λ, it takes value β_2. Formulated in this fashion, threshold CAPM treats beta as constant so long as the threshold variable remains on one side of the boundary. This discrete-time model outperforms the CAPM model, lending support for its realistic approximation of beta.

This paper examines the *Stressed-Beta model*, which can be regarded as a continuous-time threshold CAPM model using the stock market as the threshold variable. Stressed-Beta models have been introduced in the context of hedge fund risk management in discrete time in [20], and in continuous time in [21]. In fact, in a Stressed-Beta model, the volatility of the stock becomes stochastic, driven by the market. The model can be seen as a volatility regime-switching model, or a stochastic volatility model. However, the market is complete when trading in the stock and in the market, so that the risk-neutral pricing measure is unique as shown in Section 3. Also, the correlation structure (leverage effect) is very particular in this model, and induces a negative correlation ("market goes up, stock volatility goes down").

Stochastic volatility models form a rich class of models which, in particular, generate the observed skews of implied volatilities (see for instance [7]). In general, unlike in the Black-Scholes model, there is no explicit option pricing formula, one exception being the Heston model [11] for which European options are given semi-explicitly, up to an inverse Fourier transform.

Surprisingly, we show that our regime-switching stressed-beta model leads to an explicit formula for pricing a European option (up to computation of integrals of known densities). The derivation of this formula, presented in Section 4, relies on passage time analysis for Brownian motions and on the joint distribution of the triplet (terminal value, local time, occupation time) derived in [13,14].

In Section 5, we show how to incorporate in the model a fast mean-reverting stochastic volatility component in the dynamics of the market price, in order to account for the skew of implied volatility in the options on the market as well. We use a singular perturbation method introduced in [7] and also employed in [6].

In Section 6, we present implied volatility skews generated by call option prices computed with the formula (35) derived in Section 4.3, and a calibration example.

2 The Stressed-Beta Continuous-Time CAPM Model

2.1 The Continuous-Time CAPM Model

To start, we consider a simple continuous-time CAPM model in which the market price M_t and an asset price S_t evolve as follows:

$$\frac{dM_t}{M_t} = \mu dt + \sigma_m dW_t, \tag{1}$$
$$\frac{dS_t}{S_t} = \beta \frac{dM_t}{M_t} + \sigma dZ_t, \tag{2}$$

for constant positive volatilities σ_m and σ, and a *slope* β. This model is consistent with CAPM in that that the return of the asset $\frac{dS_t}{S_t}$ is a linear function of the return of the market $\frac{dM_t}{M_t}$ through the β coefficient and a Brownian-driven noise process. In this model, under the physical probability measure $I\!P$, we assume independence between the standard Brownian motions driving the market and asset price processes:

$$d\left\langle W, Z\right\rangle_t = 0. \tag{3}$$

Most importantly, the process preserves the definition of the β coefficient as the covariance of the asset and market returns divided by the market variance, that is formally:

$$\begin{aligned}\frac{Cov\left(\frac{dS_t}{S_t}, \frac{dM_t}{M_t}\right)}{Var\left(\frac{dM_t}{M_t}\right)} &= \frac{Cov\left(\beta\frac{dM_t}{M_t} + \sigma dZ_t, \frac{dM_t}{M_t}\right)}{Var\left(\frac{dM_t}{M_t}\right)} \\ &= \frac{Cov\left(\beta\frac{dM_t}{M_t}, \frac{dM_t}{M_t}\right)}{Var\left(\frac{dM_t}{M_t}\right)} = \beta, \end{aligned} \tag{4}$$

where the second equality holds due to the independence of M_t and Z_t. Observe that the evolution of S_t is given by

$$\frac{dS_t}{S_t} = \beta\mu dt + \beta\sigma_m dW_t + \sigma dZ_t\,,$$

that is a geometric Brownian motion with volatility $\sqrt{\beta^2\sigma_m^2 + \sigma^2}$.

Therefore, in terms of options on the asset, this model is nothing else than the constant volatility Black-Scholes model which cannot capture the observed skew of implied volatilities in option data.

Fouque and Kollman [6] introduced a continuous-time CAPM model with stochastic volatility in which σ_m is driven by an additional stochastic processes. Here, we generalize the continuous-time CAPM model in a different direction, namely in the case where the slope β is no longer a constant, but depends on M_t.

2.2 The Stressed-Beta Model

We extend the CAPM model by considering a piecewise-linear relationship between the asset and the market. When the market is above a given level $c > 0$ (this may be the case when the economy is in a *good* regime), the slope takes the value β, but when the market is below this level c (the *bad* regime), the slope switches to the value $\beta+\delta$. Generally, δ will be positive, thus the slope will be steeper when $M_t < c$. The slope is written as

$$\beta(M_t) = \beta + \delta \mathbf{1}_{\{M_t<c\}}, \tag{5}$$

where $\mathbf{1}_A$ denotes the indicator function of the set A.

Given (5), the model for the market price M_t and the asset price S_t evolve as follows:

$$\frac{dM_t}{M_t} = \mu dt + \sigma_m dW_t, \tag{6}$$

$$\frac{dS_t}{S_t} = \beta(M_t)\frac{dM_t}{M_t} + \sigma dZ_t, \tag{7}$$

where, as before, the volatilities σ_m and σ are constant, and W_t and Z_t are independent Brownian motions (3). This model preserves the definition of β as the covariance of the asset and market returns divided by the variance of the market, since, given M_t, the computation (4) remains the same with β replaced by $\beta(M_t)$.

Substituting the market equation (6) into the asset equation (7) yields

$$\frac{dS_t}{S_t} = \beta(M_t)\mu dt + \beta(M_t)\sigma_m dW_t + \sigma dZ_t, \tag{8}$$

which appears as a stochastic volatility model through the slope-switching mechanism $\beta(M_t)$ driven by the market price level M_t.

3 Option Pricing in the Stressed-Beta Model

Consider a European option written on the stock S with maturity date T, and payoff function $h(S)$. In this paper we assume that the risk free rate r is constant. The problem of option pricing can be approached in two different ways: risk-neutral valuation or replication. In both approaches, it is essential to keep in mind that the market M and the asset S are both tradable.

3.1 Risk-Neutral Pricing

The market (or index) and the asset being both tradable, their discounted prices need to be martingales under a risk-neutral pricing measure. Recall that (W_t, Z_t) are two independent standard Brownian motions, and rewrite the system (6, 7) as:

$$\frac{dM_t}{M_t} = r dt + \sigma_m\left(dW_t + \frac{\mu-r}{\sigma_m}dt\right),$$

$$\frac{dS_t}{S_t} = r dt + \beta(M_t)\sigma_m\left(dW_t + \frac{\mu-r}{\sigma_m}dt\right) + \sigma\left(dZ_t + \frac{(\beta(M_t)-1)r}{\sigma}dt\right).$$

We set

$$dW_t^* = dW_t + \frac{\mu-r}{\sigma_m}dt\,,$$

$$dZ_t^* = dZ_t + \frac{(\beta(M_t)-1)r}{\sigma}dt\,,$$

and we observe that the ratios $\frac{\mu - r}{\sigma_m}$ and $\frac{(\beta(M_t)-1)r}{\sigma}$ are bounded. By Girsanov theorem, there is a unique equivalent probability $I\!\!P^\star \sim I\!\!P$ such that (W_t^*, Z_t^*) are independent standard Brownian motions under $I\!\!P^\star$, called the pricing *equivalent martingale measure* or *risk-neutral measure*. Under $I\!\!P^\star$, the dynamics (6, 7) becomes:

$$\frac{dM_t}{M_t} = rdt + \sigma_m dW_t^*, \tag{9}$$

$$\frac{dS_t}{S_t} = rdt + \beta(M_t)\sigma_m dW_t^* + \sigma dZ_t^*. \tag{10}$$

By the classical no-arbitrage argument, the price of the option at time $t < T$, denoted by P_t, is then given by

$$P_t = I\!\!E^\star \left\{ e^{-r(T-t)} h(S_T) \mid \mathcal{F}_t \right\} = P(t, M_t, S_t), \tag{11}$$

where $\mathcal{F}_t$ denotes the filtration generated by (M_t, S_t), or equivalently by the two Brownian motions, and we have used the Markov property of (M_t, S_t) to write the price of the option as a function of (t, M_t, S_t).

3.2 Replication and Pricing PDE

The derivation of an adapted, replicating, and self-financing strategy follows the lines of the original Black–Scholes–Merton derivation with, in the present case, two tradable risky assets M and S. One seeks a portfolio made, at time t, of a_t shares of market M, b_t shares of asset S, and $c_t e^{rt}$ in cash, such that it replicates the price of the option

$$a_t M_t + b_t S_t + c_t e^{rt} = P(t, M_t, S_t), \quad t \le T,$$

and it is self-financing

$$a_t dM_t + b_t dS_t + r c_t e^{rt} dt = dP(t, M_t, S_t).$$

Using Itô's formula and canceling the risks from the Brownian motions W_t and Z_t, one finds that

$$a_t = \frac{\partial P}{\partial M}(t, M_t, S_t), \quad b_t = \frac{\partial P}{\partial S}(t, M_t, S_t), \tag{12}$$

where $P(t, M, S)$ satisfies the pricing partial differential equation with terminal condition:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma_m^2 M^2 \frac{\partial^2 P}{\partial M^2} + \frac{1}{2}\left(\sigma_m^2 \beta^2(M) + \sigma^2\right) S^2 \frac{\partial^2 P}{\partial S^2} + \beta(M)\sigma_m^2 MS \frac{\partial^2 P}{\partial M \partial S}$$
$$+ r\left(M\frac{\partial P}{\partial M} + S\frac{\partial P}{\partial S} - P\right) = 0, \tag{13}$$

$$P(T, M, S) = h(S). \tag{14}$$

Indeed, (11) is the Feynman–Kac representation of the solution to the problem (13–14).

4 Option Pricing Formula

In this section, we show that, surprisingly, the option price given by (11) admits a closed-form solution as an integral with respect to a multidimensional known density. Since the derivation of this formula is purely probabilistic, for simplicity we consider the case where $t = 0$, so that the time-to-maturity is simply $T - t = T$.

4.1 Log-Variables and Driftless Market

We first consider the log-variables $\xi_t = \log M_t$ and $X_t = \log S_t$, so that the risk-neutral dynamics (9–10) become:

$$d\xi_t = \left(r - \frac{\sigma_m^2}{2}\right) dt + \sigma_m dW_t^*, \tag{15}$$

$$dX_t = \left(r - \frac{1}{2}\left(\sigma_m^2 \beta^2(e^{\xi_t}) + \sigma^2\right)\right) dt + \beta(e^{\xi_t})\sigma_m dW_t^* + \sigma dZ_t^*. \tag{16}$$

In integral form, starting from the initial point $\xi_0 = \xi$, (15) becomes:

$$\xi_t = \xi + \left(r - \frac{\sigma_m^2}{2}\right) t + \sigma_m W_t^* . \tag{17}$$

In integral form, starting from the initial point $X_0 = x$ and evaluated at time T, (16) becomes:

$$\begin{aligned} X_T &= x + \left(r - \frac{\sigma^2}{2}\right) T - \frac{\sigma_m^2}{2} \int_0^T \beta^2(e^{\xi_t}) dt + \sigma_m \int_0^T \beta(e^{\xi_t}) dW_t^* + \sigma Z_T^* \\ &= x + \left(r - \frac{\sigma_m^2 \beta^2 + \sigma^2}{2}\right) T + \sigma_m \beta\, W_T^* + \sigma Z_T^* \\ &\quad -(\delta^2 + 2\delta\beta) \frac{\sigma_m^2}{2} \int_0^T \mathbf{1}_{\{\xi_t < \log c\}} dt + \sigma_m \delta \int_0^T \mathbf{1}_{\{\xi_t < \log c\}} dW_t^* , \end{aligned} \tag{18}$$

where we have used the particular form (5) for the function $\beta(M)$:

$$\beta(M) = \beta + \delta \mathbf{1}_{\{M < c\}}.$$

The expression (18) involves the integral $\int_0^T \mathbf{1}_{\{\xi_t < \log c\}} dt$, which is the occupation time of ξ_t, the Brownian motion with drift given by (17). Our next step is to remove the drift by a Girsanov change of probability. Consider the new probability measure $\widetilde{I\!P}$ defined on $\mathcal{F}_T$ by

$$\frac{d\widetilde{I\!P}}{dI\!P^\star} = \exp\left\{-\theta W_T^* - \frac{1}{2}\theta^2 T\right\},$$

$$\theta = \frac{1}{\sigma_m}\left(r - \frac{\sigma_m^2}{2}\right).$$

Setting

$$\widetilde{W}_t = W_t^* + \theta t, \qquad \widetilde{Z}_t = Z_t^*,$$

then under $\widetilde{I\!P}$, the processes $\widetilde{W}_t$ and $\widetilde{Z}_t$ are two independent standard Brownian motions, and (17) becomes the driftless Brownian motion:

$$\xi_t = \xi + \sigma_m \widetilde{W}_t . \tag{19}$$

Consequently, (18) becomes

$$\begin{aligned} X_T &= x + A_1 T + \sigma_m \beta \widetilde{W}_T + \sigma \widetilde{Z}_T \\ &\quad - A_2 \int_0^T \mathbf{1}_{\{\xi_t < \log c\}} dt + \sigma_m \delta \int_0^T \mathbf{1}_{\{\xi_t < \log c\}} d\widetilde{W}_t, \end{aligned} \tag{20}$$

where A_1 and A_2 are constants defined as

$$A_1 = r(1-\beta) - \frac{\sigma_m^2(\beta^2-\beta)+\sigma^2}{2},$$
$$A_2 = \delta(\delta+2\beta-1)\frac{\sigma_m^2}{2} + \delta r.$$

4.2 Hitting Time and Conditional Distribution of X_T

Next, we introduce the first passage time

$$\tau = \inf\{t \ge 0 : \xi_t = \log c\} = \inf\left\{t \ge 0 : \widetilde{W}_t = \tilde{c}\right\}, \tag{21}$$

where we have used (19) for ξ_t under $\widetilde{I\!P}$ and the notation

$$\tilde{c} = \frac{\log c - \xi}{\sigma_m}. \tag{22}$$

Using the stopping time $\tau \wedge T$, (20) can be rewritten

$$\begin{aligned} X_T &= x + A_1 T + \sigma_m \beta \widetilde{W}_T + \sigma \widetilde{Z}_T \\ &\quad - A_2(\tau\wedge T)\,\mathbf{1}_{\{\tilde{c}>0\}} - A_2 \int_{\tau\wedge T}^{T} \mathbf{1}_{\left\{\widetilde{W}_t<\tilde{c}\right\}} dt \\ &\quad + \sigma_m \delta\, \widetilde{W}_{\tau\wedge T}\,\mathbf{1}_{\{\tilde{c}>0\}} + \sigma_m\delta \int_{\tau\wedge T}^{T} \mathbf{1}_{\left\{\widetilde{W}_t<\tilde{c}\right\}} d\widetilde{W}_t\,. \end{aligned} \tag{23}$$

The stochastic integral appearing in (23) can be re-expressed in terms of the local time $\widetilde{L}^{\tilde{c}}$ of $\widetilde{W}$ at level $\tilde{c}$. Applying Tanaka's formula (see [14] Section 3.6) to the function $\phi(w) = (w-\tilde{c})\mathbf{1}_{\{w<\tilde{c}\}}$ between $\tau\wedge T$ and T, we get:

$$\int_{\tau\wedge T}^{T} \mathbf{1}_{\left\{\widetilde{W}_t<\tilde{c}\right\}} d\widetilde{W}_t = \phi(\widetilde{W}_T) - \phi(\widetilde{W}_{\tau\wedge T}) + \widetilde{L}_T^{\tilde{c}} - \widetilde{L}_{\tau\wedge T}^{\tilde{c}}\,. \tag{24}$$

If $\xi = \log c$, or equivalently $\tilde{c} = 0$, then $\tau = 0$.
If $\xi \neq \log c$, or equivalently $\tilde{c} \neq 0$, then the probability distribution of $\tau\wedge T$ is given by

$$p(u;\tilde{c})\mathbf{1}_{(0,T)}(u)du + \widetilde{I\!P}\{\tau > T\}\delta_T(du)\,,$$

where the density $p(u;\tilde{c})$ is given by ([14] Section 2.6.C):

$$p(u;\tilde{c}) = \frac{|\tilde{c}|}{\sqrt{2\pi u^3}} \exp\left(-\frac{\tilde{c}^2}{2u}\right), \quad u > 0, \tag{25}$$

and

$$\widetilde{I\!P}\{\tau > T\} = \int_T^\infty p(u;\tilde{c})du = 2N_T(|\tilde{c}|) - 1, \tag{26}$$

where N_T denotes the $\mathcal{N}(0,T)$-cdf.

At this point, it is convenient to treat separately the cases $\xi = \log c$, $\xi > \log c$, and $\xi < \log c$ (or equivalently $\tilde{c} = 0$, $\tilde{c} < 0$ and $\tilde{c} > 0$ respectively).

Case $\xi = \log c$

In that case, $\tau = 0$, and from (23) and (24) we get:

$$\begin{aligned} X_T &= x + A_1 T + \sigma_m \beta \widetilde{W}_T + \sigma \widetilde{Z}_T \\ &\quad - A_2 \int_0^T \mathbf{1}_{\left\{\widetilde{W}_t < 0\right\}} dt + \sigma_m \delta \left(\widetilde{W}_T \mathbf{1}_{\{\widetilde{W}_T < 0\}} + \widetilde{L}_T^0 \right) \\ &= x + (A_1 - A_2) T + \sigma \widetilde{Z}_T \\ &\quad + A_2 \int_0^T \mathbf{1}_{\left\{\widetilde{W}_t > 0\right\}} dt + \sigma_m \widetilde{W}_T \left(\beta + \delta \mathbf{1}_{\{\widetilde{W}_T < 0\}} \right) + \sigma_m \delta \widetilde{L}_T^0 \\ &=: \Psi_0(\widetilde{W}_T, \widetilde{L}_T^0, \widetilde{\Gamma}_T^+, \widetilde{Z}_T), \end{aligned} \tag{27}$$

where we have also expressed the occupation time of $(-\infty, 0)$ in terms of the occupation time of $(0, \infty)$ denoted by $\widetilde{\Gamma}_T^+ = \int_0^T \mathbf{1}_{\left\{\widetilde{W}_t > 0\right\}} dt$. It is now clear that the distribution of X_T is given explicitly in terms of the distribution of the triplet $\left(\widetilde{W}_T, \widetilde{L}_T^0, \widetilde{\Gamma}_T^+\right)$ and the independent Gaussian random variable $\widetilde{Z}_T$. The density of the triplet is derived in [13] (see also [14], Section 6.3.C):

$$\begin{aligned} & I\!P \left\{ \widetilde{W}_T \in da,\ \widetilde{L}_T^0 \in db,\ \widetilde{\Gamma}_T^+ \in d\gamma \right\} \\ &= \begin{cases} 2p(T-\gamma; b)\, p(\gamma; a+b) & \text{if } a > 0, b > 0, 0 < \gamma < T, \\ 2p(\gamma; b)\, p(T-\gamma; -a+b) & \text{if } a < 0, b > 0, 0 < \gamma < T, \end{cases} \end{aligned} \tag{28}$$

where $p(u; \cdot)$ is given by (25).

Case $\xi < \log c$

In that case, $\tilde{c} > 0$, and from (23) and (24) we get:

$$\begin{aligned} X_T &= x + A_1 T + \sigma_m \beta \widetilde{W}_T + \sigma \widetilde{Z}_T \\ &\quad - A_2 (\tau \wedge T) - A_2 \int_{\tau \wedge T}^T \mathbf{1}_{\left\{\widetilde{W}_t < \tilde{c}\right\}} dt + \sigma_m \delta \widetilde{W}_{\tau \wedge T} \\ &\quad + \sigma_m \delta \left[\left(\widetilde{W}_T - \tilde{c} \right) \mathbf{1}_{\{\widetilde{W}_T < \tilde{c}\}} - \left(\widetilde{W}_{\tau \wedge T} - \tilde{c} \right) \mathbf{1}_{\{\widetilde{W}_{\tau \wedge T} < \tilde{c}\}} + \widetilde{L}_T^{\tilde{c}} - \widetilde{L}_{\tau \wedge T}^{\tilde{c}} \right]. \end{aligned}$$

– On $\{\tau > T\}$, we have:

$$\begin{aligned} X_T &= x + (A_1 - A_2) T + \sigma_m (\beta + \delta) \widetilde{W}_T + \sigma \widetilde{Z}_T \\ &=: \Psi_{T^+}^-(\widetilde{W}_T, \widetilde{Z}_T), \end{aligned} \tag{29}$$

where the upper index "−" stands for ξ below the level $\log c$, and the lower index T^+ stands for $\tau > T$.
Therefore, in this case, the distribution of X_T is given by the distribution of the independent Gaussian random variable $\widetilde{Z}_T$, and the conditional distribution of $\widetilde{W}_T$ given that $\{\tau > T\}$. From [14] Section 2.8.A, one easily obtains:

$$\begin{aligned} I\!P \left\{ \widetilde{W}_T \in da, \tau > T \right\} &= \frac{1}{\sqrt{2\pi T}} \left(e^{-\frac{a^2}{2T}} - e^{-\frac{(2\tilde{c}-a)^2}{2T}} \right) da, \quad a < \tilde{c}, \\ &=: q_T(a; \tilde{c})\, da. \end{aligned} \tag{30}$$

– On $\{\tau = u\}$ with $u \le T$, we have $\widetilde{W}_u = \tilde{c}$, and X_T is given by

$$\begin{aligned}X_T &= x + (A_1 - A_2)T + \sigma_m(\beta+\delta)\tilde{c} + \sigma_m\beta(\widetilde{W}_T - \widetilde{W}_u) + \sigma\widetilde{Z}_T \\ &+A_2\int_u^T \mathbf{1}_{\left\{\widetilde{W}_t - \widetilde{W}_u > 0\right\}} dt \\ &+\sigma_m\delta\left[\left(\widetilde{W}_T - \widetilde{W}_u\right)\mathbf{1}_{\{\widetilde{W}_T - \widetilde{W}_u < 0\}} + \widetilde{L}_T^{\tilde{c}} - \widetilde{L}_u^{\tilde{c}}\right].\end{aligned}$$

Therefore, in this case, the distribution of X_T is given by the distribution of $\widetilde{Z}_T$ and an independent triplet $\left(B_{T-u}, L^0_{T-u}, \Gamma^+_{T-u}\right)$ formed by the value, the local time at 0, and the occupation time of the positive half-space, at time $T-u$, of a standard Brownian motion B. That is, in distribution:

$$\begin{aligned}X_T &= x + (A_1 - A_2)T + \sigma_m(\beta+\delta)\tilde{c} + \sigma_m B_{T-u}\left(\beta + \delta\,\mathbf{1}_{\{B_{T-u}<0\}}\right) + \sigma\widetilde{Z}_T \\ &+A_2\,\Gamma^+_{T-u} + \sigma_m\delta L^0_{T-u} \\ &=: \Psi^-_{T^-}(B_{T-u}, L^0_{T-u}, \Gamma^+_{T-u}, \widetilde{Z}_T). \end{aligned} \tag{31}$$

The distribution of the triplet $\left(B_{T-u}, L^0_{T-u}, \Gamma^+_{T-u}\right)$ is given by (28) with T replaced by $T-u$.

Case $\xi > \log c$

In that case, $\tilde{c} < 0$, and from (23) and (24) we get:

$$\begin{aligned}X_T &= x + A_1T + \sigma_m\beta\widetilde{W}_T + \sigma\widetilde{Z}_T - A_2\int_{\tau\wedge T}^T \mathbf{1}_{\left\{\widetilde{W}_t < \tilde{c}\right\}} dt \\ &+\sigma_m\delta\left[\left(\widetilde{W}_T - \tilde{c}\right)\mathbf{1}_{\{\widetilde{W}_T < \tilde{c}\}} - \left(\widetilde{W}_{\tau\wedge T} - \tilde{c}\right)\mathbf{1}_{\{\widetilde{W}_{\tau\wedge T} < \tilde{c}\}} + \widetilde{L}_T^{\tilde{c}} - \widetilde{L}_{\tau\wedge T}^{\tilde{c}}\right].\end{aligned}$$

– On $\{\tau > T\}$, we have:

$$\begin{aligned}X_T &= x + A_1T + \beta\sigma_m\widetilde{W}_T + \sigma\widetilde{Z}_T \\ &=: \Psi^+_{T^+}(\widetilde{W}_T, \widetilde{Z}_T),\end{aligned} \tag{32}$$

where $\widetilde{Z}_T$ and $\widetilde{W}_T$ are independent and the distribution of $\widetilde{W}_T$ is given by (30) with $a > \tilde{c}$ in this case.

– On $\{\tau = u\}$ with $u \le T$, we have $\widetilde{W}_u = \tilde{c}$, and X_T is given by

$$\begin{aligned}X_T &= x + A_1T + \sigma_m\beta\tilde{c} + \sigma_m\beta\left(\widetilde{W}_T - \widetilde{W}_u\right) + \sigma\widetilde{Z}_T \\ &-A_2\int_u^T \mathbf{1}_{\left\{\widetilde{W}_t - \widetilde{W}_u < 0\right\}} dt + \sigma_m\delta\left[(\widetilde{W}_T - \widetilde{W}_u)\mathbf{1}_{\{\widetilde{W}_T - \widetilde{W}_u < 0\}} + \widetilde{L}_T^{\tilde{c}} - \widetilde{L}_u^{\tilde{c}}\right].\end{aligned}$$

Therefore, in this case the distribution of X_T is given by the distribution of $\widetilde{Z}_T$ and an independent triplet $\left(B_{T-u}, L^0_{T-u}, \Gamma^-_{T-u}\right)$ formed by the value, the local time at 0, and the occupation time of the negative half-space, at time $T-u$, of a standard Brownian motion B. That is, in distribution:

$$\begin{aligned} X_T \;&=\; x + A_1 T + \sigma_m \beta \tilde{c} + + \sigma_m B_{T-u}\left(\beta + \delta\, \mathbf{1}_{\{B_{T-u}<0\}}\right) + \sigma \widetilde{Z}_T \\ &\quad - A_2\, \Gamma^-_{T-u} + \sigma_m \delta L^0_{T-u} \\ &=: \Psi^+_{T^-}(B_{T-u}, L^0_{T-u}, \Gamma^-_{T-u}, \widetilde{Z}_T). \end{aligned} \tag{33}$$

The distribution of the triplet $\left(B_{T-u}, L^0_{T-u}, \Gamma^-_{T-u}\right)$ is the same as the distribution of the triplet $\left(-B_{T-u}, L^0_{T-u}, T-u-\Gamma^-_{T-u}\right)$, given by (28) with (a,T,γ) replaced by $(-a, T-u, T-u-\gamma)$.

4.3 Pricing Formula

From (11) and the change of measure introduced in Section 4.1, the price at time $t=0$, starting from $(M_0, S_0) = (e^\xi, e^x)$, of a European option with payoff $h(S_T)$ at maturity T is given by

$$\begin{aligned} P_0 &= \mathbb{E}^\star\left\{e^{-rT} h(S_T)\right\} \\ &= \widetilde{\mathbb{E}}\left\{e^{-rT} h(e^{X_T}) \frac{d\mathbb{P}^\star}{d\widetilde{\mathbb{P}}}\right\} \\ &= \widetilde{\mathbb{E}}\left\{e^{-rT} h(e^{X_T}) e^{\theta \widetilde{W}_T - \frac{1}{2}\theta^2 T}\right\} \\ &= e^{-rT} e^{-\frac{1}{2}\theta^2 T}\, \widetilde{\mathbb{E}}\left\{h(e^{X_T}) e^{\theta \widetilde{W}_T}\right\}. \end{aligned}$$

Decomposing the expectation on $\{\tau \le T\}$ and $\{\tau > T\}$, one obtains:

$$\begin{aligned} P_0 = e^{-(r+\frac{1}{2}\theta^2)T} &\left[\int_0^T \widetilde{\mathbb{E}}\left\{h(e^{X_T}) e^{\theta(\widetilde{W}_T - \widetilde{W}_u) + \theta \tilde{c}} \mid \tau = u\right\} p(u;\tilde{c})\, du \right. \\ &\left. + \widetilde{\mathbb{E}}\left\{h(e^{X_T}) e^{\theta \widetilde{W}_T} \mathbf{1}_{\{\tau > T\}}\right\}\right], \end{aligned}$$

where the density $p(u;\tilde{c})$ is given by (25) and the degenerate case $\tilde{c} = 0$ (or $\xi = \log c$) corresponds to $p(u;0)du = \delta_0(du)$.

Using the conditional distributions derived in Section 4.2 and denoting by $n_T(z)$ the $\mathcal{N}(0,T)$ density, we get:

$$\begin{aligned} P_0 = e^{-(r+\frac{1}{2}\theta^2)T} &\left[e^{\theta \tilde{c}} \int_{-\infty}^{\infty} \int_0^T \int_0^{T-u} \int_0^\infty \int_{-\infty}^{\infty} h(e^{\Psi^\pm_{T^-}(a,b,\gamma,z)}) e^{\theta a} \right. \\ &\qquad \times g(a,b,\gamma; T-u)\, da\, db\, d\gamma\, p(u;\tilde{c}) du\, n_T(z) dz \\ &\left. + \left(\int_{-\infty}^{\infty} \int_{D^\pm} h(e^{\Psi^\pm_{T^+}(a,z)}) e^{\theta a} q_T(a;\tilde{c}) da\, n_T(z) dz\right)\right], \end{aligned}$$

where the densities $p(u;\tilde{c})$ and $q_T(a;\tilde{c})$ are given by (25) and (30) respectively, the functions $\Psi^\pm_{T^\pm}$ are given by $(29, 31, 32, 33)$, $g(a,b,\gamma,T-u)$ represents the density of the triplet involved in each case, and the domain $D^\pm$ is $(-\infty, \tilde{c})$ (resp. $(\tilde{c}, \infty)$) if $\tilde{c} > 0$ (resp. $\tilde{c} < 0$).

Using Fubini between u and γ, the fact that $\Psi^{\pm}_{T-}$ is independent of u, and the convolution relation

$$\begin{aligned}&\int_0^{T-\gamma} g(a,b,\gamma;T-u)p(u;\tilde{c})du\\&=\begin{cases}2p(\gamma;a+b)\,p(T-\gamma;b+|\tilde{c}|) & \text{if } a>0,\\ 2p(\gamma;b)\,p(T-\gamma;-a+b+|\tilde{c}|) & \text{if } a<0,\end{cases}\\&=: G(a,b,\gamma;T),\end{aligned}\tag{34}$$

we arrive at the pricing formula:

$$\begin{aligned}P_0=e^{-(r+\frac{1}{2}\theta^2)T}\Bigg[&e^{\theta\tilde{c}}\int_{-\infty}^{\infty}\int_0^T\int_0^{\infty}\int_{-\infty}^{\infty} h(e^{\Psi^{\pm}_{T-}(a,b,\gamma,z)})e^{\theta a}\\&\qquad\qquad\times G(a,b,\gamma;T)\,da\,db\,d\gamma\,n_T(z)dz\\&+\left(\int_{-\infty}^{\infty}\int_{D^{\pm}} h(e^{\Psi^{\pm}_{T+}(a,z)})e^{\theta a}q_T(a;\tilde{c})da\,n_T(z)dz\right)\Bigg].\end{aligned}\tag{35}$$

Numerically, one has to consider separately the two cases $\tilde{c}>0$ and $\tilde{c}<0$ which determine the choice of functions $\Psi^{\pm}$ given by $(29,31,32,33)$. One also must be careful in each case to decompose the integral with respect to a over $a>0$ and $a<0$ since the function G given by (34) depends on it. Recall that the densities p and q_T are given by (25) and (30) respectively, and

$$D^{\pm}=\begin{cases}(-\infty,\tilde{c}) & \text{if } \tilde{c}>0,\\(\tilde{c},\infty) & \text{if } \tilde{c}<0.\end{cases}$$

Remarks

1. If $\tilde{c}=0$ (equivalently $\xi=\log c$ or $M_0=c$), then $\tau=0$ and the pricing formula (35) reduces to:

$$\begin{aligned}P_0&=e^{-(r+\frac{1}{2}\theta^2)T}\\&\times\int_{-\infty}^{\infty}\int_0^T\int_0^{\infty}\int_{-\infty}^{\infty} h(e^{\Psi_0(a,b,\gamma,z)})e^{\theta a}G(a,b,\gamma;T)\,da\,db\,d\gamma\,n_T(z)dz,\end{aligned}\tag{36}$$

   where the function $\Psi_0(a,b,\gamma,z)$ was introduced in (27).
2. If $\delta=0$ (for any value of $\tilde{c}$), the pricing formula (35) reduces to the Black–Scholes pricing formula with square-volatility $\sigma_m^2\beta^2+\sigma^2$ as it should be since that case corresponds to the linear CAPM model described in Section 2.1. This reduction is actually not straightforward. One needs to remark that $A_2=0$, and $a+\tilde{c}$ recombines, so that $\Psi_{\pm}$ becomes $x+A_1T+\sigma_m w+\sigma z$.
3. For hedging purposes, one needs to compute the *Deltas* given by (12). They are obtained by taking derivatives of the option price (35) with respect to ξ and x. Note that x appears only in the payoff function h (specifically in the functions $\Psi^{\pm}$), while ξ appears also in the density functions G and q_T, and in the domain of integration, making the corresponding formula more complicated and numerically involved.

5 Market Volatility

In the model described by the dynamics (1)–(2) under the physical measure, the market volatility σ_m is assumed to be constant. This is indeed not realistic both from the point of view of the market returns distribution and the market skews of implied volatilities. In this section, we propose to introduce stochastic volatility in the market model. We will follow the approach taken in [7] and also used recently in [6] in the CAPM context. In this generalized CAPM model, the market volatility is driven by a fast mean-reverting factor according to

$$\frac{dM_t}{M_t} = \mu dt + f(Y_t)dW_t, \tag{37}$$

$$dY_t = \frac{1}{\epsilon}(m - Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}dW_t^y\,, \tag{38}$$

$$\frac{dS_t}{S_t} = \beta(M_t)\frac{dM_t}{M_t} + \sigma dZ_t\,, \tag{39}$$

where Y_t is an Ornstein-Uhlenbeck (OU) process with large rate of mean-reversion $1/\varepsilon$, that is ε is a small positive parameter, and which admits the Gaussian invariant distribution $\mathcal{N}(m, \nu^2)$. The function f is positive increasing, which can be assumed smooth bounded and bounded away from zero for technical simplicity. The Brownian motions W_t and W_t^y are correlated according to $d\langle W, W^y\rangle_t = \rho\, dt$ where ρ is constant with $|\rho| < 1$. The equation for the asset price S_t is unchanged, and the Brownian motion Z_t is independent of W_t and W_t^y. As before, the function $\beta(M)$ is given by (5).

5.1 Risk-Neutral Pricing Measure

As before, the market and the asset being both tradable, their discounted prices need to be martingales under a risk-neutral pricing measure. In order to achieve that, we first write

$$W_t^y = \rho\, dW_t + \sqrt{1-\rho^2}\, dW_t^\perp,$$

with now $(W_t, W_t^\perp, Z_t)$ being three independent standard Brownian motions, and then we rewrite the system (37, 38, 39) as:

$$\begin{aligned}
\frac{dM_t}{M_t} &= rdt + f(Y_t)\left(dW_t + \frac{\mu - r}{f(Y_t)}dt\right),\\
dY_t &= \frac{1}{\epsilon}(m - Y_t)dt - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(Y_t)dt\\
&\quad + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\left[\rho\left(dW_t + \frac{\mu - r}{f(Y_t)}dt\right) + \sqrt{1-\rho^2}\left(dW_t^\perp + \gamma(Y_t)dt\right)\right],\\
\frac{dS_t}{S_t} &= rdt + \beta(M_t)f(Y_t)\left(dW_t + \frac{\mu - r}{f(Y_t)}dt\right) + \sigma\left(dZ_t + \frac{(\beta - 1)r}{\sigma}dt\right),
\end{aligned}$$

where $\gamma(Y_t)$ is a market price of volatility risk, which we suppose to depend on Y_t only, and we define

$$\Lambda(Y_t) = \rho\,\frac{\mu - r}{f(Y_t)} + \sqrt{1-\rho^2}\,\gamma(Y_t).$$

Setting

$$dW_t^* = dW_t + \frac{\mu - r}{f(Y_t)}dt\,,$$
$$dW_t^{\perp *} = dW_t^{\perp} + \gamma(Y_t)dt\,,$$
$$dZ_t^* = dZ_t + \frac{(\beta - 1)r}{\sigma}dt\,,$$

by Girsanov theorem, there is an equivalent probability $I\!P^{\star(\gamma)}$ such that $(W_t^*, W_t^{\perp *}, Z_t^*)$ are independent standard Brownian motions under $I\!P^{\star(\gamma)}$, called the pricing equivalent martingale measure and determined by the market price of volatility risk γ. We assume here that both the Sharpe ratio $\frac{\mu - r}{f(Y_t)}$ and $\gamma(Y_t)$ are bounded, which, depending on the choice of function f, may require that μ depends on Y_t. Finally, under $I\!P^{\star(\gamma)}$, the dynamics (37, 38, 39) becomes:

$$\frac{dM_t}{M_t} = rdt + f(Y_t)dW_t^*, \tag{40}$$
$$dY_t = \frac{1}{\epsilon}(m - Y_t)dt - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\Lambda(Y_t)dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}dW_t^{y*}\,, \tag{41}$$
$$W_t^{y*} = \rho W_t^* + \sqrt{1-\rho^2}\,W_t^{\perp *}\,,$$
$$\frac{dS_t}{S_t} = rdt + \beta(M_t)f(Y_t)dW_t^* + \sigma dZ_t^*\,. \tag{42}$$

In what follows, we take the point of view that by pricing options on the index M and on the particular asset S, the market is "completing" itself and indirectly choosing the market price of volatility risk γ.

5.2 Market Option Prices

In looking at option prices on the market index we only focus on the autonomous evolution of (M_t, Y_t) described by equations $(40, 41)$ under the risk-neutral pricing measure. A singular perturbation approach to option pricing on the model described in $(40, 41)$ was developed in [7]. Here we use this approximation technique but with an additional parameter reduction to allow for parameter estimation using option data only. The details of this derivation can be found in the appendix A of [6], and lead to the following price approximation for call options on the market. Let $P^\epsilon = P(t, M, y; T, K)$ denote the price of a European call option written on the market index M, with maturity T and strike K, evaluated at time $t < T$ with current values $M_t = M$ and $Y_t = y$, where we explicitly show the dependence on the small volatility mean-reversion time ε. Then, we have the following approximation

$$P^\epsilon \sim P^* + (T-t)V_3^\epsilon M\frac{\partial}{\partial M}\left(M^2\frac{\partial^2 P^*}{\partial M^2}\right), \tag{43}$$

where $P^* = P_{BS}(\sigma^*)$ is the corresponding Black Scholes call price with constant volatility equal to the *adjusted effective volatility* σ^*. Here

$$\sigma^* = \sqrt{\bar{\sigma}^2 + 2V_2^\varepsilon}, \tag{44}$$

where $\bar{\sigma}$ is the *effective volatility* defined by

$$\bar{\sigma}^2 = \langle f^2 \rangle \equiv \int f(y)^2 \frac{1}{\sqrt{2\pi}\nu} e^{-\frac{(y-m)^2}{2\nu^2}} dy, \tag{45}$$

with the average being taken with respect to the invariant distribution of the OU process Y. The small parameter V_2^ε – which is proportional to $\sqrt{\epsilon}$ and arises in the asymptotic analysis – accounts for a volatility adjustment due to the market price of volatility risk. The small parameter V_3^ε appearing in (43) is proportional to $\sqrt{\epsilon}$ and to the correlation coefficient ρ, and it accounts for the skew of implied volatility. It is shown in [8] that the accuracy of the approximation (43) is $\mathcal{O}(\varepsilon \log|\varepsilon|)$.

5.3 Market Implied Volatilities

Following [7] and [6], we introduce the *Log-Moneyness to Maturity Ratio* ($LMMR$)

$$LMMR = \frac{log(K/x)}{T}, \tag{46}$$

and for calibration purposes, we use the *affine LMMR formula*

$$I \sim b^* + a^\epsilon LMMR\,, \tag{47}$$

with the intercept b^* and the slope a^ε to be fitted to the skew of option data. We then use the estimators derived in [6]:

$$\sigma^* \sim b^* + a^\epsilon (r - \frac{b^{*2}}{2}) \equiv \widehat{\sigma^*}, \tag{48}$$

$$V_3^\epsilon = a^\epsilon \sigma^{*3} \sim a^\epsilon b^{*3} \equiv \widehat{V_3^\varepsilon}. \tag{49}$$

5.4 Effect on Asset Options

Indeed, the introduction of market stochastic volatility in the model has also an effect on the dynamics of the asset price (42) where the constant volatility σ_m in (10) has been replaced by $f(Y_t)$. However, the asymptotic analysis performed on asset option prices as in [6] reveals that, to the leading order, these prices are given by $(9, 10, 11)$ with σ_m replaced by σ^* given by (44) and calibrated on market skews using (48). Therefore, in what follows, we simply use our pricing formula (35) with σ_m replaced by σ^*.

Note that one could derive a formula for the first order correction due to market stochastic volatility which would involve the parameter V_3^ε appearing in (43) and calibrated using (49). However, this formula is quite complicated and numerically involved, and here, for the purpose of calibration of β and δ (the main goal of this paper), we restrict ourselves to the leading order.

6 Numerical Results and Calibration

6.1 Asset Skews of Implied Volatilities

This section presents implied volatility skews produced from the Stressed-Beta model. Additionally, a sensitivity analysis examines how the implied volatility skew responds to a change in the model parameter δ or the initial value M_0. For each of these studies, a European call option is priced, and the following parameter settings are used: $c = 1000$, $S_0 = 100$, $r = 0.01$, $\beta = 1$, $\sigma_m = 0.30$, $\sigma = 0.01$, and $T = 1$. Strike prices of $70, 71, \ldots, 150$ are used to build the implied volatility curves.

For the sensitivity analysis with respect to δ, we consider the case when $M_0 = c$, that is when the first passage time occurs at the start. In this case, the log-stock price X_T at terminal time is given by (27) and call options are priced according to the simplified formula (36).

For the sensitivity with respect to M_0, we examine the implied volatility skew produced from a starting market price that is (1) far below the boundary c, (2) below the boundary, (3) at the boundary, (4) above the boundary, and (5) far above the boundary. The value of δ is set to 0.5 for this analysis.

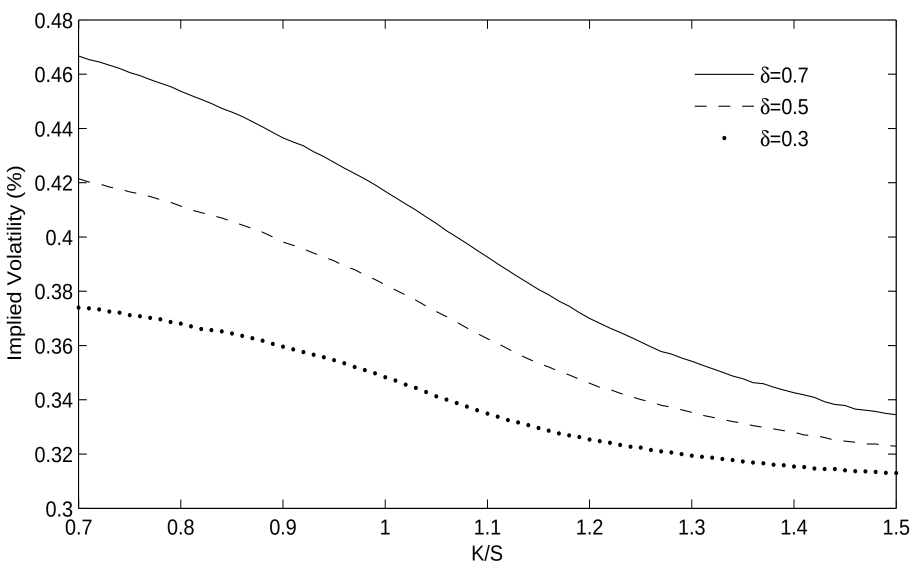

Fig. 1 Implied volatility versus moneyness of a European call option for different values of δ. The level c is fixed at 1000, $S_0 = 100$, $T = 1$, $r = 0.01$, $\beta = 1$, $\sigma_m = 0.30$, and $\sigma = 0.01$. The starting market price M_0 is set equal to $c = 1000$.

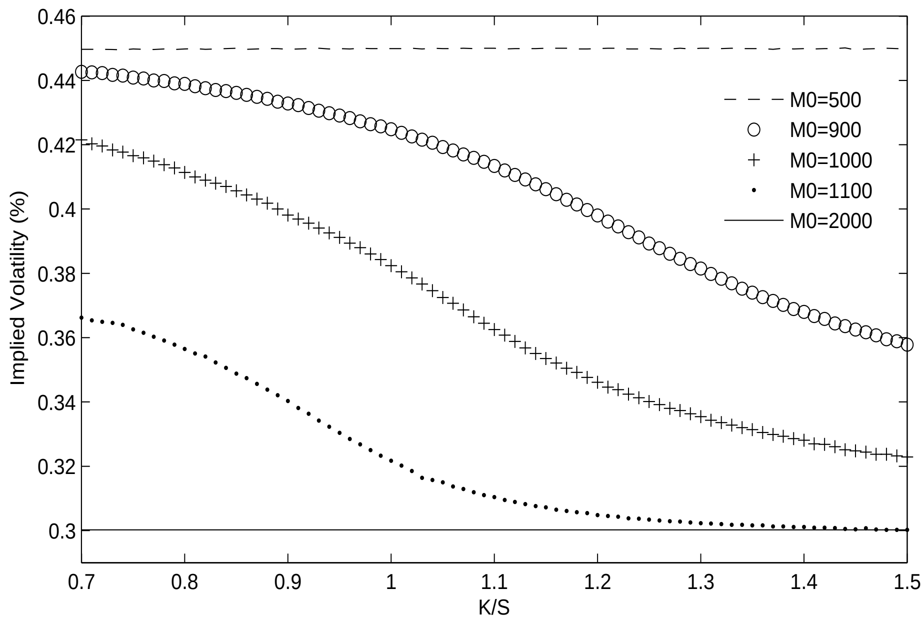

Fig. 2 Implied volatility versus moneyness of a European call option for different values of the starting market price, M_0. The level c is fixed at 1000, $S_0 = 100$, $T = 1$, $r = 0.01$, $\beta = 1$, $\delta = 0.5$, $\sigma_m = 0.30$, and $\sigma = 0.01$.

Figure 1 illustrates that for each setting of δ, the implied volatility curve exhibits a skew. Implied volatility is an increasing function of δ for each moneyness, and the slope of the volatility skew is an increasing function of δ as well.

Next, consider the results from the M_0 study, shown in Figure 2. When the market price starts far below the boundary ($M_0 = 500$), it is very unlikely that it will cross above c, and thus asset volatility will most likely remain at the high setting through expiry. This is the Black-Scholes model with volatility $\sqrt{(\beta+\delta)^2\sigma_m^2 + \sigma^2} = 0.4501$, which is an upper bound. Note that the implied volatility curve for $M_0 = 500$ is approximately equal to this value. As M_0 decreases further, the Stressed-Beta price of the option will converge to the Black-Scholes price with high volatility. Next, consider the case when the market price starts far above the boundary ($M_0 = 2000$). Now, it is very unlikely that the market price will cross below c, and so the asset volatility will most likely remain at the low setting through expiry. This is the Black-Scholes model with volatility $\sqrt{\beta^2\sigma_m^2 + \sigma^2} = 0.3002$. The implied volatility curve for $M_0 = 2000$ is equal to this value, which forms the lower bound on implied volatility. For a value of M_0 closer to c, the implied volatility curve exhibits a skew, and it falls within the interval [0.3002, 0.4501]. This skew, in fact, will be greatest at $M_0 = c$, and will flatten as M_0 is moved away from the boundary.

6.2 Calibration to Data

Next, we calibrate the Stressed-Beta model to Amgen options with October 2009 expiry. We consider options with $LMMR$ (46) between -1 and 1, using closing mid-prices as of May 26, 2009. For simplicity, here we set to zero the asset-specific volatility σ. The market volatility σ^* is estimated from option data on the S&P 500 Index. We use options with strike prices between 800 and 1200, and expiry closest to October 2009 (these are the September-expiry options). Next, the affine $LMMR$ formula (47) is fit to the skew of the option data as shown in Figure 3. Given the slope, intercept, and 4-month-interpolated $LIBOR$ rate, $\sigma^* = 0.2549$ is obtained from (48). There are now three free parameters: c, β, and δ. These parameters are set to the values which minimize the sum of squared errors between the option model prices and market prices. As this procedure is computationally intensive, we limit the search across c to the following set of values: 900, 925, 950, 975, 1000. The closing level of the S&P 500 Index as of May 26, 2009 was 910.33.

Figure 4 shows the fit of our model to this set of option data. The estimated parameters are: $c = 925$, $\beta = 1.17$, and $\delta = 0.65$.

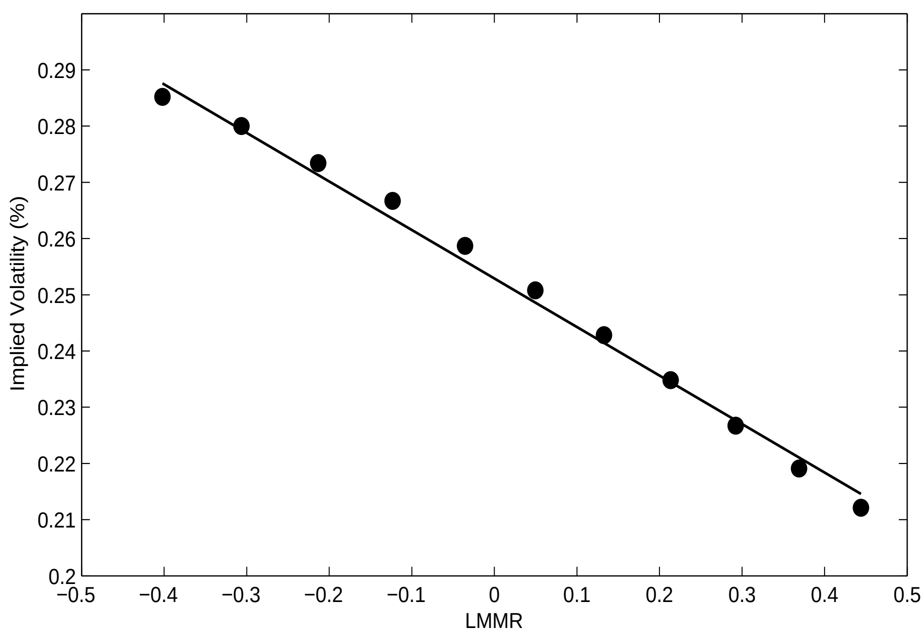

Fig. 3 Affine $LMMR$ fit to S&P 500 Index options expiring September 18, 2009 based on May 26, 2009 market prices. The estimated slope is $a^\varepsilon = -0.086$, the estimated intercept is $b^* = 0.25$, and the R-squared is 0.9741.

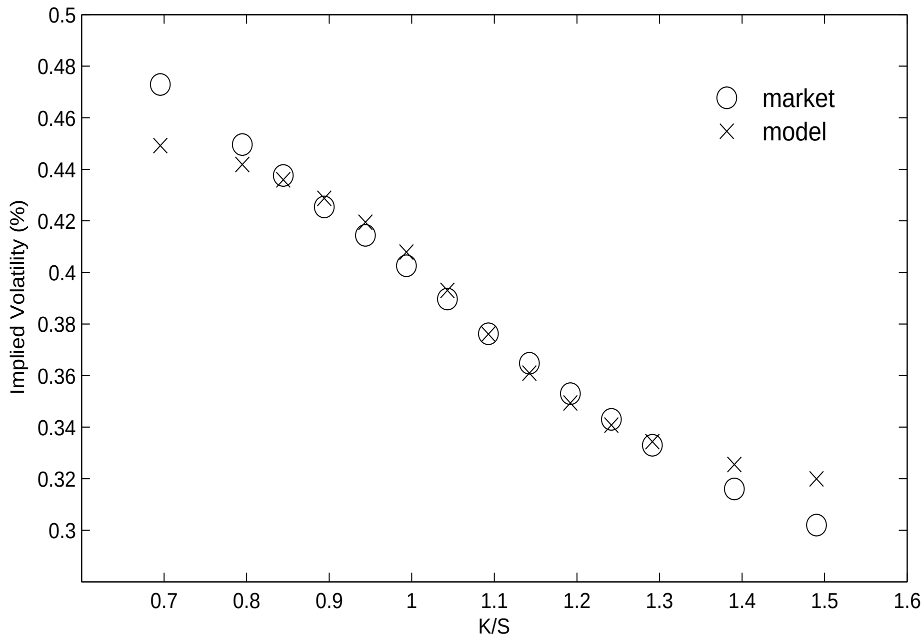

Fig. 4 Volatility skews for Amgen call options expiring October 2009 based on (1) market prices as of May 26, 2009 and (2) the Stressed-Beta model. The model parameters are the following: $c = 925$, $\beta = 1.17$, $\delta = 0.65$, $\sigma^* = 0.2549$, and $\sigma = 0$.

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