

# Portfolio Optimization Under a Stressed-Beta Model

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## Abstract

This paper presents a closed-form solution to the portfolio optimization problem where an agent wishes to maximize expected terminal wealth, trading continuously between a risk-free bond and a risky stock following Stressed-Beta dynamics specified in Fouque and Tashman (2010). The agent has a finite horizon and a utility of the Constant Relative Risk Aversion type. The model for stock dynamics is an extension of the Capital Asset Pricing Model (CAPM); it is expressed in continuous-time, and the slope relating excess stock returns to excess market returns switches between two values. This mechanism reflects the fact that the slope may steepen during periods of stress, a feature which has been demonstrated to better model stock dynamics than CAPM. An asymptotic expansion technique is used to write an explicit expression for the agent's optimal strategy. Lastly, the optimization approach is illustrated with market data, and its outperformance versus the Merton approach is demonstrated.

**Keywords** portfolio optimization, Stressed-Beta model, CAPM, regime-switching, calibration

## 1 Introduction

This paper is a contribution to the theory of portfolio optimization. An agent trades in a riskless bond and a risky stock, and the innovation is the solution to the portfolio problem given the model specification for the stock. The bond price is deterministic, while the stock price follows the Stressed-Beta model first appearing in Fouque and Tashman (2010). This model is a continuous-time Capital Asset Pricing Model (CAPM) with the linearity assumption relaxed. In particular, the slope coefficient can switch between two values depending on the market level. These two slopes capture the phenomenon of distinct relationships between the excess return of the stock and the excess return of the market in normal and stress states.

The agent notes the market level when executing his strategy, but does not trade in the market index. The investor's utility is of the Constant Relative Risk Aversion (CRRA) type, and the goal is to specify the trading strategy which maximizes the expected terminal utility. The agent is allowed to short the stock, and to buy or sell fractional shares. The only restriction on the trading strategy is that it must be self-financing, and the total capital must be invested at all times.

The value function in this portfolio problem solves a nonlinear partial differential equation which cannot be reduced to a linear equation using the power transformation appearing in Zariphopoulou (2001). Instead, we apply an asymptotic expansion, expressing the solution in terms of powers of the stock-specific variance. Asymptotic expansion techniques have been widely used in finance (see for example Fouque et al. (2000), Fouque et al. (2003), Fouque and Kollman (2010), Fouque and Tashman (2010)). Given this technique, we are able to

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derive a closed-form expression for the optimal strategy and the value function. Interestingly, the formula for the optimal strategy consists of two parts: the optimal strategy for the Merton problem (with the constant slope replaced by the slope-switching mechanism), and a correction to the Merton solution which accounts for the slope-switching mechanism.

The paper is organized as follows: in Section 2, we review the Stressed-Beta model. Section 3 discusses portfolio optimization, starting with the Merton problem, and then extending to the case when the risky asset follows Stressed-Beta dynamics. Details of the extension appear in the Appendix. Section 4 presents numerical results. Section 5 concludes and mentions future directions of research.

## 2 The Stressed-Beta Model

Consider the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , and all Brownian motions are adapted to the filtration  $\{\mathcal{F}\}_{0 \leq t \leq T}$  where  $T$  is a fixed time. The probability  $\mathbb{P}$  is the physical measure. At time  $t$ , an agent faces the stock market with level  $M_t$ , the risky asset with price  $S_t$  (the *stock*), and the riskless asset with constant interest rate  $r > 0$  (the *bond*). We consider a model for the excess return of the stock expressed as the excess return of the market multiplied by a *slope* plus a noise term. This slope will be two-valued, depending on the level of the market. As such, it is a function which we call the *slope-switching mechanism*, and it has the following definition:

$$\beta(M_t) = \beta + \delta \mathbf{1}_{\{M_t < c\}}, \quad (1)$$

where  $\mathbf{1}_A$  denotes the indicator function of the set  $A$ . This mechanism has the following interpretation: when the market is above a given level  $c > 0$  (this may be the case of a *normal* regime), the slope takes the value  $\beta$ , but when the market is below this level (the *stress* regime), the slope switches to the value  $\beta + \delta$ . Generally,  $\beta$  and  $\delta$  will both be positive, thus the slope will be steeper when the market is below  $c$ . The market level  $M_t$  and the stock price  $S_t$  evolve as follows:

$$\frac{dM_t}{M_t} = \mu dt + \Sigma dW_t, \quad (2)$$

$$\frac{dS_t}{S_t} = r dt + \beta(M_t) \left( \frac{dM_t}{M_t} - r dt \right) + \sigma dZ_t, \quad (3)$$

where the volatilities  $\Sigma$  and  $\sigma$  are constant scalars, and  $W_t$  and  $Z_t$  are independent Brownian motions. Substituting the market equation (2) into the stock equation (3) yields

$$\frac{dS_t}{S_t} = r(1 - \beta(M_t)) dt + \beta(M_t) \mu dt + \beta(M_t) \Sigma dW_t + \sigma dZ_t. \quad (4)$$

Given this model for the stock dynamics, as well as parameter estimates, we can apply it in a portfolio optimization framework.

**Remark 2.1** For the case  $\sigma = 0$ , there is only one source of uncertainty in the model, namely the Brownian motion  $W_t$  appearing in the market and stock dynamics. Furthermore, since the stock volatility is a function of  $W_t$  through  $\beta(M_t)$ , (4) has a form which is similar to the complete model with stochastic volatility in Hobson and Rogers (1998).

### 3 The Portfolio Optimization Problem

An agent observes the level of the stock market, but will invest only in the stock or the bond. The individual has a finite horizon  $T$  and a constant relative risk averse (CRRA) utility function  $\mathcal{U}(y) = y^\lambda/\lambda$ , where  $y$  is the level of wealth and  $\lambda \in (0, 1)$  is the risk aversion parameter. The agent's objective is to maximize expected terminal utility. Let  $\pi_t$  and  $1 - \pi_t$  represent the fraction of wealth invested in the stock and bond, respectively, at time  $t$ . Denote by  $\mathcal{A}$  the set of admissible strategies. The following constraints form the space  $\mathcal{A}$ :

- Budget constraint: All capital must be invested at all times  $t \in [0, T]$
- Only self-financing trading strategies can be used

We will consider two optimization problems, the first being the Merton portfolio problem (that is, the stock follows a geometric Brownian motion), and the second assuming that the stock follows Stressed-Beta dynamics. Specifically, we wish to know the agent's optimal trading strategy  $\pi_t^*$  in each case.

#### 3.1 Merton Problem

We assume a model of the same form as (2,3) but where  $\beta(M_t) = \beta^H$ , a constant. The superscript  $H$  represents the fact that this parameter will be estimated from historical data. This model has the form of CAPM but in continuous time. The investor's optimal strategy is the following:

$$\pi^* = \frac{(\mu - r)\beta^H}{(1 - \lambda)((\beta^H)^2\Sigma^2 + \sigma^2)}. \quad (5)$$

Note that the optimal fraction of wealth invested in the stock is a constant which depends only on the model parameters and the level of risk aversion. This equation will appear again in similar form when we study Stressed-Beta optimization.

#### 3.2 Stressed-Beta Problem

We continue the development of Stressed-Beta optimization. That is, we assume that the risky stock follows (4). Let  $a_t$  and  $b_t$  represent the units of the stock and bond held by the agent at time  $t$ , respectively. Then the wealth equation at time  $t$  is the following:

$$Y_t = a_t S_t + b_t e^{rt}.$$

Since the portfolio is self-financing,

$$dY_t = \frac{\pi_t Y_t}{S_t} dS_t + \frac{(1 - \pi_t) Y_t}{e^{rt}} r e^{rt} dt. \quad (6)$$

where we have used  $a_t = \pi_t Y_t / S_t$  and  $b_t = (1 - \pi_t) Y_t / e^{rt}$ . Substituting the stock equation (4) into (6) yields

$$\begin{aligned} dY_t &= \pi_t Y_t [r(1 - \beta(M_t)) dt + \beta(M_t) \mu dt + \beta(M_t) \Sigma dW_t + \sigma dZ_t] + (1 - \pi_t) r Y_t dt, \\ &= Y_t [(r + (\mu - r)\beta(M_t)\pi_t) dt + \beta(M_t) \Sigma \pi_t dW_t + \sigma \pi_t dZ_t]. \end{aligned}$$

Notice that there is no dependency on  $S$ , but there is a dependency on  $M$  through the slope-switching mechanism. Thus, we are dealing with a two-dimensional Markov process  $(Y, M)$ .

Define the expected terminal utility given starting conditions:

$$J(t, y, m, \pi) = \mathbb{E} \{ Y_T^\lambda / \lambda \mid Y_t = y, M_t = m \}, \quad (7)$$

where  $y$  represents the initial endowment. Define the value function

$$V(t, y, m) = \sup_{\pi \in \mathcal{A}} J(t, y, m, \pi). \quad (8)$$

Assuming  $V(t, y, m) \in C_0^{1,2,2}$ , it satisfies the stochastic Hamilton-Jacobi-Bellman (HJB) partial differential equation

$$\begin{aligned} V_t + \sup_{\pi \in \mathcal{A}} \{ & y[r + (\mu - r)\beta(m)\pi]V_y + m\mu V_m + \frac{1}{2}y^2\pi^2(\beta^2(m)\Sigma^2 + \sigma^2)V_{yy} \\ & + \frac{1}{2}m^2\Sigma^2V_{mm} + my\beta(m)\Sigma^2\pi V_{my} \} = 0, \end{aligned} \quad (9)$$

with terminal condition  $V(T, y, m) = y^\lambda / \lambda$ . As usual, the validity of the HJB equation relies on a verification step assuming regularity of the function  $V(t, y, m)$ . Since we specialize below to the case of power utility functions, and after separation of variables we are back to a classical one-dimensional problem with respect to the  $m$ -variable.

Write the value function in separable form

$$V(t, y, m) = \frac{y^\lambda}{\lambda} u(t, m). \quad (10)$$

Substituting (10) into the HJB equation (9) gives

$$\begin{aligned} u_t + \sup_{\pi \in \mathcal{A}} \{ & [r + (\mu - r)\beta(m)\pi]\lambda u + m\mu u_m + \frac{1}{2}\pi^2(\beta^2(m)\Sigma^2 + \sigma^2)\lambda(\lambda - 1)u \\ & + \frac{1}{2}m^2\Sigma^2u_{mm} + m\beta(m)\Sigma^2\pi\lambda u_m \} = 0, \end{aligned} \quad (11)$$

with terminal condition  $u(T, m) = 1$ . The quadratic form in  $\pi$  attains a maximum at

$$\pi^*(m) = \frac{\beta(m)\Sigma^2}{(1 - \lambda)(\beta^2(m)\Sigma^2 + \sigma^2)} \frac{mu_m}{u} + \frac{(\mu - r)\beta(m)}{(1 - \lambda)(\beta^2(m)\Sigma^2 + \sigma^2)}. \quad (12)$$

Notice the similarity between the second term in (12) and equation (5), where the only difference is that in (12) the slope is a function of the market level. Given this structure, we refer to the second term as the Merton solution, while the first term is a correction to the Merton solution.

Substituting (12) into (11) and simplifying produces the following partial differential equation in  $u$ :

$$\begin{aligned}
u_t &+ \frac{1}{2}\Sigma^2 m^2 u_{mm} + \left( m\mu + \frac{m\lambda(\mu - r)}{(1 - \lambda)\left(1 + \frac{\sigma^2}{\beta^2(m)\Sigma^2}\right)} \right) u_m \\
&+ \left( r\lambda + \frac{\lambda(\mu - r)^2}{2(1 - \lambda)\Sigma^2\left(1 + \frac{\sigma^2}{\beta^2(m)\Sigma^2}\right)} \right) u + \frac{\lambda m^2 \Sigma^2}{2(1 - \lambda)\left(1 + \frac{\sigma^2}{\beta^2(m)\Sigma^2}\right)} \frac{u_m^2}{u} = 0.
\end{aligned} \tag{13}$$

The form of this equation motivates an approach taken in Zariphopoulou (2001) whereby the solution was posed in the form of  $u^\psi$ , with  $\psi$  referred to as the distortion power. In this case, the approach doesn't work, as the value of  $\psi$  needed to cause the nonlinear term to vanish depends on the market level. As an alternative approach, we consider an asymptotic expansion, expressing the solution in terms of powers of  $\sigma^2$  as

$$u = u_0 + \sigma^2 u_1 + \mathcal{O}(\sigma^4), \tag{14}$$

and using the first two terms  $u_0 + \sigma^2 u_1$  to approximate the solution.

We now present the main results (see the Appendix for the full derivation). Setting  $t = 0$ , the solution to the zero-order equation has the form

$$u_0(0, m) = e^{RT}, \tag{15}$$

where the constant rate  $R$  is given by

$$R = r\lambda + \frac{\lambda(\mu - r)^2}{2(1 - \lambda)\Sigma^2}.$$

The correction term is  $u_1(0, m) = w_1(0, m)e^{RT}$ , where  $w_1(0, m)$  is computed as (32), (34), or (35) given below, depending on whether  $m = c$ ,  $m < c$ , or  $m > c$ , respectively.

Next we expand in the optimal fraction of wealth  $\pi^*(m)$  and substitute the approximation  $u = u_0 + \sigma^2 u_1$  to write

$$\pi^*(m) = \pi_0^*(m) + \sigma^2 \pi_1^*(m) + \mathcal{O}(\sigma^4),$$

where

$$\begin{aligned}
\pi_0^*(m) &= \frac{\mu - r}{(1 - \lambda)\beta(m)\Sigma^2}, \\
\pi_1^*(m) &= \frac{1}{\beta(m)(1 - \lambda)} \frac{m u_{1m}}{u_0} - \frac{\mu - r}{(1 - \lambda)\beta^3(m)\Sigma^4}.
\end{aligned}$$

The first-order term  $\pi_0^*(m)$  represents the optimal fraction of wealth invested in the stock for the case  $\sigma = 0$ . The term  $\pi_1^*(m)$  represents the correction for the case  $\sigma$  nonzero. Given expressions for  $u_0$  and  $u_{1m}$  (the latter quantity is developed in the Appendix) and ignoring

higher-order terms,  $\pi^*(m)$  may be estimated as

$$\pi^*(m) = \pi_0^*(m) + \sigma^2 \pi_1^*(m). \quad (16)$$

## 4 Numerical Results

Since a goal of the Stressed-Beta model is to better capture stock dynamics during market upheaval, we provide an illustration during a major stress period: the wake of the Lehman Brothers failure. We will consider portfolios optimized according to the Merton framework and the Stressed-Beta framework. The stock used as an example is Cisco Systems (ticker: CSCO), a member of the Dow Jones Industrial Average as well as the S&P 500. Each portfolio will begin September 26, 2008 and end one month later. This start date was selected because it marked the beginning of a major decline in Cisco's stock price, and it also coincided with high trading volume of short-dated call options used for model calibration. The portfolios are rebalanced daily.

There are some parameters which need to be estimated from historical data; the period used for estimation is July 1, 2008 – September 26, 2008. For the Merton strategy, we will need the drift and volatility of the market,  $\mu$  and  $\Sigma$ , respectively. In this analysis, the S&P 500 is used as a proxy for the market. We also need the slope parameter  $\beta^H$  and the stock-specific volatility  $\sigma$ . The former parameter is estimated from a regression of the excess stock returns on the excess market returns, while the latter parameter is estimated as the root mean squared error. The risk-free rate used throughout the analysis is one-month *LIBOR*. The risk-aversion parameter  $\lambda$  is set to 0.5.

Our application of the Stressed-Beta model uses historically-estimated parameters ( $\mu$ ,  $\Sigma$ ,  $\sigma$ ), as well as forward-looking parameters calibrated from option data ( $c$ ,  $\beta$ ,  $\delta$ ). The calibration uses call options with an expiry of January 2009. Please refer to Fouque and Tashman (2010) for details on calibrating the Stressed-Beta model. The parameter estimates are shown in Table 1.

The optimal fraction of wealth invested in the stock is calculated as (5) and (16) for the Merton strategy and Stressed-Beta strategy, respectively. This fraction is constant for the Merton strategy, and its value is  $-3.12$ . For the Stressed-Beta strategy, the fraction depends on the level of the market only through the slope-switching mechanism  $\beta(m)$  (although an  $m$  appears explicitly in (16), it cancels with an  $m$  in the term  $u_{1m}$ ). For the period of analysis,  $\beta(m)$  is constant at  $\hat{\beta} + \hat{\delta} = 1.30$  since the market remains below  $\hat{c} = 1250$ . Thus, the fraction is constant; its value is  $-3.77$ . Each strategy indicates that the optimal portfolio should remain short the stock, while the Stressed-Beta strategy employs more leverage than the Merton strategy. Given the negative drift of the market of  $-19.26\%$  and the positive slope, it is expected that the stock will depreciate. Therefore, it makes intuitive sense that the investor should be short the stock.

Figure 1 shows the portfolio utility over time for each of the strategies. The utility curve for the Stressed-Beta strategy dominates the curve for the Merton strategy. Additionally, the terminal wealth of the Stressed-Beta portfolio is 6% higher than the terminal wealth of the Merton portfolio.

Table 1: Parameter estimates; 7/1/2008 – 9/26/2008

$\hat{\mu}$	$\hat{\Sigma}$	$\hat{\sigma}$	$\hat{\beta}^H$	$\hat{\beta}$	$\hat{\delta}$	$\hat{c}$
-19.26%	29.79%	21.34%	1.11	0.40	0.90	1250

## 5 Conclusion

This analysis put forth a portfolio optimization technique which allows for the stock to follow Stressed-Beta dynamics. By including these dynamics, we are extending the CAPM model to allow for a nonlinear relationship between excess stock returns and excess market returns. Specifically, we took the view that the slope relating excess stock returns to excess market returns steepens when the market falls below a critical threshold. The ultimate goal was to write the agent's optimal trading strategy in closed form, and we have done this using an asymptotic expansion. The trading strategy consists of a term in the form of the Merton solution, and a correction term which accounts for the slope-switching mechanism. The technique was demonstrated to outperform the Merton approach given a sample of market data from the period following the Lehman Brothers failure.

The portfolio problem considered was limited to a single risky stock; an obvious extension we will explore is to consider a set of  $n$  stocks following Stressed-Beta dynamics. Additionally, the agent might also wish to trade in the market index. The Stressed-Beta model is general in the sense that each stock has its own threshold  $c$ , slope in the normal state  $\beta$ , and change in slope  $\delta$ . This means that at any given time, some stocks may be in normal states while others are in stress states. Given this structure, we will explore different trading strategies.

## 6 Appendix

We begin with the partial differential equation in  $u$ :

$$\begin{aligned}
u_t + \frac{1}{2}\Sigma^2 m^2 u_{mm} + \left( m\mu + \frac{m\lambda(\mu - r)}{(1 - \lambda)\left(1 + \frac{\sigma^2}{\beta^2(m)\Sigma^2}\right)} \right) u_m \\
+ \left( r\lambda + \frac{\lambda(\mu - r)^2}{2(1 - \lambda)\Sigma^2\left(1 + \frac{\sigma^2}{\beta^2(m)\Sigma^2}\right)} \right) u + \frac{\lambda m^2 \Sigma^2}{2(1 - \lambda)\left(1 + \frac{\sigma^2}{\beta^2(m)\Sigma^2}\right)} \frac{u_m^2}{u} = 0.
\end{aligned}$$

Express the solution in terms of powers of  $\sigma^2$  as

$$u = u_0 + \sigma^2 u_1 + \mathcal{O}(\sigma^4),$$

and use the first two terms  $u_0 + \sigma^2 u_1$  to approximate the solution. Defining

$$\mathcal{V} := \frac{\sigma^2}{\beta^2(m)\Sigma^2},$$

so long as  $\mathcal{V} < 1$ , we can use the geometric series expansion to write

$$\frac{1}{1 + \mathcal{V}} = 1 - \mathcal{V} + \mathcal{O}(\sigma^4).$$

The condition  $\mathcal{V} < 1$  must be verified before proceeding with this methodology.

**Remark A.1** One possible cause for the exception  $\mathcal{V} \geq 1$  is that  $\beta(m) = 0$ , indicating that the stock return and market return are uncorrelated. We would expect that for this case, CAPM-based models would not be appropriate.

**Remark A.2** Another cause would be that  $\beta \neq 0$  but  $\delta = 0$ , indicating that there is really one state. In this case, it makes sense to treat this as the Merton problem where the stock variance is  $\beta^2 \Sigma^2 + \sigma^2$ .

Substituting the first two terms of (6) into (17) and ignoring the higher-order terms  $\mathcal{O}(\sigma^4)$  yields

$$\mathcal{L}_0 u_0 + \sigma^2 (\mathcal{L}_1 u_1 + s_0) = 0, \quad (17)$$

where the order-1 terms are

$$\begin{aligned} \mathcal{L}_0 u_0 &= u_{0t} + \frac{1}{2} \Sigma^2 m^2 u_{0mm} + \left( m\mu + \frac{m\lambda(\mu - r)}{1 - \lambda} \right) u_{0m} \\ &+ \left( r\lambda + \frac{\lambda(\mu - r)^2}{2(1 - \lambda)\Sigma^2} \right) u_0 + \frac{\lambda m^2 \Sigma^2}{2(1 - \lambda)} \frac{u_{0m}^2}{u_0}, \end{aligned}$$

the order- $\sigma^2$  terms in  $u_1$  are

$$\mathcal{L}_1 u_1 = u_{1t} + \frac{1}{2} \Sigma^2 m^2 u_{1mm} + \left( m\mu + \frac{m\lambda(\mu - r)}{1 - \lambda} \right) u_{1m} + \left( r\lambda + \frac{\lambda(\mu - r)^2}{2(1 - \lambda)\Sigma^2} \right) u_1,$$

and the source term is

$$s_0 = -\frac{m\lambda(\mu - r)}{(1 - \lambda)\beta^2(m)\Sigma^2} u_{0m} - \frac{\lambda(\mu - r)^2}{2(1 - \lambda)\Sigma^4 \beta^2(m)} u_0.$$

We set

$$\mathcal{L}_0 u_0 = 0, \quad (18)$$

subject to the terminal condition  $u_0(T, m) = 1$ , and

$$\mathcal{L}_1 u_1 + s_0 = 0, \quad (19)$$

subject to the terminal condition  $u_1(T, m) = 0$ . Since (18) has a terminal condition which doesn't depend on  $m$ , and there are terms involving derivatives with respect to  $m$ , the solution is given by

$$u_0(t, m) = e^{R(T-t)}, \quad (20)$$

where the constant rate  $R$  is given by

$$R = r\lambda + \frac{\lambda(\mu - r)^2}{2(1 - \lambda)\Sigma^2}.$$



As the solution to (18) doesn't depend on  $m$ , it can be written  $u_0(t, m) = u_0(t)$ . It is worth noting that (18) is the equation that would result if we assumed  $\sigma = 0$  from the beginning. Turning to (19), the term  $u_{0m}$  in the source vanishes and we have

$$u_{1t} + \frac{1}{2}\Sigma^2 m^2 u_{1mm} + \left( m\mu + \frac{m\lambda(\mu - r)}{1 - \lambda} \right) u_{1m} + Ru_1 - \frac{\lambda(\mu - r)^2}{2(1 - \lambda)\Sigma^4 \beta^2(m)} u_0 = 0. \quad (21)$$

Using the transformation  $u_1(t, m) = w_1(t, m)e^{R(T-t)}$ ,

$$w_{1t} + \frac{1}{2}\Sigma^2 m^2 w_{1mm} + Bmw_{1m} + \frac{C}{\beta^2(m)} = 0, \quad (22)$$

where

$$\begin{aligned} B &= \mu + \frac{\lambda(\mu - r)}{1 - \lambda}, \\ C &= -\frac{\lambda(\mu - r)^2}{2(1 - \lambda)\Sigma^4}, \end{aligned}$$

and the terminal condition is  $w_1(T, m) = 0$ . Use the Feynman-Kac formula to write  $w_1(t, m)$  as the stochastic representation

$$w_1(t, m) = \mathbb{E} \left\{ \int_t^T \frac{C}{\beta^2(X_s)} ds \mid X_t = m \right\}, \quad (23)$$

where  $X_t$  is a stochastic process with the following dynamics:

$$dX_t = BX_t dt + \Sigma X_t dW_t. \quad (24)$$

Using the slope definition (1) in (23),

$$w_1(t, m) = C \mathbb{E}_m \left\{ \int_t^T \frac{1}{\beta^2 + \delta(2\beta + \delta) \mathbf{1}_{\{X_s < c\}}} ds \right\},$$

where  $\mathbb{E}_m$  is shorthand for  $\mathbb{E} \{ \cdot \mid X_t = m \}$ . Next, use the fact that

$$\frac{1}{\beta^2 + \delta(2\beta + \delta) \mathbf{1}_{\{X_s < c\}}} = \frac{1}{\beta^2} (1 - D \mathbf{1}_{\{X_s < c\}}),$$

where

$$D = \frac{\delta(2\beta + \delta)}{\beta^2 + \delta(2\beta + \delta)}.$$

Then write

$$\begin{aligned} w_1(t, m) &= \frac{C}{\beta^2} \mathbb{E}_m \left\{ \int_t^T (1 - D \mathbf{1}_{\{X_s < c\}}) ds \right\}, \\ &= \frac{C}{\beta^2} \left[ T - t - D \mathbb{E}_m \left\{ \int_t^T \mathbf{1}_{\{X_s < c\}} ds \right\} \right]. \end{aligned}$$

The term in the expectation is the occupation time of the geometric Brownian motion given by (24). We consider the log-process, and then apply Girsanov's Theorem to remove the drift. Since the derivation going forward is purely probabilistic, for simplicity we consider the case where the starting time is zero, and we denote the running time by  $t$ . Let  $\xi_t = \log M_t$ , and  $\xi_0 = \xi = \log m$ . Then the dynamics for  $\xi_t$  are

$$d\xi_t = \left( B - \frac{1}{2}\Sigma^2 \right) dt + \Sigma dW_t. \quad (25)$$

Consider the new probability measure  $\tilde{\mathbb{P}}$  defined on  $\mathcal{F}_T$  by

$$\begin{aligned} \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} &= \exp \left\{ -\theta W_T - \frac{1}{2}\theta^2 T \right\}, \\ \theta &= \frac{1}{\Sigma} \left( B - \frac{1}{2}\Sigma^2 \right). \end{aligned}$$

Setting

$$\tilde{W}_t = W_t + \theta t,$$

then under  $\tilde{\mathbb{P}}$ , the process  $\tilde{W}_t$  is a standard Brownian motion, and  $\xi_t$  is a martingale:

$$\xi_t = \xi + \Sigma \tilde{W}_t. \quad (26)$$

Changing measure, the solution  $w_1$  at time zero becomes

$$w_1(0, m) = \frac{C}{\beta^2} \left[ T - D e^{-\frac{1}{2}\theta^2 T} \tilde{\mathbb{E}}_m \left\{ e^{\theta \tilde{W}_T} \int_0^T \mathbf{1}_{\{\tilde{W}_s < \tilde{c}\}} ds \right\} \right], \quad (27)$$

where

$$\tilde{c} = \frac{\log c - \xi}{\Sigma}. \quad (28)$$

Next, we will work with the occupation time. Introduce the first passage time

$$\tau = \inf \{ t \geq 0 : \xi_t = \log c \} = \inf \{ t \geq 0 : \tilde{W}_t = \tilde{c} \}, \quad (29)$$

where we have used (26) for  $\xi_t$  under  $\tilde{\mathbb{P}}$ . If  $m = c$ , or equivalently  $\tilde{c} = 0$ , then  $\tau = 0$ . If  $m \neq c$ , or equivalently  $\tilde{c} \neq 0$ , then the probability distribution of  $\tau \wedge T$  is given by

$$p(u; \tilde{c}) \mathbf{1}_{(0, T)}(u) du + \tilde{\mathbb{P}}\{\tau > T\} \delta_T(du),$$

where the density  $p(u; \tilde{c})$  is given by Karatzas and Shreve (1991) in Section 2.6.C:

$$p(u; \tilde{c}) = \frac{|\tilde{c}|}{\sqrt{2\pi u^3}} \exp \left( -\frac{\tilde{c}^2}{2u} \right), \quad u > 0. \quad (30)$$

At this point, it is convenient to treat separately the cases  $m = c$ ,  $m < c$ , and  $m > c$  (or equivalently  $\tilde{c} = 0$ ,  $\tilde{c} > 0$ , and  $\tilde{c} < 0$  respectively). Each of the cases involves the expectation

of a function of the level of the Brownian motion at terminal time,  $\widetilde{W}_T$  (introduced by the change of measure), and the occupation time. As our occupation time is on the negative half-interval and most formulas for occupation times are on the positive half-interval, we make use of the following:

$$\int_0^T \mathbf{1}_{\{\widetilde{W}_s < 0\}} ds = T - \widetilde{\Gamma}_T^+,$$

where  $\widetilde{\Gamma}_T^+ := \int_0^T \mathbf{1}_{\{\widetilde{W}_s > 0\}} ds$ . We will need the joint density of the level of the Brownian motion and its occupation time. The density of the triplet  $(\widetilde{W}_T, \widetilde{L}_T^0, \widetilde{\Gamma}_T^+)$  where  $\widetilde{L}_T^0$  is the local time of Brownian motion at the origin is derived in Karatzas and Shreve (1984) (see also Karatzas and Shreve (1991), Section 6.3.C). The density has the following form:

$$\begin{aligned} & \widetilde{\mathbb{P}} \left\{ \widetilde{W}_T \in da, \widetilde{L}_T^0 \in db, \widetilde{\Gamma}_T^+ \in d\gamma \right\} \\ &= \begin{cases} 2p(T - \gamma; b) p(\gamma; a + b) & \text{if } a > 0, b > 0, 0 < \gamma < T, \\ 2p(\gamma; b) p(T - \gamma; -a + b) & \text{if } a < 0, b > 0, 0 < \gamma < T, \end{cases} \end{aligned}$$

where  $p(u; \cdot)$  is given by (30). Since we need the joint density of the level of the Brownian motion and its occupation time, we will integrate out the local time to arrive at the necessary joint density:

$$\begin{aligned} & \widetilde{\mathbb{P}} \left\{ \widetilde{W}_T \in da, \widetilde{\Gamma}_T^+ \in d\gamma \right\} \\ &= \begin{cases} \int_0^\infty 2p(T - \gamma; b) p(\gamma; a + b) db & \text{if } a > 0, b > 0, 0 < \gamma < T, \\ \int_0^\infty 2p(\gamma; b) p(T - \gamma; -a + b) db & \text{if } a < 0, b > 0, 0 < \gamma < T, \end{cases} \\ &=: g(a, \gamma; T). \end{aligned} \tag{31}$$

Next, we analyze the three cases.

**Case  $m = c$**

In this case,  $\tilde{c} = 0$ , and so

$$\int_0^T \mathbf{1}_{\{\widetilde{W}_s < \tilde{c}\}} ds = \int_0^T \mathbf{1}_{\{\widetilde{W}_s < 0\}} ds = T - \widetilde{\Gamma}_T^+,$$

and thus we have

$$\begin{aligned} w_1(0, m) &= \frac{C}{\beta^2} \left[ T - D e^{-\frac{1}{2}\theta^2 T} \widetilde{\mathbb{E}}_m \left\{ e^{\theta \widetilde{W}_T} \int_0^T \mathbf{1}_{\{\widetilde{W}_s < \tilde{c}\}} ds \right\} \right], \\ &= \frac{C}{\beta^2} \left[ T - D e^{-\frac{1}{2}\theta^2 T} \widetilde{\mathbb{E}}_m \left\{ e^{\theta \widetilde{W}_T} (T - \widetilde{\Gamma}_T^+) \right\} \right], \\ &= \frac{C}{\beta^2} \left[ T - D e^{-\frac{1}{2}\theta^2 T} \int_{-\infty}^\infty \int_0^T e^{\theta a} (T - \gamma) g(a, \gamma; T) d\gamma da \right]. \end{aligned} \tag{32}$$

**Case  $m < c$**

When  $\tau > T$ , the full period of consideration  $[0, T]$  counts as occupation time. The only random variable in the expectation is the terminal value of the Brownian motion. Thus, we need the distribution of  $\widetilde{W}_T$  conditional on  $\{\tau > T\}$ . From Karatzas and Shreve (1991) Section 2.8.A, one easily obtains:

$$\begin{aligned} \widetilde{\mathbb{P}} \left\{ \widetilde{W}_T \in da, \tau > T \right\} &= \frac{1}{\sqrt{2\pi T}} \left( e^{-\frac{a^2}{2T}} - e^{-\frac{(2\tilde{c}-a)^2}{2T}} \right) da, \quad a < \tilde{c}, \\ &=: q_T(a; \tilde{c}) da. \end{aligned} \quad (33)$$

On  $\{\tau = u\}$  with  $u \leq T$ , we have  $\widetilde{W}_u = \tilde{c}$ . The time interval  $[0, u]$  counts as occupation time. Working with the argument in the expectation,

$$e^{\theta \widetilde{W}_T} \int_0^T \mathbf{1}_{\{\widetilde{W}_s < \tilde{c}\}} ds = e^{\theta \widetilde{W}_T} \left( T - \int_u^T \mathbf{1}_{\{\widetilde{W}_s - \widetilde{W}_u > 0\}} ds \right).$$

Define the standard Brownian motion  $B_{t-u} = \widetilde{W}_t - \widetilde{W}_u$ . Then

$$e^{\theta \widetilde{W}_T} \left( T - \int_u^T \mathbf{1}_{\{\widetilde{W}_s - \widetilde{W}_u > 0\}} ds \right) = e^{\theta \tilde{c}} e^{\theta B_{T-u}} \left( T - \int_u^T \mathbf{1}_{\{B_{s-u} > 0\}} ds \right).$$

Thus, the expectation involves the joint density of a standard Brownian motion  $B$  and its occupation time. This density is given by (31) with  $T$  replaced by  $T - u$ . Next, decompose the expectation on  $\{\tau \leq T\}$  and  $\{\tau > T\}$ , applying the density  $p(u; \tilde{c})$  given by (30); the degenerate case  $\tilde{c} = 0$  (or  $m = c$ ) corresponds to  $p(u; 0)du = \delta_0(du)$ .

Then

$$\begin{aligned} w_1(0, m) &= \frac{C}{\beta^2} \left[ T - D e^{-\frac{1}{2}\theta^2 T} \widetilde{\mathbb{E}}_m \left\{ e^{\theta \widetilde{W}_T} \int_0^T \mathbf{1}_{\{\widetilde{W}_s < \tilde{c}\}} ds \right\} \right], \\ &= \frac{C}{\beta^2} \left[ T - D e^{-\frac{1}{2}\theta^2 T} \right. \\ &\quad \times \left. \left\{ e^{\theta \tilde{c}} \int_0^T \int_{-\infty}^{\infty} \int_0^{T-u} e^{\theta a} (T - \gamma) g(a, \gamma; T - u) d\gamma da p(u; \tilde{c}) du + \int_{-\infty}^{\tilde{c}} e^{\theta a} T q_T(a; \tilde{c}) da \right\} \right]. \end{aligned} \quad (34)$$

**Case  $m > c$**

When  $\tau > T$ , the occupation time is zero.

On  $\{\tau = u\}$  with  $u \leq T$ , we have  $\widetilde{W}_u = \tilde{c}$ . The time interval  $[0, u]$  does not count as occupation time. Once again working with the argument in the expectation,

$$\begin{aligned} e^{\theta \widetilde{W}_T} \int_0^T \mathbf{1}_{\{\widetilde{W}_s < \tilde{c}\}} ds &= e^{\theta \widetilde{W}_T} \int_u^T \mathbf{1}_{\{\widetilde{W}_s - \widetilde{W}_u < 0\}} ds, \\ &= e^{\theta \tilde{c}} e^{\theta B_{T-u}} \int_u^T \mathbf{1}_{\{B_{s-u} < 0\}} ds. \end{aligned}$$

Thus, the expectation involves the joint density of a standard Brownian motion  $B$  and its occupation time in the negative half-interval. The distribution of  $(B_{T-u}, \Gamma_{T-u}^-)$  is the same as the distribution of  $(-B_{T-u}, T-u-\Gamma_{T-u}^-)$ , given by (31) with  $(a, \gamma)$  replaced by  $(-a, T-u-\gamma)$ . . Next, decompose the expectation on  $\{\tau \leq T\}$  and  $\{\tau > T\}$ , recalling that the integral in the latter case vanishes. Then

$$\begin{aligned}
w_1(0, m) &= \frac{C}{\beta^2} \left[ T - D e^{-\frac{1}{2}\theta^2 T} \tilde{\mathbb{E}}_m \left\{ e^{\theta \tilde{W}_T} \int_0^T \mathbf{1}_{\{\tilde{W}_s < \tilde{c}\}} ds \right\} \right], \\
&= \frac{C}{\beta^2} \left[ T - D e^{-\frac{1}{2}\theta^2 T} \right. \\
&\quad \times \left. \left\{ e^{\theta \tilde{c}} \int_0^T \int_{-\infty}^{\infty} \int_0^{T-u} e^{\theta a} \gamma g(a, \gamma; T-u) d\gamma da p(u; \tilde{c}) du \right\} \right].
\end{aligned} \tag{35}$$

Now that we have all of the pieces, we may compute  $u = u_0 + \sigma^2 u_1$ . From (20),  $u_0(0) = e^{RT}$ . The correction term is  $u_1(0, m) = w_1(0, m)e^{RT}$ , where  $w_1(0, m)$  is computed as (32), (34), or (35) depending on whether  $m = c$ ,  $m < c$ , or  $m > c$ , respectively.

Returning to the optimal fraction of wealth  $\pi^*(m)$  and substituting the approximation  $u = u_0 + \sigma^2 u_1$ , we have

$$\pi^*(m) = \pi_0^*(m) + \sigma^2 \pi_1^*(m) + \mathcal{O}(\sigma^4),$$

where

$$\begin{aligned}
\pi_0^*(m) &= \frac{\mu - r}{(1 - \lambda)\beta(m)\Sigma^2}, \\
\pi_1^*(m) &= \frac{1}{\beta(m)(1 - \lambda)} \frac{m u_{1m}}{u_0} - \frac{\mu - r}{(1 - \lambda)\beta^3(m)\Sigma^4}.
\end{aligned}$$

Then  $\pi^*(m)$  may be estimated as

$$\pi^*(m) = \pi_0^*(m) + \sigma^2 \pi_1^*(m).$$

To compute  $u_{1m}$ , recall that  $u_1 = w_1 e^{RT}$ . Then  $u_{1m} = w_{1m} e^{RT}$ , and  $w_{1m}$  may be computed directly. We write

$$w_{1m} = \frac{\partial w_1}{\partial \xi} \frac{\partial \xi}{\partial M} = \frac{1}{M_t} \frac{\partial w_1}{\partial \xi}.$$

We need to evaluate two cases:  $m < c$  and  $m > c$  ( $w_1$  is not differentiable when  $m = c$  due to the point discontinuity).

Case  $m < c$

$$\begin{aligned}
\frac{\partial w_1}{\partial \xi} &= -\frac{CD}{\beta^2} e^{-\frac{1}{2}\theta^2 T} \\
&\times \left( e^{\theta \tilde{c}} \int_0^T \int_{-\infty}^{\infty} \int_0^{T-u} e^{\theta a} (T - \gamma) g(a, \gamma; T - u) d\gamma da \frac{\partial p(u; \tilde{c})}{\partial \xi} du \right. \\
&- \frac{\theta e^{\theta \tilde{c}}}{\Sigma} \int_0^T \int_{-\infty}^{\infty} \int_0^{T-u} e^{\theta a} (T - \gamma) g(a, \gamma; T - u) d\gamma da p(u; \tilde{c}) du \\
&\left. + \int_{-\infty}^{\tilde{c}} e^{\theta a} T \frac{\partial q_T(a; \tilde{c})}{\partial \xi} da \right),
\end{aligned} \tag{36}$$

where

$$\begin{aligned}
\frac{\partial p(u; \tilde{c})}{\partial \xi} &= \frac{e^{-\frac{\tilde{c}^2}{2u}} |\tilde{c}| \tilde{c}}{\sqrt{2\pi u^{5/2} \Sigma}} + \frac{e^{-\frac{\tilde{c}^2}{2u}}}{\sqrt{2\pi u^3}} \frac{\partial |\tilde{c}|}{\partial \xi}, \\
\frac{\partial q_T(a; \tilde{c})}{\partial \xi} &= -\frac{e^{-\frac{(2\tilde{c}-a)^2}{2T}} \sqrt{\frac{2}{\pi}} (2\tilde{c} - a)}{T^{3/2} \Sigma}.
\end{aligned}$$

Case  $m > c$

$$\begin{aligned}
\frac{\partial w_1}{\partial \xi} &= -\frac{CD}{\beta^2} e^{-\frac{1}{2}\theta^2 T} \\
&\times e^{\theta \tilde{c}} \left( \int_0^T \int_{-\infty}^{\infty} \int_0^{T-u} e^{\theta a} \gamma g(a, \gamma; T - u) d\gamma da \frac{\partial p(u; \tilde{c})}{\partial \xi} du \right. \\
&\left. - \frac{\theta}{\Sigma} \int_0^T \int_{-\infty}^{\infty} \int_0^{T-u} e^{\theta a} \gamma g(a, \gamma; T - u) d\gamma da p(u; \tilde{c}) du \right).
\end{aligned} \tag{37}$$

We now have all of the pieces needed to compute  $\pi^*(m)$ .

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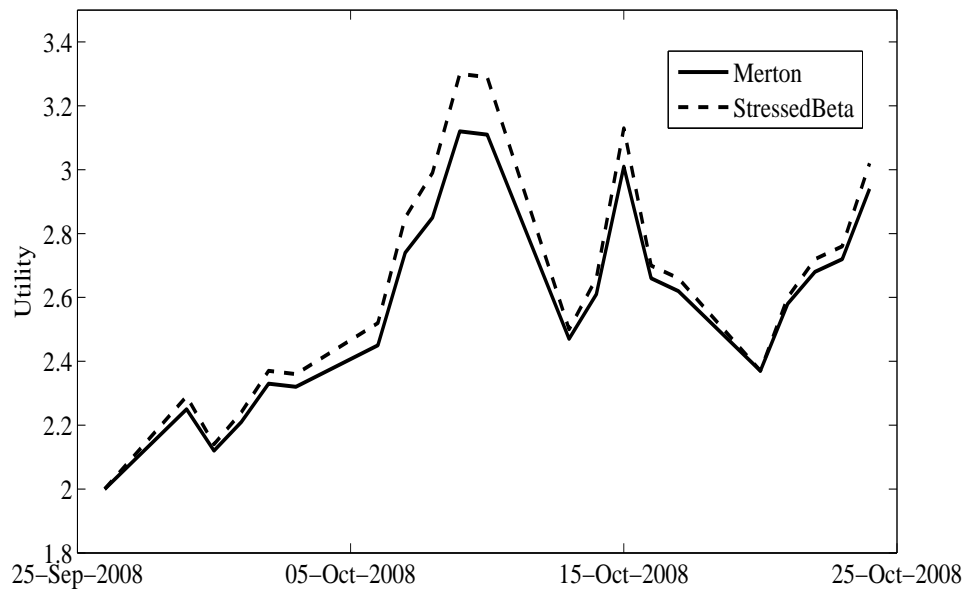


Figure 1: Portfolio utility for each of the strategies. Note that utility is defined as  $y^\gamma / \gamma$ , where  $y$  represents wealth and  $\lambda = 0.5$  is the risk-aversion coefficient. For different levels of  $\lambda$ , the pattern remains the same; increasing (decreasing)  $\lambda$  results in a wider (narrower) spread between the curves.