A Unified Approach to Systemic Risk Measures via Acceptance Sets

Francesca Biagini* Jean-Pierre Fouque † Marco Frittelli‡
Thilo Meyer-Brandis§

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Abstract

The financial crisis has dramatically demonstrated that the traditional approach to apply univariate monetary risk measures to single institutions does not capture sufficiently the perilous systemic risk that is generated by the interconnectedness of the system entities and the corresponding contagion effects. This has brought awareness of the urgent need for novel approaches that capture systemic riskiness. The purpose of this paper is to specify a general methodological framework that is flexible enough to cover a wide range of possibilities to design systemic risk measures via multi-dimensional acceptance sets and aggregation functions, and to study corresponding examples. Existing systemic risk measures can usually be interpreted as the minimal amount of cash needed to secure the system *after aggregating* individual risks. In contrast, our approach also includes systemic risk measures that can be interpreted as the minimal amount of cash that secures the aggregated system by allocating capital to the single institutions

*Department of Mathematics, University of Munich, Theresienstraße 39, 80333 Munich, Germany. *francesca.biagini@math.lmu.de*.

†Department of Statistics & Applied Probability, University of California, Santa Barbara, CA 93106-3110, *fouque@pstat.ucsb.edu*. Work supported by NSF grants DMS-1107468 and DMS-1409434.

‡Dipartimento di Matematica, Università degli Studi di Milano, Via Saldini 50, 20133 Milano, Italy, *marco.frittelli@unimi.it*.

§Department of Mathematics, University of Munich, Theresienstraße 39, 80333 Munich, Germany.*meyerbr@math.lmu.de*. Francesca Biagini and Thilo Meyer-Brandis thank the Frankfurt Institute for Risk Management and Regulation and the Europlace Institute of Finance for support.

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before aggregating the individual risks. This allows for a possible ranking of the institutions in terms of systemic riskiness measured by the optimal allocations. Another important feature of our approach is the possibility of allocating cash according to the future state of the system (scenario-dependent allocation). We illustrate with several examples the advantages of this feature. We also provide conditions which ensure monotonicity, convexity, or quasi-convexity properties of our systemic risk measures.

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1 Introduction

A large part of the current literature on systemic financial risk is concerned with the modeling structure of financial networks and the analysis of the contagion and the spread of a potential exogenous (or even endogenous) shock into the system. For a given financial (possibly random) network and a given random shock one then determines the "cascade" mechanism which generates possibly many defaults. This mechanism often requires a detailed description of the balance sheet of each institution; assumptions on the interbank network and exposures, on the recovery rate at default, on the liquidation policy; the analysis of direct liabilities, bankruptcy costs, cross-holdings, leverage structures, fire sales, and liquidity freezes.

Among the many contributions we mention here the classical contagion model proposed by Eisenberg and Noe [2001], the default model of Gai and Kapadia [2010a], the illiquidity cascade models of Gai and Kapadia [2010b], Hurd et al. [2014] and Lee [2013], the asset fire sale cascade model by Cifuentes et al. [2005] and Caccioli et al. [2012], as well as the model in Awiszus and Weber [2015] that additionally includes cross-holdings. For an exhaustive reference on the literature we defer the reader to the recent volume: "Contagion! The Spread of Systemic Risk in Financial Networks", Hurd [2015].

These approaches may be relevant also from the viewpoint of a policy maker that has to intervene and regulate the banking system to reduce the risk that, in case of an adverse (local) shock, a substantial part or even the complete system breaks down.

However, once such a model for the financial network has been identified and the mechanism for the spread of the contagion determined, one still has

to understand how to compare the possible final outcomes in a reasonable way or, in other words, how to measure the risk carried by the global financial system. This is the focus of our approach, as we measure the risk embedded in a financial system taking as primitive a vector $\mathbf{X} = (X^1, \dots, X^N)$ of positions, where X^i represents the position of institution i. Our approach is very close in spirit to the "classical" conceptual framework initiated by the seminal paper by Artzner et al. [1999] and that has been recently adopted also to analyze systemic risk by Chen et al. [2013], Kromer et al. [2013] and Hoffmann et al. [2014].
We recall this classical approach, in the case of one single institution, by the following two quotes from Artzner et al. [1999]:

"*The basic objects of our study shall therefore be the random variables on the set of states of nature at a future date, interpreted as possible future values of positions or portfolios currently held.*"

...

"These measures of risk can be used as (extra) capital requirements to regulate the risk assumed by market participants, traders, and insurance underwriters, as well as to allocate existing capital."

Of course one main difference is that we have to take into consideration not just one single institution but the global system and in this paper we will illustrate how to achieve this in an appropriate way. We interpret X^i as the profits and losses of institution i at a future time T, precisely as the gain if X^i is positive or as the loss if X^i is negative. Such profit and loss is typically uncertain and therefore it will be modeled by a random variable $X^i(\omega)$ on some space of possible scenarios $\omega \in \Omega$.

To summarize, we consider the random vector $\mathbf{X} = (X^1, \dots, X^N)$ as primitive: One may interpret it as a "reduced form model" of a complex financial system without reference to a specific structural network model, and consequently $\mathbf{X}$ may already comprehend the potential risk of a contagion spread into the system. Otherwise one may also interpret $\mathbf{X}$ as the net worth of the positions before the contagion takes place and then the contagion mechanism will be embedded in the risk measure via an aggregation function (as in the model of Eisenberg and Noe [2001]). Either way, our scope is to provide a consistent criterion to asses wether one possible vector $\mathbf{X}$ is riskier than another.

1.1 From one-dimensional to N-dimensional risk profiles

In this subsection we review the literature on risk measurement based on acceptable sets, both in the traditional one-dimensional setting as well as in the case of N interacting financial institutions. Here we denote with $\mathcal{L}^0(\mathbb{R}^N) := \mathcal{L}^0(\Omega, \mathcal{F}; \mathbb{R}^N)$, $N \in \mathbb{N}$, the space of $\mathbb{R}^N$-valued random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Traditional risk management strategies of financial systems have predominantly focused on the solvency of individual institutions as if they were in isolation. A typical approach is to evaluate the risk $\eta(X^i)$ of each institution $i \in \{1, ...N\}$ by applying a *univariate monetary risk measure* η to the single financial positions. A monetary risk measure (see Föllmer and Schied [2004]) is a map $\eta : \mathcal{L}^0(\mathbb{R}) \to \mathbb{R}$ that can be interpreted as the minimal capital needed to secure a financial position with payoff $X \in \mathcal{L}^0(\mathbb{R})$, i.e. the minimal amount $m \in \mathbb{R}$ that must be added to X in order to make the resulting (discounted) payoff at time T acceptable:

$$\eta(X) := \inf\{m \in \mathbb{R} \mid X + m \in \mathbb{A}\}, \tag{1.1}$$

where the acceptance set $\mathbb{A} \subseteq \mathcal{L}^0(\mathbb{R})$ is assumed to be monotone, i.e. $X \geq Y \in \mathbb{A}$ implies $X \in \mathbb{A}$. In addition to decreasing monotonicity, the characterizing feature of these maps is the cash additivity property:

$$\eta(X + m) = \eta(X) - m, \text{ for all } m \in \mathbb{R}. \tag{1.2}$$

Under the assumption that the set $\mathbb{A}$ is convex (resp. is a convex cone) the maps in (1.1) are convex (resp. convex and positively homogeneous) and are called *convex* (resp. coherent) *risk measures*, see Artzner et al. [1999], Föllmer and Schied [2002], Frittelli and Rosazza Gianin [2002]. The principle that diversification should not increase the risk is mathematically translated not necessarily with the convexity property but with the weaker condition of quasiconvexity:

$$\eta(\lambda X + (1 - \lambda)Y) \leq \eta(X) \vee \eta(Y).$$

As a result, in Cerreia-Vioglio et al. [2010] and Frittelli and Maggis [2014], the only properties assumed in the definition of a *quasi-convex risk measure* are monotonicity and quasiconvexity. Such risk measures can always be written as:

$$\eta(X) := \inf\{m \in \mathbb{R} \mid X \in \mathbb{A}^m\}, \tag{1.3}$$

where each set $\mathbb{A}^m \subseteq \mathcal{L}^0(\mathbb{R})$ is monotone and convex, for each m. Here $\mathbb{A}^m$ is interpreted as the class of payoffs carrying the same risk level m.

Contrary to the convex, cash additive case where each random variable is binary cataloged as acceptable or as not acceptabe, in the quasi-convex case one admits various degrees of acceptability, described by the risk level m, see Cherny and Madan [2009]. Furthermore, in the quasi-convex case the cash additivity property will not hold in general and one looses a direct interpretation of m as the minimal capital required to secure the payoff X, but preserves the interpretation of $\mathbb{A}^m$ as the set of positions acceptable for the given risk level m. By selecting $\mathbb{A}^m := \mathbb{A}-m$, the risk measure in (1.1) is clearly a particular case of the one in (1.3).

However, the financial crisis has dramatically demonstrated that the traditional approach to apply univariate monetary risk measures to single institutions does not capture sufficiently the perilous systemic risk that is generated by the interconnectedness of the system entities and the corresponding contagion effects. This has brought awareness of the urgent need for novel approaches that capture systemic riskiness, and a rapidly growing literature is concerned with designing more appropriate risk measures for financial systems. A *systemic risk measure* is then a map $\rho : \mathcal{L}^0(\mathbb{R}^N) \to \mathbb{R}$ that evaluates the risk $\rho(\mathbf{X})$ of the complete system $\mathbf{X}$ of financial positions. Most of the systemic risk measures in the existing literature are of the form

$$\rho(\mathbf{X}) = \eta(\Lambda(\mathbf{X})), \tag{1.4}$$

where $\eta : \mathcal{L}^0(\mathbb{R}) \to \mathbb{R}$ is a univariate risk measure and

$$\Lambda : \mathbb{R}^N \to \mathbb{R}$$

is an aggregation rule that aggregates the N-dimensional risk factor $\mathbf{X}$ into a univariate risk factor $\Lambda(\mathbf{X})$ representing the total risk in the system. Some examples of aggregation rules found in the literature are the following:

- In general, one of the most common ways to aggregate multivariate risk is to simply sum the single risk factors: $\Lambda(\mathbf{x}) = \sum_{i=1}^N x_i$, $\mathbf{x} = (x_1, ..., x_N) \in \mathbb{R}^N$. Also in the literature on systemic risk measures there are examples using this aggregation rule, like for example the *Systemic Expected Shortfall* introduced in Acharya et al. [2010], or the *Contagion Value at Risk (CoVaR)* introduced in Adrian and Brunnermeier [2011]. However, while summing up profit and loss positions might be reasonable from the viewpoint of a portfolio manager where the portfolio components compensate each other, this aggregation rule seems inappropriate for a financial system where cross-subsidization between institutions is rather unrealistic. Further, if the sum was a

suitable aggregation of risk in financial systems, then the traditional approach of applying a univariate coherent risk measure η to the single risk factors would be sufficiently prudential in the sense that by sub-linearity it holds that $\eta(\sum_{i=1}^N X_i) \leq \sum_{i=1}^N \eta(X_i)$.

- One possible aggregation that takes the lack of cross-subsidization between financial institutions into account is to sum up losses only: $\Lambda(\mathbf{x}) = \sum_{i=1}^N -x_i^-$. This kind of aggregation is for example used in Huang et al. [2009], Lehar [2005]. See also Brunnermeier and Cheridito [2013] for an extension of this type of aggregation rule that also considers a certain effect of gains, as $\Lambda(\mathbf{x}) = \sum_{i=1}^N -\alpha_i x_i^- + \sum_{i=1}^N \beta_i (x_i - v_i)^+$ for some $\alpha_i, \beta_i, v_i \in \mathbb{R}^+$, $i = 1, \cdots, N$.

- Beside the lack of cross-subsidization in a financial system, the aggregation rule may also accounts for contagion effects that can considerably accelerate systemwide losses resulting from an initial shock. Motivated by the structural contagion model of Eisenberg and Noe [2001], in Chen et al. [2013] they introduce an aggregation function that explicitly models the net systemic cost of the contagion in a financial system by defining the aggregation rule

$$\Lambda_{CM}(x) = \min_{y_i \geq x_i + \sum_{j=1}^N \Pi_{ij} y_j, \forall i=1,\cdots,N,\, y \in \mathbb{R}_+^N} \left\{ \sum_{i=1}^N y_i \right\}.$$

  Here, $\Pi = (\Pi_{ij})_{i,j=1,\cdots,N}$ represents the relative liability matrix, i.e.firm i has to pay the proportion Π_{ij} of its total liabilities to firm j.
  In the literature there are various extensions of the structural contagion model of Eisenberg and Noe [2001] and the corresponding aggregation rule that take into account further contagion channels of systemic risk such as effects from firesales or liquidity freezes, see e.g. Amini et al. [2013], Awiszus and Weber [2015], Cifuentes et al. [2005], Gai and Kapadia [2010a].

An axiomatic characterization of systemic risk measures of the form (1.4) on a finite state space is provided in Chen et al. [2013], see also Kromer et al. [2013] for the extension to a general probability space and Hoffmann et al. [2014] for a further extension to a conditional setting. Also, in these references further examples of possible aggregation functions can be found. Our framework may accommodate also such aggregation functions, provided these satisfy the (simple) conditions outlined in Section 3.

If η in (1.4) is a monetary risk measure it follows from (1.1) that we can rewrite the systemic risk measure ρ in (1.4) as

$$\rho(\mathbf{X}) := \inf\{m \in \mathbb{R} \mid \Lambda(\mathbf{X}) + m \in \mathbb{A}\}. \tag{1.5}$$

Thus, presuming $\Lambda(\mathbf{X})$ represents some loss, systemic risk can again be interpreted as the minimal cash amount that secures the system when it is added to the total aggregated system loss $\Lambda(\mathbf{X})$. If $\Lambda(\mathbf{X})$ does not allow for an interpretation as cash, the risk measure in (1.5) has to be understood as some general risk level of the system rather than some capital requirement. Similarly, if η is a quasi-convex risk measure the systemic risk measure ρ in (1.4) can be rewritten as

$$\rho(\mathbf{X}) := \inf\{m \in \mathbb{R} \mid \Lambda(\mathbf{X}) \in \mathbb{A}^m\}. \tag{1.6}$$

Again one first aggregates the risk factors via the function Λ and in a second step one computes the minimal risk level associated to $\Lambda(\mathbf{X})$.
While the approach prescribed in (1.5) and (1.6) defines an interesting class of systemic risk measures, one could think of meaningful alternative or extended procedures of measuring systemic risk not captured by (1.5) or (1.6). The purpose of this paper is to specify a general methodological framework that is flexible enough to cover a wide range of possibilities to design systemic risk measures via acceptance sets and aggregation functions and to study corresponding examples. In the following subsections we extend the conceptual framework for systemic risk measures via acceptance sets step by step in order to gradually include certain novel key features of our approach.

1.2 First add capital, then aggregate

The interpretation of (1.5) to measure systemic risk as minimal capital needed to secure the system *after aggregating individual risks* is for example meaningful in the situation where some kind of rescue fund shall be installed to repair damage from systemic loss. However, for instance from the viewpoint of a regulator that has the possibility to intervene on the level of the single institutions before contagion effects generate further losses it might be more relevant to measure systemic risk as the minimal capital that secures the aggregated system by injecting the capital into the single institutions *before aggregating the individual risks*. This way of measuring systemic risk can be expressed by

$$\rho(\mathbf{X}) := \inf\{\sum_{i=1}^{N} m_i \mid \mathbf{m} = (m_1, ..., m_N) \in \mathbb{R}^N,\, \Lambda(\mathbf{X} + \mathbf{m}) \in \mathbb{A}\}. \tag{1.7}$$

Here, the amount m_i is added to the financial position X^i of institution $i \in \{1, ..., N\}$ before the corresponding total loss $\Lambda(\mathbf{X}+\mathbf{m})$ is computed. For example, considering the aggregation function Λ_{CM} from above it becomes clear that injecting cash first might prevent further losses that would be generated by contagion effects. The systemic risk is then measured as the minimal total amountl $\sum_{i=1}^{N} m_i$ injected into the institutions to secure the system. [1]

Another interesting feature of the approach in (1.7) is that it delivers at the same time a measure of total systemic risk as well as a potential ranking of the institutions in terms of systemic riskiness. Indeed, for $\mathbf{X}$ given, let $\mathbf{m}^* = (m_1^*, ..., m_N^*)$ be such that $\rho(\mathbf{X}) = \sum_{i=1}^{N} m_i^*$ and denote the ordered cash allocations by $m_{i_1}^* \geq ... \geq m_{i_N}^*$. Then, one could argue that the risk factor X^{i_1} that requires the biggest cash allocation $m_{i_1}^*$ corresponds to the systemic riskiest institution, X^{i_2} corresponds to the systemic second riskiest institution, and so on. Of course, such allocation $\mathbf{m}^*$ does not need to be unique, in which case one has to discuss criteria that justify the choice of a specific allocation.

1.3 First add scenario-dependent allocation, then aggregate

One main novelty of this paper is that we want to allow for the possibility of adding to $\mathbf{X}$ not merely a vector $\mathbf{m} = (m_1, ..., m_N) \in \mathbb{R}^N$ of cash but a random vector

$$\mathbf{Y} \in \mathcal{C} \subseteq \mathcal{L}^0(\mathbb{R}^N)$$

which represents admissible assets with possibly random payoffs at time T, in the spirit of Frittelli and Scandolo [2006]. To each $\mathbf{Y} \in \mathcal{C}$ we assign a measure $\pi(\mathbf{Y})$ of the risk (or cost) associated to $\mathbf{Y}$ determined by a monotone increasing map

$$\pi : \mathcal{C} \to \mathbb{R}\,. \tag{1.8}$$

This leads to the following extension of (1.7):

$$\rho(\mathbf{X}) := \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C},\, \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}\}\,. \tag{1.9}$$

Note that in order to establish a ranking of the institutions in a system $\mathbf{X}$ in terms of systemic riskiness implied by a $\mathbf{Y}^* = (Y_1^*, ..., Y_N^*) \in \mathcal{C}$ with $\rho(\mathbf{X}) = \pi(\mathbf{Y}^*)$ in analogy to the ranking process described above implied by a deterministic $\mathbf{m}^* = (m_1^*, ..., m_N^*)$ one now first has to introduce an ordering of the $Y_1^*, ..., Y_N^*$. For example, one could say X^i is systemic riskier

[1] Independently a related concept in the context of set-valued systemic risk measures has been developed in Feinstein et al. [2015].

than X^j if $E[Y_i^*] > E[Y_j^*]$, presumed the expectations $E[Y_i^*]$, $i = 1, ..., N$ are well defined.

Considering a general set $\mathcal{C}$ in (1.9) allows for more general measurement of systemic risk than the cash needed today for each institution to secure the system. For example, $\mathcal{C}$ could be a set of (vectors of) general admissible financial assets that can be used to secure a system by adding $\mathbf{Y}$ to $\mathbf{X}$ component-wise, and $\pi(\mathbf{Y})$ is a valuation of $\mathbf{Y}$. Another example that we focus on in this paper and which is particularly interesting from the viewpoint of a lender of last resort is the following class of sets $\mathcal{C}$:

$$\mathcal{C} \subseteq \{\mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^N) \mid \sum_{n=1}^{N} Y^n \in \mathbb{R}\} =: \mathcal{C}_{\mathbb{R}}, \tag{1.10}$$

and $\pi(\mathbf{Y}) = \sum_{n=1}^{N} Y^n$. Here the notation $\sum_{n=1}^{N} Y^n \in \mathbb{R}$ means that $\sum_{n=1}^{N} Y^n$ is equal to some deterministic constant in $\mathbb{R}$, even though each single Y^n, $n = 1, \cdots, N$, is a random variable. Then, as in (1.7) the systemic risk measure

$$\rho(\mathbf{X}) := \inf\{\sum_{n=1}^{N} Y^n \mid \mathbf{Y} \in \mathcal{C},\, \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}\} \tag{1.11}$$

can still be interpreted as the minimal total cash amount $\sum_{n=1}^{N} Y^n \in \mathbb{R}$ needed today to secure the system by distributing the cash at the future time T among the components of the risk vector $\mathbf{X}$. However, contrary to (1.7), in general the allocation $Y^i(\omega)$ to institution i does not need to be decided today but depends on the scenario ω that has been realized at time T. This corresponds to the situation of a lender of last resort who is equipped with a certain amount of cash today and who will allocate it according to where it serves the most depending on the scenario that has been realized. Restrictions on the possible distributions of cash are given by the set $\mathcal{C}$. For example, for $\mathcal{C} = \mathbb{R}^N$ the situation corresponds to (1.7) where the distribution is already determined today, while for $\mathcal{C} = \mathcal{C}_{\mathbb{R}}$ the distribution can be chosen completely freely depending on the scenario ω that has been realized (including negative amounts, i.e. withdrawals of cash from certain components).

Section 4, 5, and 6 will be devoted to the analysis and concrete examples of the class of systemic risk measures using a set $\mathcal{C}$ as in (1.10). We will see that in the case $\mathcal{C} = \mathcal{C}_{\mathbb{R}}$ where unrestricted cross-subsidization is possible the canonical way of measuring systemic risk measure is of the form (1.4) with aggregation rule $\Lambda(\mathbf{x}) = \sum_{i=1}^{N} x_i$, $\mathbf{x} \in \mathbb{R}^N$, i.e. to apply a univariate risk measure to the sum of the risk factors. Another interesting feature

of allowing scenario depending allocations of cash $\mathbf{Y} \in \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$ is that in general the systemic risk measure will take the dependence structure of the components of $\mathbf{X}$ into account even though acceptable positions might be defined in terms of the marginal distributions of X^i, $i = 1, ..., N$ only. For instance, the example in Section 5 employs the aggregation rule $\Lambda(\mathbf{x}) = \sum_{i=1}^{N} -x_i^-$, $\mathbf{x} \in \mathbb{R}^N$, and the acceptance set $\mathbb{A}_\gamma := \{Z \in \mathcal{L}^0(\mathbb{R}) \mid E[Z] \geq \gamma\}, \gamma \in \mathbb{R}$. Then a risk vector $\mathbf{Z} = (Z_1, ..., Z_N) \in \mathcal{L}^0(\mathbb{R}^N)$ is acceptable if and only if $\Lambda(\mathbf{Z}) \in \mathbb{A}$, i.e.

$$\sum_{i=1}^{N} -E[Z_i^-] \geq \gamma\,,$$

which only depends on the marginal distributions of $\mathbf{Z}$. Thus, if we choose $\mathcal{C} = \mathbb{R}^N$ then it is obvious that in this case also the systemic risk measure $\rho(\mathbf{X})$ in (1.11) depends on the marginal distributions of $\mathbf{X}$ only. If, however, one allows for more general allocations $\mathbf{Y} \in \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$ that might differ from scenario to scenario the systemic risk measure will in general depend on the multivariate distribution of $\mathbf{X}$ since it can play on the dependence of the components of $\mathbf{X}$ to minimize the costs.

1.4 Multi-dimensional Acceptance Sets

Until now we have always defined systemic risk measures in terms of acceptability of an aggregated, one-dimensional loss figure. However, not necessarily every relevant systemic risk measure is of this aggregated type. Consider for instance the popular approach (though possibly problematic for financial systems as explained above) to add single univariate monetary risk measures η_i, $i = 1, ..., N$, i.e.

$$\rho(\mathbf{X}) := \sum_{i=1}^{N} \eta_i(X^i)\,. \tag{1.12}$$

In general, the systemic risk measure in (1.12) cannot be expressed in the form (1.9). Denoting by $\mathbb{A}_i \subseteq \mathcal{L}^0(\mathbb{R})$ the acceptance set of η_i, $i = 1, ..., N$, one easily sees from (1.1), however, that ρ in (1.12) can be written in terms of the multivariate acceptance set $\mathbb{A}_1 \times ... \times \mathbb{A}_N$:

$$\rho(\mathbf{X}) := \inf\{\sum_{i=1}^{N} m_i \mid \mathbf{m} = (m_1, ..., m_N) \in \mathbb{R}^N,\ \mathbf{X} + \mathbf{m} \in \mathbb{A}_1 \times ... \times \mathbb{A}_N\}\,.$$

Motivated by this example, we extend (1.9) to the formulation of systemic risk measures as the minimal cost of admissible asset vectors $\mathbf{Y} \in \mathcal{C}$ that, when added to the vector of financial positions $\mathbf{X}$, makes the augmented

financial positions $\mathbf{X}+\mathbf{Y}$ acceptable in terms of a general multidimensional acceptance set $\mathcal{A} \subseteq \mathcal{L}^0(\mathbb{R}^N)$:

$$\rho(\mathbf{X}) := \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C},\, \mathbf{X}+\mathbf{Y} \in \mathcal{A}\}\,. \tag{1.13}$$

Note that by putting $\mathcal{A} := \left\{\mathbf{Z} \in \mathcal{L}^0(\mathbb{R}^N) \mid \Lambda(\mathbf{Z}) \in \mathbb{A}\right\}$ Definition (1.9) is a special case of (1.13). Also, in analogy to (1.2), we remark that for linear valuation rules π the systemic risk measure given in (1.13) exhibits an extended type of cash invariance in the sense that

$$\rho(\mathbf{X}+\mathbf{Y}) = \rho(\mathbf{X}) + \pi(\mathbf{Y}) \tag{1.14}$$

for $\mathbf{Y} \in \mathcal{C}$ such that $\mathbf{Y}' \pm \mathbf{Y} \in \mathcal{C}$ for all $\mathbf{Y}' \in \mathcal{C}$, see Frittelli and Scandolo [2006].

1.5 Degree of Acceptability

In order to reach the final, most general formulation of systemic risk measures, we assign, in analogy to (1.3), to each $\mathbf{Y} \in \mathcal{C}$ a set $\mathcal{A}^{\mathbf{Y}} \subseteq \mathcal{L}^0(\mathbb{R}^N)$ of risk vectors that are acceptable for the given (random) vector $\mathbf{Y}$, and define the systemic risk measure by:

$$\rho(\mathbf{X}) := \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C},\, \mathbf{X} \in \mathcal{A}^{\mathbf{Y}}\}\,. \tag{1.15}$$

Note that analogously to the one-dimensional quasi-convex case (1.3), the systemic risk measures (1.15) cannot necessarily be interpreted as cash added to the system but in general represents some minimal aggregated risk level $\pi(\mathbf{Y})$ at which the system $\mathbf{X}$ is acceptable. The approach in (1.15) is very flexible and unifies a variety of different features in the design of systemic risk measures. In particular, it includes all previous cases if we set

$$\mathcal{A}^{\mathbf{Y}} := \mathcal{A} - \mathbf{Y},$$

where the set $\mathcal{A} \subseteq \mathcal{L}^0(\mathbb{R}^N)$ represents acceptable risk vectors. Then obviously (1.13) is obtained from (1.15).

Another advantage of the formulation in terms of general acceptance sets is the possibility to design systemic risk measures via general aggregation rules. Indeed the formulation (1.15) includes the case

$$\rho(\mathbf{X}) := \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C},\, \Theta(\mathbf{X},\mathbf{Y}) \in \mathbb{A}\}, \tag{1.16}$$

where $\Theta : \mathcal{L}^0(\mathbb{R}^N) \times \mathcal{C} \to \mathcal{L}^0(\mathbb{R})$ denotes some aggregation function jointly in $\mathbf{X}$ and $\mathbf{Y}$. Just select $\mathcal{A}^{\mathbf{Y}} := \left\{\mathbf{Z} \in \mathcal{L}^0(\mathbb{R}^N) \mid \Theta(\mathbf{Z},\mathbf{Y}) \in \mathbb{A}\right\}$. In particular, (1.16) includes both the case "injecting capital before aggregation"as in

(1.7) and (1.9) by putting $\Theta(\mathbf{X}, \mathbf{Y}) = \Lambda(\mathbf{X} + \mathbf{Y})$, and the case "aggregation before injecting capital"as in (1.5) by putting $\Theta(\mathbf{X}, \mathbf{Y}) := \Lambda_1(\mathbf{X}) + \Lambda_2(\mathbf{Y})$, where $\Lambda_1 : \mathcal{L}^0(\mathbb{R}^N) \to \mathcal{L}^0(\mathbb{R})$ is an aggregation function and $\Lambda_2 : \mathcal{C} \to \mathcal{L}^0(\mathbb{R})$ could be, for example, the discounted cost of $\mathbf{Y}$.
Also, again in analogy to the one-dimensional case (1.3), the more general dependence of the acceptance set on $\mathbf{Y}$ in (1.15) allows for multi-dimensional quasi-convex risk measures. Note that the cash additivity property (1.14) is then lost in general.
The remainder of the paper is organized as follows. In the next Section we structure and lay the theoretical foundations of the approach to systemic risk measures motivated and outlined above. In particular, we provide reasonable conditions on the ingredients $\mathcal{C}$, π, $\mathcal{A}$, $\mathbb{A}$, and Λ such that the above definitions of ρ are well posed and ρ has the natural properties of decreasing monotonicity and quasi-convexity (or convexity). In Section 3 we analyze the situation and give various families of systemic risk measures when the risk measurement is defined in terms of the natural approach to apply some kind of aggregation to risk factors and test acceptability with respect to some one-dimensional acceptance set as in (1.16) above. Section 4 investigates the interesting class of systemic risk measures that are defined in terms of a set $\mathcal{C}$ of scenario-dependent allocations as in (1.10). Then we present two concrete examples within this class of systemic risk measures in Sections 5 and 6. In Section 5 we look Gaussian systems and consider both deterministic cash allocations as well as a certain class of random cash allocations. Further, we apply the results to a particular Gaussian system where the flow of money between the institutions (borrowing and lending) is modeled by a system of interacting diffusions (see Carmona et al. [2015]). In Section 6 we introduce an example on a finite probability space. As a consequence of the finite probability space we are able to compute systemic risk measures for very general random cash allocations $\mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$.

2 Definition of Systemic Risk Measures and Properties

In this Section we provide the definitions and properties of the systemic risk measures in our setting. As in the Introduction, we consider the set of random vectors

$$\mathcal{L}^0(\mathbb{R}^N) := \{\mathbf{X} = (X^1, \dots, X^N) \mid X^n \in \mathcal{L}^0(\Omega, \mathcal{F}, \mathbb{P}),\ n = 1, \cdots, N\},$$

on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that $\mathcal{L}^0(\mathbb{R}^N)$ is equipped with an order relation $\succeq$ such that it is a vector lattice. One such example

is provided by the order relation: $\mathbf{X}_1 \succeq \mathbf{X}_2$ if $X_1^i \geq X_2^i$ for all components $i = 1, ..., N$, where for random variables on $\mathcal{L}^0(\mathbb{R})$, the order relation is determined by $\mathbb{P}-$a.s inequality.

Definition 2.1 *Let* $\mathbf{X}_1, \mathbf{X}_2 \in \mathcal{L}^0(\mathbb{R}^N)$.

1. *A set* $\mathcal{A} \subset \mathcal{L}^0(\mathbb{R}^N)$ *is* $\succeq$ *-monotone if* $\mathbf{X}_1 \in \mathcal{A}$ *and* $\mathbf{X}_2 \succeq \mathbf{X}_1$ *implies* $\mathbf{X}_2 \in \mathcal{A}$.

2. *A map* $f : \mathcal{L}^0(\mathbb{R}^N) \to \mathcal{L}^0(\mathbb{R})$ *is* $\succeq$*-monotone decreasing if* $\mathbf{X}_2 \succeq \mathbf{X}_1$ *implies* $f(\mathbf{X}_1) \geq f(\mathbf{X}_2)$. *Analogously for functions* $f : \mathcal{L}^0(\mathbb{R}^N) \to \overline{\mathbb{R}}$.

3. *A map* $f : \mathcal{L}^0(\mathbb{R}^N) \to \overline{\mathbb{R}}$ *is quasi-convex if*

$$f(\lambda \mathbf{X}_1 + (1-\lambda)\mathbf{X}_2) \leq f(\mathbf{X}_1) \vee f(\mathbf{X}_2).$$

A vector $\mathbf{X} = (X^1, \ldots, X^N) \in \mathcal{L}^0(\mathbb{R}^N)$ denotes a configuration of risky factors at a future time T associated to a system of N entities. Let

$$\mathcal{C} \subseteq \mathcal{L}^0(\mathbb{R}^N).$$

To each $\mathbf{Y} \in \mathcal{C}$ we assign a set $\mathcal{A}^{\mathbf{Y}} \subseteq \mathcal{L}^0(\mathbb{R}^N)$. The set $\mathcal{A}^{\mathbf{Y}}$ represents the risk vectors $\mathbf{X}$ that are acceptable for the given random vector $\mathbf{Y}$. Let also consider a map

$$\pi : \mathcal{C} \to \mathbb{R}\,,$$

so that $\pi(\mathbf{Y})$ represents the risk (or cost) associated to $\mathbf{Y}$.
We now introduce the concept of monotone and (quasi-) convex systemic risk measure.

Definition 2.2 *The systemic risk measure associated with* $\mathcal{C}, \mathcal{A}^{\mathbf{Y}}$ *and* π *is a map* $\rho : \mathcal{L}^0(\mathbb{R}^N) \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$, *defined by:*

$$\rho(\mathbf{X}) := \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C},\, \mathbf{X} \in \mathcal{A}^{\mathbf{Y}}\}\,. \tag{2.1}$$

Moreover ρ *is called a quasi-convex (resp. convex) systemic risk measure if it is* $\succeq$*-monotone decreasing and quasi-convex (resp. convex on* $\{\rho(\mathbf{X}) < +\infty\}$*).*

In other words, the systemic risk of a random vector $\mathbf{X}$ is measured by the minimal risk (cost) of those random vectors $\mathbf{Y}$ that make $\mathbf{X}$ acceptable.
As already sketched in the Introduction, we now focus on several examples of systemic risk measures of the type (2.1). To guarantee that such maps are finite valued one could consider their restriction to some vector subspaces of $\mathcal{L}^0(\mathbb{R}^N)$ (for examples $\mathcal{L}^p(\mathbb{R}^N)$, $p \in [1, \infty]$) and impose further conditions

on the defining ingredients $(\pi,\, \mathcal{C},\, \mathcal{A}^{\mathbf{Y}})$ of ρ. For example, suppose that $\mathcal{C}$ and $\mathcal{A}^{\mathbf{Y}}$ satisfy the two conditions

$$\begin{aligned}&\left\{m\mathbf{1} \in \mathbb{R}^{\mathbf{N}} \mid m \in \mathbb{R}_+,\, \mathbf{1} := (1,...,1)\right\} \subseteq \mathcal{C},\\ &-m\mathbf{1} \in \mathcal{A}^{m\mathbf{1}} \text{ and } \mathcal{A}^{m\mathbf{1}} \text{ is a monotone set for each } m \in \mathbb{R}_+,\end{aligned}$$

then $\rho : \mathcal{L}^\infty(\mathbb{R}^N) \to \overline{\mathbb{R}}$ defined by (2.1) satisfies $\rho(\mathbf{X}) < +\infty$ for all $\mathbf{X} \in \mathcal{L}^\infty(\mathbb{R}^N)$. Indeed, for $m := \max_i \|X^i\|_\infty$, $\mathbf{X} \geq -m\mathbf{1} \in \mathcal{A}^{m\mathbf{1}}$ implies that $\mathbf{X} \in \mathcal{A}^{m\mathbf{1}}$ and $\pi(m\mathbf{1}) < +\infty$.
Clearly, other sufficient conditions may be obtained in each specific example of systemic risk measures considered in the subsequent Sections.
We opt to accept the possibility that such maps ρ may assume values $\pm\infty$. However, it is not difficult to find simple sufficient conditions assuring that the systemic risk measure in (2.1) is proper (not identically equal to $+\infty$). One such example is the condition:

$$\text{if } \mathbf{0} \in \mathcal{C} \text{ and } \mathbf{0} \in \mathcal{A}^{\mathbf{0}} \text{ then } \rho(\mathbf{0}) \leq \pi(\mathbf{0}) < +\infty.$$

We now consider the "structural properties" (i.e. monotonicity, quasiconvexity, convexity) of our systemic risk measures and introduce two sets of conditions (properties (P1), (P2) and (P3) below and the alternative properties (P2a) and (P3a)) that guarantee that the map in (2.1) is a quasi-convex (or convex) risk measure. In Section 3 we show that these sets of conditions can be easily checked in some relevant examples of maps in the form (2.1), where the set $\mathcal{A}^{\mathbf{Y}}$ is determined from aggregation and one-dimensional acceptance sets.
We introduce the following properties:

(P1) For all $\mathbf{Y} \in \mathcal{C}$ the set $\mathcal{A}^{\mathbf{Y}} \subset \mathcal{L}^0(\mathbb{R}^N)$ is $\succeq$-monotone.

(P2) For all $m \in \mathbb{R}$, for all $\mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{C}$ such that $\pi(\mathbf{Y}_1) \leq m$ and $\pi(\mathbf{Y}_2) \leq m$ and for all $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and all $\lambda \in [0,1]$ there exists $\mathbf{Y} \in \mathcal{C}$ such that $\pi(\mathbf{Y}) \leq m$ and $\lambda\mathbf{X}_1 + (1-\lambda)\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}}$.

(P3) For all $\mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{C}$ and all $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and all $\lambda \in [0,1]$ there exists $\mathbf{Y} \in \mathcal{C}$ such that $\pi(\mathbf{Y}) \leq \lambda\pi(\mathbf{Y_1}) + (1-\lambda)\pi(\mathbf{Y_2})$ and $\lambda\mathbf{X}_1 + (1-\lambda)\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}}$.

It is clear that property (P3) implies property (P2). Moreover, we have:

Lemma 2.3 *i) If the systemic risk measure ρ defined in (2.1) satisfies the properties (P1) and (P2), then ρ is $\succeq$-monotone decreasing and quasi-convex.*

ii) If the systemic risk measure ρ defined in (2.1) satisfies the properties (P1) and (P3), then ρ is $\succeq$-monotone decreasing and convex on $\{\rho(\mathbf{X}) < +\infty\}$.

Proof. Set

$$B(\mathbf{X}) := \left\{\mathbf{Y} \in C \mid \mathbf{X} \in \mathcal{A}^{\mathbf{Y}}\right\}.$$

First assume that property (P1) holds and w.l.o.g. suppose $\mathbf{X}_2 \succeq \mathbf{X}_1$ and $B(\mathbf{X}_1) \neq \varnothing$. Then property (P1) implies that if $\mathbf{X}_1 \in \mathcal{A}^Y$ and $\mathbf{X}_2 \succeq \mathbf{X}_1$ then: $B(\mathbf{X}_1) \subseteq B(\mathbf{X}_2)$. Hence:

$$\rho(\mathbf{X}_1) = \inf\{\pi(\mathbf{Y}) \mid \mathbf{Y} \in B(\mathbf{X}_1)\} \geq \inf\{\pi(\mathbf{Y}) \mid \mathbf{Y} \in B(\mathbf{X}_2)\} = \rho(\mathbf{X}_2)$$

so that ρ is $\succeq$-monotone decreasing.

i) Now assume that property (P2) holds and let $\mathbf{X}_1, \mathbf{X}_2 \in \mathcal{L}^0(\mathbb{R}^N)$ be arbitrarily chosen. For the quasi-convexity we need to prove, for any $m \in \mathbb{R}$, that:

$$\rho(\mathbf{X}_1) \leq m \text{ and } \rho(\mathbf{X}_2) \leq m \Rightarrow \rho(\lambda\mathbf{X}_1 + (1-\lambda)\mathbf{X}_2) \leq m.$$

By definition of the infimum in the definition of $\rho(\mathbf{X}_i)$, $\forall\varepsilon > 0$ there exist $\mathbf{Y}_i \in \mathcal{C}$ such that $\mathbf{X}_i \in \mathcal{A}^{\mathbf{Y}_i}$ and

$$\pi(\mathbf{Y}_i) \leq \rho(\mathbf{X}_i) + \varepsilon \leq m + \varepsilon, \ i = 1, 2.$$

Take any $\lambda \in [0, 1]$. Property (P2) guarantees the existence of $\mathbf{Z} \in \mathcal{C}$ such that $\pi(\mathbf{Z}) \leq m + \varepsilon$ and $\lambda\mathbf{X}_1 + (1-\lambda)\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Z}}$. Hence

$$\begin{aligned}\rho(\lambda\mathbf{X}_1 + (1-\lambda)\mathbf{X}_2) &= \inf\{\pi(\mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}, \lambda\mathbf{X}_1 + (1-\lambda)\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}}\} \\ &\leq \pi(\mathbf{Z}) \leq m + \varepsilon.\end{aligned}$$

As this holds for any $\varepsilon > 0$, we obtain the quasi-convexity.

ii) Assume that property (P3) holds and that $\mathbf{X}_1, \mathbf{X}_2 \in \mathcal{L}^0(\mathbb{R}^N)$ satisfy $\rho(\mathbf{X}_i) < +\infty$. Then $B(\mathbf{X}_i) \neq \varnothing$ and, as before, $\forall\varepsilon > 0$ there exists $\mathbf{Y}_i \in \mathcal{L}^0(\mathbb{R}^N)$ such that: $\mathbf{Y}_i \in \mathcal{C}$, $\mathbf{X}_i \in \mathcal{A}^{\mathbf{Y}_i}$ and

$$\pi(\mathbf{Y}_i) \leq \rho(\mathbf{X}_i) + \varepsilon, \ i = 1, 2. \tag{2.2}$$

By property (P3) there exists $\mathbf{Z} \in \mathcal{C}$ such that $\pi(\mathbf{Z}) \leq \lambda\pi(\mathbf{Y_1}) + (1-\lambda)\pi(\mathbf{Y_2})$ and $\lambda\mathbf{X}_1 + (1-\lambda)\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Z}}$. Hence

$$\begin{aligned}\rho(\lambda\mathbf{X}_1 + (1-\lambda)\mathbf{X}_2) &= \inf\{\pi(\mathbf{Y}) \mid \mathbf{Y} \in \mathcal{C}, \lambda\mathbf{X}_1 + (1-\lambda)\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}}\} \\ &\leq \pi(\mathbf{Z}) \leq \lambda\pi(\mathbf{Y}_1) + (1-\lambda)\pi(\mathbf{Y}_2) \\ &\leq \lambda\rho(\mathbf{X}_1) + (1-\lambda)\rho(\mathbf{X}_2) + \epsilon,\end{aligned}$$

from (2.2). As this holds for any $\varepsilon > 0$, the map ρ is convex on $\{\rho(\mathbf{X}) < +\infty\}$.

■

We now consider the following alternative properties:

(P2a) For all $\mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{C}$, $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and $\lambda \in [0,1]$ there exists $\alpha \in [0,1]$ such that $\lambda \mathbf{X}_1 + (1-\lambda)\mathbf{X}_2 \in \mathcal{A}^{\alpha \mathbf{Y}_1 + (1-\alpha)\mathbf{Y}_2}$.

(P3a) For all $\mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{C}$, $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and $\lambda \in [0,1]$ it holds: $\lambda \mathbf{X}_1 + (1-\lambda)\mathbf{X}_2 \in \mathcal{A}^{\lambda \mathbf{Y}_1 + (1-\lambda)\mathbf{Y}_2}$.

It is clear that property (P3a) implies property (P2a). Furthermore we introduce the following properties for $\mathcal{C}$ and π:

(P4) $\mathcal{C}$ is convex,

(P5) π is quasi-convex,

(P6) π is convex.

We have the following:

Lemma 2.4 *i) Under the conditions (P1), (P2a), (P4), and (P5) the map ρ defined in (2.1) is a quasi-convex systemic risk measure.*

ii) Under the conditions (P1), (P3a) , (P4) and (P6) the map ρ defined in (2.1) is a convex systemic risk measure.

Proof.

i) It follows from Lemma 2.3 and the fact that the properties (P2a), (P4), and (P5) imply (P2). Indeed, let $\mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{C}$ such that $\pi(\mathbf{Y}_1) \leq m$, $\pi(\mathbf{Y}_2) \leq m$ and let $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and $\lambda \in [0,1]$. Then there exists $\alpha \in [0,1]$ such that $\lambda \mathbf{X}_1 + (1-\lambda)\mathbf{X}_2 \in \mathcal{A}^{\alpha \mathbf{Y}_1 + (1-\alpha)\mathbf{Y}_2}$. If we set: $\mathbf{Y} := \alpha \mathbf{Y}_1 + (1-\alpha)\mathbf{Y}_2 \in \mathcal{C}$ then $\lambda \mathbf{X}_1 + (1-\lambda)\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}}$ and $\pi(\alpha \mathbf{Y}_1 + (1-\alpha)\mathbf{Y}_2) \leq \max(\pi(\mathbf{Y}_1), \pi(\mathbf{Y}_2)) \leq m$.

ii) It follows from Lemma 2.3 and the fact that the properties (P3a) , (P4), and (P6) imply (P3). Indeed, let $\mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{C}$ and let $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and $\lambda \in [0,1]$. If we set: $\mathbf{Y} := \lambda \mathbf{Y}_1 + (1-\lambda)\mathbf{Y}_2 \in \mathcal{C}$ then $\lambda \mathbf{X}_1 + (1-\lambda)\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}}$ and $\pi(\mathbf{Y}) \leq \lambda \pi(\mathbf{Y_1}) + (1-\lambda)\pi(\mathbf{Y_2})$.

■

3 Systemic Risk Measures via Aggregation and One-dimensional Acceptance Sets

In this Section we study four classes of systemic risk measures in the form (2.1), which differ from each other by the definition of their aggregation functions and their acceptance sets. However, these four classes, defined in equations (3.2), (3.3), (3.6), (3.7), all satisfy the structural properties of monotonicity and quasi-convexity (or convexity). We consider the following definitions and assumptions, which will hold true throughout this Section:

1. the aggregation functions are

$$\begin{aligned} \Lambda &: \mathcal{L}^0(\mathbb{R}^N) \times \mathcal{C} \to \mathcal{L}^0(\mathbb{R}), \\ \Lambda_1 &: \mathcal{L}^0(\mathbb{R}^N) \to \mathcal{L}^0(\mathbb{R}), \end{aligned}$$

   and we assume that Λ_1 is $\succeq$-increasing and concave;

2. the acceptance family

$$(\mathcal{B}^x)_{x\in\mathbb{R}}$$

   is an increasing family with respect to x and each set $\mathcal{B}^x \subseteq \mathcal{L}^0(\mathbb{R})$ is assumed monotone and convex;

3. the acceptance subset

$$\mathbb{A} \subseteq \mathcal{L}^0(\mathbb{R}).$$

   is assumed monotone and convex.

The convexity of the acceptance set $\mathbb{A} \subseteq \mathcal{L}^0(\mathbb{R})$ (or of the acceptance family $(\mathcal{B}^x)_{x\in\mathbb{R}}$) are the standard conditions which have been assumed since the origin of the theory of risk measures. The concavity of the aggregation functions is justified, not only from the many relevant examples in literature, but also by the preservation of the convexity from one dimensional acceptance sets to multi-dimensional ones. Indeed, let $\Theta : \mathcal{L}^0(\mathbb{R}^N) \to \mathcal{L}^0(\mathbb{R})$ be an aggregation function, $\mathbb{A} \subseteq \mathcal{L}^0(\mathbb{R})$ a one dimensional acceptance set and define $\mathcal{A} \subseteq \mathcal{L}^0(\mathbb{R}^N)$ as the inverse image $\mathcal{A} := \Theta^{-1}(\mathbb{A})$. Suppose that Θ is increasing and concave. Then one may easily check that if $\mathbb{A}$ is monotone and convex, then $\mathcal{A}$ is monotone and convex.
We note that in all results of this Section, the selection of the set $\mathcal{C} \subseteq \mathcal{L}^0(\mathbb{R}^N)$ of permitted vectors is left as general as possible (in some cases we require the convexity of $\mathcal{C}$ and only in Proposition 3.6 we further ask that $\mathcal{C} + \mathbb{R}^N_+ \in \mathcal{C}$). Therefore, we are very flexible in the choice of $\mathcal{C}$ and we may interpret its elements as vectors of admissible or safe financial assets, or merely as cash

vectors. Only in the next Section we attribute a particular structure to $\mathcal{C}$. In the conclusive statements of the following propositions in this section we apply Lemma 2.3 and Lemma 2.4 without explicit mention.

Proposition 3.1 *Let*

$$\mathcal{A}^{\mathbf{Y}} := \left\{\mathbf{Z} \in \mathcal{L}^0(\mathbb{R}^N) \mid \Lambda(\mathbf{Z}, \mathbf{Y}) \in \mathbb{A}\right\}, \quad \mathbf{Y} \in \mathcal{C}, \tag{3.1}$$

Λ be concave, and $\Lambda(\cdot, \mathbf{Y})$ be $\succeq$-increasing for all $\mathbf{Y} \in \mathcal{C}$. Then $\mathcal{A}^{\mathbf{Y}}$ satisfies properties (P1) and (P3a) (and (P2a)). The map ρ defined in (2.1) is given by

$$\rho(\mathbf{X}) := \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C},\, \Lambda(\mathbf{X}, \mathbf{Y}) \in \mathbb{A}\}, \tag{3.2}$$

and is a quasi-convex systemic risk measure, under the assumptions (P4) and (P5); it is a convex systemic risk measure under the assumptions (P4) and (P6).

Proof. Property (P1): Let $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}}$ and $\mathbf{X}_2 \succeq \mathbf{X}_1$. Note that $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}}$ implies $\Lambda(\mathbf{X}_1, \mathbf{Y}) \in \mathbb{A}$ and $\mathbf{X}_2 \succeq \mathbf{X}_1$ implies $\Lambda(\mathbf{X}_2, \mathbf{Y}) \geq \Lambda(\mathbf{X}_1, \mathbf{Y})$. Since $\mathbb{A}$ is monotone we have $\Lambda(\mathbf{X}_2, \mathbf{Y}) \in \mathbb{A}$ and $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}}$.
Property (P3a) : Let $\mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{C}$, $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and $\lambda \in [0, 1]$. Then $\Lambda(\mathbf{X}_1, \mathbf{Y}_1) \in \mathbb{A}$ and $\Lambda(\mathbf{X}_2, \mathbf{Y}_2) \in \mathbb{A}$ and the convexity of $\mathbb{A}$ guarantees:

$$\lambda\Lambda(\mathbf{X}_1, \mathbf{Y}_1) + (1-\lambda)\Lambda(\mathbf{X}_2, \mathbf{Y}_2) \in \mathbb{A}.$$

From the concavity of $\Lambda(\cdot, \cdot)$ we obtain:

$$\Lambda(\lambda(\mathbf{X}_1, \mathbf{Y}_1) + (1-\lambda)(\mathbf{X}_2, \mathbf{Y}_2)) \geq \lambda\Lambda(\mathbf{X}_1, \mathbf{Y}_1) + (1-\lambda)\Lambda(\mathbf{X}_2, \mathbf{Y}_2) \in \mathbb{A}.$$

The monotonicity of $\mathbb{A}$ implies:

$$\Lambda(\lambda\mathbf{X}_1 + (1-\lambda)\mathbf{X}_2, \lambda\mathbf{Y}_1 + (1-\lambda)\mathbf{Y}_2) = \Lambda(\lambda(\mathbf{X}_1, \mathbf{Y}_1) + (1-\lambda)(\mathbf{X}_2, \mathbf{Y}_2)) \in \mathbb{A},$$

and therefore, $\lambda\mathbf{X}_1 + (1-\lambda)\mathbf{X}_2 \in \mathcal{A}^{\lambda\mathbf{Y}_1 + (1-\lambda)\mathbf{Y}_2}$. ■

The class of systemic risk measures defined in (3.2) is a fairly general representation since the aggregation function Λ needs only to be concave and increasing in one of its arguments and the acceptance set $\mathbb{A}$ is only required to be monotone and convex. As shown in the following Corollary 3.2, such a risk measure may describe either the possibility of "first aggregate and second add the capital" (for example if $\Lambda(\mathbf{X}, \mathbf{Y}) := \Lambda_1(\mathbf{X}) + \Lambda_2(\mathbf{Y})$, where $\Lambda_2(\mathbf{Y})$ could be interpreted as the discounted cost of $\mathbf{Y}$) or the case of "first add and second aggregate" (for example if $\Lambda(\mathbf{X}, \mathbf{Y}) := \Lambda_1(\mathbf{X} + \mathbf{Y})$).

Corollary 3.2 *Let $\Lambda_2 : \mathcal{C} \to \mathcal{L}^0(\mathbb{R})$ be concave, let $\mathcal{A}^{\mathbf{Y}}$ be defined in (3.1), where the function Λ has one of the following forms:*

$$\Lambda(\mathbf{Z}, \mathbf{Y}) = \Lambda_1(\mathbf{Z}) + \Lambda_2(\mathbf{Y}),$$
$$\Lambda(\mathbf{Z}, \mathbf{Y}) = \Lambda_1(\mathbf{Z} + \mathbf{Y}).$$

Then, $\mathcal{A}^{\mathbf{Y}}$ fulfills properties (P1) and (P3a) . Therefore, the map ρ defined in (3.2) is a quasi-convex systemic risk measure under the assumptions (P4) and (P5); it is a convex systemic risk measure under the assumptions (P4) and (P6).

We now turn to the class of truly quasi-convex systemic risk measures defined by (3.3), which represents the generalization of the quasi-convex risk measure in (1.6) in the one-dimensional case.

Proposition 3.3 *Let $\theta : \mathcal{C} \to \mathbb{R}$. Then the set*

$$\mathcal{A}^{\mathbf{Y}} := \left\{ \mathbf{Z} \in \mathcal{L}^0(\mathbb{R}^N) \mid \Lambda_1(\mathbf{Z}) \in \mathcal{B}^{\theta(\mathbf{Y})} \right\}, \quad \mathbf{Y} \in \mathcal{C},$$

satisfies properties (P1) and (P2a). The map ρ defined in (2.1) is given by

$$\rho(\mathbf{X}) := \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C},\, \Lambda_1(\mathbf{X}) \in \mathcal{B}^{\theta(\mathbf{Y})}\}, \tag{3.3}$$

and under the assumptions (P4) and (P5) is a quasi-convex systemic risk measure.

Proof. Property (P1): Let $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}}$ and $\mathbf{X}_2 \succeq \mathbf{X}_1$. Note that $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}}$ implies $\Lambda_1(\mathbf{X}_1) \in \mathcal{B}^{\theta(\mathbf{Y})}$ and $\mathbf{X}_2 \succeq \mathbf{X}_1$ implies $\Lambda_1(\mathbf{X}_2) \geq \Lambda_1(\mathbf{X}_1)$. Since $\mathcal{B}^x$ is a monotone set for all x, we have $\Lambda_1(\mathbf{X}_2) \in \mathcal{B}^{\theta(\mathbf{Y})}$ and $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}}$.
Property (P2a): Fix $\mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{C}$, $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and $\lambda \in [0, 1]$. Then $\Lambda_1(\mathbf{X}_1) \in \mathcal{B}^{\theta(\mathbf{Y}_1)}$ and $\Lambda_1(\mathbf{X}_2) \in \mathcal{B}^{\theta(\mathbf{Y}_2)}$. From the concavity of Λ_1 we obtain:

$$\Lambda_1(\lambda\mathbf{X}_1 + (1-\lambda)\mathbf{X}_2) \geq \lambda\Lambda_1(\mathbf{X}_1) + (1-\lambda)\Lambda_1(\mathbf{X}_2) \in \lambda\mathcal{B}^{\theta(\mathbf{Y}_1)} + (1-\lambda)\mathcal{B}^{\theta(\mathbf{Y}_2.)}$$

Since $(\mathcal{B}^x)_{x\in\mathbb{R}}$ is an increasing family and each $\mathcal{B}^x$ is convex, we deduce

$$\lambda\mathcal{B}^{\theta(\mathbf{Y}_1)} + (1-\lambda)\mathcal{B}^{\theta(\mathbf{Y}_2)} \subseteq \mathcal{B}^{\max\{\theta(\mathbf{Y}_1), \theta(\mathbf{Y}_2)\}}. \tag{3.4}$$

Suppose that $\max(\theta(\mathbf{Y}_1), \theta(\mathbf{Y}_2)) = \theta(\mathbf{Y}_1)$, using the monotonicity of the set $\mathcal{B}^{\theta(\mathbf{Y}_1)}$ we deduce

$$\Lambda_1(\lambda\mathbf{X}_1 + (1-\lambda)\mathbf{X}_2) \in \mathcal{B}^{\theta(\mathbf{Y}_1)},$$

and $\lambda\mathbf{X}_1+(1-\lambda)\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_1}$, so that property (P2a) is satisfied with $\alpha = 1$. ■

In the risk measure (3.3) we are not allowing to add capital to $\mathbf{X}$ before the aggregation takes place**,** as the quasi-convexity property of ρ would be lost in general. Next, we contemplate this possibility (i.e. we consider conditions of the type $\Lambda(\mathbf{X}, \mathbf{Y}) \in \mathcal{B}^{\theta(\mathbf{Y})}$) in the systemic risk measures (3.6) and (3.7) but only under some non trivial restrictions: for the case (3.6) we impose conditions on the aggregation function Λ that are made explicit in equation (3.5) and in the Example 3.5. In contrast, for the case (3.7) we consider a general aggregation function Λ, but we restrict the family of acceptance sets to $\mathcal{B}^{\pi(\mathbf{Y})}$, where π is positively linear and represents the risk level of the acceptance family.

Proposition 3.4 *Let $\theta : \mathcal{C} \to \mathbb{R}$ and*

$$\mathcal{A}^{\mathbf{Y}} := \left\{ \mathbf{Z} \in \mathcal{L}^0(\mathbb{R}^N) \mid \Lambda(\mathbf{Z}, \mathbf{Y}) \in \mathcal{B}^{\theta(\mathbf{Y})} \right\}, \quad \mathbf{Y} \in \mathcal{C},$$

where $\Lambda(\cdot, \mathbf{Y}) : \mathcal{L}^0(\mathbb{R}^N) \to \mathcal{L}^0(\mathbb{R})$ is $\succeq$-increasing and concave for all $\mathbf{Y} \in \mathcal{C}$. Assume in addition that:

$$\theta(\mathbf{Y}_2) \geq \theta(\mathbf{Y}_1) \Rightarrow \Lambda(\mathbf{X}, \mathbf{Y}_2) \geq \Lambda(\mathbf{X}, \mathbf{Y}_1) \text{ for all } \mathbf{X} \in \mathcal{L}^0(\mathbb{R}^N). \tag{3.5}$$

Then properties (P1) and (P2) hold. The map ρ defined in (2.1) is given by

$$\rho(\mathbf{X}) := \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C},\ \Lambda(\mathbf{X}, \mathbf{Y}) \in \mathcal{B}^{\theta(\mathbf{Y})}\} \tag{3.6}$$

and is a quasi-convex systemic risk measure.

Proof. Property (P1): Let $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}}$ and $\mathbf{X}_2 \succeq \mathbf{X}_1$. Note that $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}}$ implies $\Lambda(\mathbf{X}_1, \mathbf{Y}) \in \mathcal{B}^{\theta(\mathbf{Y})}$ and $\mathbf{X}_2 \succeq \mathbf{X}_1$ implies $\Lambda(\mathbf{X}_2, \mathbf{Y}) \geq \Lambda(\mathbf{X}_1, \mathbf{Y})$. Since $\mathcal{B}^{\theta(\mathbf{Y})}$ is a monotone set, we have $\Lambda(\mathbf{X}_2, \mathbf{Y}) \in \mathcal{B}^{\theta(\mathbf{Y})}$ and $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}}$.
Property (P2): Fix $m \in \mathbb{R}$, $\mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{C}$ such that $\pi(\mathbf{Y}_1) \leq m$ and $\pi(\mathbf{Y}_2) \leq m$, $\lambda \in [0, 1]$ and take $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$ and $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$. Then $\Lambda(\mathbf{X}_1, \mathbf{Y}_1) \in \mathcal{B}^{\theta(\mathbf{Y}_1)}$ and $\Lambda(\mathbf{X}_2, \mathbf{Y}_2) \in \mathcal{B}^{\theta(\mathbf{Y}_2)}$. Then, w.l.o.g. we may assume that $\theta(\mathbf{Y}_2) \geq \theta(\mathbf{Y}_1)$. Since $(\mathcal{B}^x)_{x\in\mathbb{R}}$ is an increasing family, we have $\mathcal{B}^{\theta(\mathbf{Y}_1)} \subseteq \mathcal{B}^{\theta(\mathbf{Y}_2)}$. Condition (3.5) implies $\Lambda(\mathbf{X}_1, \mathbf{Y}_2) \geq \Lambda(\mathbf{X}_1, \mathbf{Y}_1) \in \mathcal{B}^{\theta(\mathbf{Y}_1)} \subseteq \mathcal{B}^{\theta(\mathbf{Y}_2)}$, so that $\Lambda(\mathbf{X}_1, \mathbf{Y}_2) \in \mathcal{B}^{\theta(\mathbf{Y}_2)}$. From the concavity of $\Lambda(\cdot, \mathbf{Y}_2)$ and the convexity of $\mathcal{B}^{\theta(\mathbf{Y}_2)}$ we obtain:

$$\Lambda(\lambda\mathbf{X}_1+(1-\lambda)\mathbf{X}_2, \mathbf{Y}_2) \geq \lambda\mathbf{\Lambda}(\mathbf{X}_1, \mathbf{Y}_2) + (1-\lambda)\mathbf{\Lambda}(\mathbf{X}_2, \mathbf{Y}_2) \in \mathcal{B}^{\theta(\mathbf{Y}_2)}.$$

Hence $\Lambda(\lambda\mathbf{X}_1+(1-\lambda)\mathbf{X}_2, \mathbf{Y}_2) \in \mathcal{B}^{\theta(\mathbf{Y}_2)}$ which means: $\lambda\mathbf{X}_1+(1-\lambda)\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$. Since $\pi(\mathbf{Y}_2) \leq m$, property (P2) holds with $\mathbf{Y} = \mathbf{Y}_2$. ■

Example 3.5 *Let $\theta : \mathcal{C} \to \mathbb{R}$ and let Λ be defined by*

$$\Lambda(\mathbf{X}, \mathbf{Y}) = g(\mathbf{X}, \theta(\mathbf{Y})),$$

where $g(\cdot, z) : \mathbb{R}^N \to \mathbb{R}$ is increasing and concave for all $z \in \mathbb{R}$ and $g(x, \cdot) : \mathbb{R} \to \mathbb{R}$ is increasing for all $x \in \mathbb{R}^N$. Then Λ satisfies all the assumptions in Proposition 3.4. Examples of functions g satisfying these conditions are:

$$g(x, z) = f(x) + h(z),$$

with f increasing and concave and h increasing, or

$$g(x, z) = f(x)h(z)$$

with f increasing, concave and positive and h increasing and positive.

Proposition 3.6 *Suppose that $\mathcal{C} \subseteq \mathcal{L}^0(\mathbb{R}^N)$ is a convex set such $\mathbf{0} \in \mathcal{C}$ and $\mathcal{C} + \mathbb{R}^N_+ \in \mathcal{C}$. Assume in addition that $\pi : \mathcal{C} \to \mathbb{R}$ satisfies $\pi(u) = 1$ for a given $u \in \mathbb{R}^N_+$, $u \neq \mathbf{0}$, and*

$$\pi(\alpha_1 Y_1 + \alpha_2 Y_2) = \alpha_1 \pi(Y_1) + \alpha_2 \pi(Y_2)$$

for all $\alpha_i \in \mathbb{R}_+$ and $Y_i \in \mathcal{C}$. Let

$$\mathcal{A}^{\mathbf{Y}} := \left\{ \mathbf{Z} \in \mathcal{L}^0(\mathbb{R}^N) \mid \Lambda(\mathbf{Z}, \mathbf{Y}) \in \mathcal{B}^{\pi(\mathbf{Y})} \right\},$$

where Λ is concave and $\Lambda(\mathbf{X}, \cdot) : \mathcal{C} \to \mathcal{L}^0(\mathbb{R})$ is increasing (with respect to the componentwise ordering) for all $\mathbf{X} \in \mathcal{L}^0(\mathbb{R}^N)$. Then the family of sets $\mathcal{A}^{\mathbf{Y}}$ fulfill properties (P1) and (P2). The map ρ defined in (2.1) is given by

$$\rho(\mathbf{X}) = \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C},\, \Lambda(\mathbf{X}, \mathbf{Y}) \in \mathcal{B}^{\pi(\mathbf{Y})}\} \tag{3.7}$$

and is a quasi-convex systemic risk measure.

Proof. Property (P1): it follows immediately from the monotonicity of $\mathcal{B}^x$, $x \in \mathbb{R}$.
Property (P2): Let $\mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{C}$, $m \in \mathbb{R}$ and assume w.l.o.g. that $\pi(\mathbf{Y}_1) \leq \pi(\mathbf{Y}_2) \leq m$. Let $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and $\lambda \in [0, 1]$. Then $\Lambda(\mathbf{X}_1, \mathbf{Y}_1) \in \mathcal{B}^{\pi(\mathbf{Y}_1)}$ and $\Lambda(\mathbf{X}_2, \mathbf{Y}_2) \in \mathcal{B}^{\pi(\mathbf{Y}_2)}$. Because $(\mathcal{B}^x)_{x \in \mathbb{R}}$ is increasing, we get $\Lambda(\mathbf{X}_1, \mathbf{Y}_1) \in \mathcal{B}^{\pi(\mathbf{Y}_2)}$. Set

$$\hat{\mathbf{Y}}_1 := \mathbf{Y}_1 + (\pi(\mathbf{Y}_2) - \pi(\mathbf{Y}_1))u \in \mathcal{C}.$$

Then $\hat{\mathbf{Y}}_1 \geq \mathbf{Y}_1$ and, since $\Lambda(\mathbf{X}, \cdot)$ is increasing, $\Lambda(\mathbf{X}_1, \hat{\mathbf{Y}}_1) \geq \Lambda(\mathbf{X}_1, \mathbf{Y}_1) \in \mathcal{B}^{\pi(\mathbf{Y}_2)}$ and

$$\Lambda(\mathbf{X}_1, \hat{\mathbf{Y}}_1) \in \mathcal{B}^{\pi(\mathbf{Y}_2)}$$

because of the monotonicity of $\mathcal{B}^{\pi(\mathbf{Y}_2)}$. Letting

$$\mathbf{Y} := \lambda\hat{\mathbf{Y}}_1 + (1-\lambda)\mathbf{Y}_2 \in \mathcal{C}\,,$$

and using the properties of π we obtain:

$$\begin{aligned}\pi(\mathbf{Y}) &= \pi(\lambda[\mathbf{Y}_1 + (\pi(\mathbf{Y}_2) - \pi(\mathbf{Y}_1))u] + (1-\lambda)\mathbf{Y}_2) \\ &= \lambda\pi(\mathbf{Y}_1 + (\pi(\mathbf{Y}_2) - \pi(\mathbf{Y}_1))u) + (1-\lambda)\pi(\mathbf{Y}_2) = \pi(\mathbf{Y_2}) \leq m.\end{aligned}$$

From the concavity of $\Lambda(\cdot,\cdot)$ and the convexity of $\mathcal{B}^{\pi(\mathbf{Y}_2)}$ we obtain:

$$\begin{aligned}\Lambda(\lambda\mathbf{X}_1+(1-\lambda)\mathbf{X}_2, \mathbf{Y}) &= \Lambda(\lambda\mathbf{X}_1+(1-\lambda)\mathbf{X}_2, \lambda\hat{\mathbf{Y}}_1 + (1-\lambda)\mathbf{Y}_2) \\ &= \Lambda(\lambda(\mathbf{X}_1, \hat{\mathbf{Y}}_1) + (1-\lambda)(\mathbf{X}_2, \mathbf{Y}_2)) \\ &\geq \lambda\Lambda(\mathbf{X}_1, \hat{\mathbf{Y}}_1) + (1-\lambda)\mathbf{\Lambda}(\mathbf{X}_2, \mathbf{Y}_2) \\ &\quad \in \mathcal{B}^{\pi(\mathbf{Y}_2)} = \mathcal{B}^{\pi(\mathbf{Y})},\end{aligned}$$

and the monotonicity of $\mathcal{B}^{\pi(\mathbf{Y})}$ implies:

$$\Lambda(\lambda\mathbf{X}_1+(1-\lambda)\mathbf{X}_2, \mathbf{Y}) \in \mathcal{B}^{\pi(\mathbf{Y})},$$

which means: $\lambda\mathbf{X}_1+(1-\lambda)\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}}$. Hence, property (P2) is satisfied. ■

4 Scenario-dependent Allocations

We will now focus on the particularly interesting family of sets $\mathcal{C}$ of risk level vectors $\mathbf{Y}$ defined by

$$\mathcal{C} \subseteq \{\mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^N) \mid \sum_{n=1}^{N} Y^n \in \mathbb{R}\} =: \mathcal{C}_{\mathbb{R}}. \tag{4.1}$$

A vector $\mathbf{Y} \in \mathcal{C}$ as in (4.1) can be interpreted as cash amount $\sum_{n=1}^{N} Y^n \in \mathbb{R}$ (which is known today because it is deterministic) that at the future time horizon T is allocated to the financial institutions according to the realized scenario. That is, for $i = 1, ..., N$, $Y^i(\omega)$ is allocated to institution i in case scenario ω has been realized at T, but the total allocated cash amount $\sum_{n=1}^{N} Y^n$ stays constant over the different scenarios. One could think of a lender of last resort or a regulator who at time T has a certain amount of cash at disposal to distribute among financial institutions in the most efficient way (with respect to systemic risk) according to the scenario that has been realized. Restrictions on the admissible distributions of cash are implied by the choice of set $\mathcal{C}$. For example, choosing $\mathcal{C} = \mathbb{R}^N$ corresponds to the fact that the distribution is deterministic, i.e. the allocation to each

institution is already determined today, whereas for $\mathcal{C} = \mathcal{C}_{\mathbb{R}}$ the distribution can be chosen completely freely depending on the scenario ω that has been realized. Note that the latter case includes potential negative cash allocations, i.e. withdrawals of cash from certain components which allows for cross-subsidization between financial institutions. The (more realistic) situation of scenario-dependent cash distribution without cross-subsidization is represented by the set

$$\mathcal{C} := \{\mathbf{Y} \in \mathcal{C}_{\mathbb{R}} \mid Y^i \geq 0, i = 1, ...N\}.$$

In this section we give some structural results and examples concerning systemic risk measures defined in terms of sets $\mathcal{C}$ as in (4.1). In Section 5 and 6 we then present two more extensive examples of systemic risk measures that employ specific sets $\mathcal{C}$ of type (4.1).
In the following we always assume the componentwise order relation on $\mathcal{L}^0(\mathbb{R}^N)$, i.e. $\mathbf{X}_1 \succeq \mathbf{X}_2$ if $\mathbf{X}_1^i \geq \mathbf{X}_2^i$ for all components $i = 1, ..., N$, and we start by specifying a general class of quasi-convex systemic risk measures that allow the interpretation of the minimal total amount needed to secure the system by scenario-dependent cash allocations as described above. To this end let $\mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}$ be such that

$$\mathcal{C} + \mathbb{R}_+^N \in \mathcal{C}. \tag{4.2}$$

Let the valuation $\pi(\mathbf{Y})$ of a $\mathbf{Y} \in \mathcal{C}$ be given by $\widetilde{\pi}(\sum_{n=1}^N Y^n)$ for $\widetilde{\pi} : \mathbb{R} \to \mathbb{R}$ increasing (for example the present value of the total cash amount $\sum_{n=1}^N Y^n$ at time T). Further, let $(\mathcal{A}^x)_{x\in\mathbb{R}}$ be an increasing family (w.r.t. x) of monotone, convex subsets $\mathcal{A}^x \subseteq \mathcal{L}^0(\mathbb{R}^{\mathbb{N}})$, and let $\theta : \mathbb{R} \to \mathbb{R}$ be an increasing function. We can then define the following family of systemic risk measures

$$\rho(\mathbf{X}) := \inf\{\pi(\mathbf{Y}) \in \mathbb{R} \mid \mathbf{Y} \in \mathcal{C}, \mathbf{X} + \mathbf{Y} \in \mathcal{A}^{\theta(\sum Y^n)}\}, \tag{4.3}$$

i.e. the risk measure can be interpreted as the valuation of the minimal total amount needed at time T to secure the system by distributing the cash in the most effective way among institutions. Note that here the criteria whether a system is safe or not after injecting a vector $\mathbf{Y}$ is given by the acceptance set $\mathcal{A}^{\theta(\sum Y^n)}$ which itself depends on the total amount $\sum_{n=1}^N Y^n$. This gives, for example, the possibility of modeling an increasing level of prudence when defining safe systems for higher amounts of the required total cash. This effect will lead to truly quasi-convex systemic risk measures as the next proposition shows:

Proposition 4.1 *The family of sets*

$$\mathcal{A}^{\mathbf{Y}} := \mathcal{A}^{\theta(\sum Y^n)} - \mathbf{Y}, \ \mathbf{Y} \in \mathcal{C},$$

fulfills properties (P1) and (P2) with respect to the componentwise order relation on $\mathcal{L}^0(\mathbb{R}^N)$*. Hence the map* (4.3) *is a quasi-convex risk measure. If further* $\widetilde{\pi}$ *is convex and* θ *is constant then the map* (4.3) *is even a convex risk measure.*

Proof. Property (P1) follows immediately from the monotonicity of $\mathcal{A}^x, x \in \mathbb{R}$. To show Property (P2) let $\mathbf{Y}_1, \mathbf{Y}_2 \in \mathcal{C}$, $m \in \mathbb{R}$, and $\widetilde{\pi}(\sum_{n=1}^N Y_1^n) \leq \widetilde{\pi}(\sum_{n=1}^N Y_2^n) \leq m$, where w.l.o.g. $\sum Y_1^n \leq \sum Y_2^n$. Further, let $\mathbf{X}_1 \in \mathcal{A}^{\mathbf{Y}_1}$, $\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}_2}$ and $\lambda \in [0,1]$. Because $(\mathcal{A}^x)_{x\in\mathbb{R}}$ and θ are increasing we get $\mathbf{X}_1 + \mathbf{Y}_1 \in \mathcal{A}^{\theta(\sum Y_2^n)}$. Set

$$\hat{\mathbf{Y}}_1 := \mathbf{Y}_1 + (\sum Y_2^n - \sum Y_1^n, 0, ..., 0) \in \mathcal{C}\,.$$

Then

$$\mathbf{X}_1 + \hat{\mathbf{Y}}_1 \in \mathcal{A}^{\theta(\sum Y_2^n)}$$

because of the monotonicity of $\mathcal{A}^{\theta(\sum Y_2^n)}$, and

$$\lambda(\mathbf{X}_1 + \hat{\mathbf{Y}}_1) + (1-\lambda)(\mathbf{X}_2 + \mathbf{Y}_2) \in \mathcal{A}^{\theta(\sum Y_2^n)}$$

because of the convexity of $\mathcal{A}^{\theta(\sum Y_2^n)}$. Furthermore, with

$$\mathbf{Y} := \lambda\hat{\mathbf{Y}}_1 + (1-\lambda)\mathbf{Y}_2\,,$$

we get $\lambda\mathbf{X}_1 + (1-\lambda)\mathbf{X}_2 \in \mathcal{A}^{\mathbf{Y}}$ and $\pi(\mathbf{Y}) = \pi(\mathbf{Y_2}) \leq m$ since $\sum_{n=1}^N Y^n = \sum_{n=1}^N Y_2^n$. Hence, property (P2) is satisfied. The final statement follows from Frittelli and Scandolo [2006]. ■

Note that the quasi-convex risk measures in (4.3) are obtained in a similar way as the ones in (3.6), the main difference being that the risk measures in (3.6) are defined on an aggregated level in terms of one-dimensional acceptance sets while the ones in (4.3) are defined in terms of general multi-dimensional acceptance sets. However, in the case $\mathcal{C} = \mathcal{C}_{\mathbb{R}}$ the next proposition shows that every systemic risk measure of type (4.3) can be written as a univariate quasi-convex risk measure applied to the sum of the risk factors. That is, when free scenario-dependent allocations with unlimited cross-subsidization between the financial institutions are possible, the sum as aggregation rule might not only be acceptable as mentioned in the introduction but is the canonical way of aggregation and the canonical way of measuring systemic risk is of type (1.4). However, while this situation and insight is relevant for a portfolio manager, the typical financial systems does not allow for unlimited cross-subsidization and more restricted sets $\mathcal{C}$ together with more appropriate aggregation rules have to be considered.

Proposition 4.2 *Let $\mathcal{C} = \mathcal{C}_{\mathbb{R}}$. Then ρ in (4.3) is of the form*

$$\rho(\mathbf{X}) = \widetilde{\rho}(\sum_{n=1}^{N} X^n) \tag{4.4}$$

for some quasi-convex risk measure

$$\widetilde{\rho} : \mathcal{L}^0(\mathbb{R}) \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\} \cup \{\infty\} \ .$$

Proof. Let $\mathbf{X}_1, \mathbf{X}_1 \in \mathcal{L}^0(\mathbb{R}^N)$ be such that $\sum_{n=1}^N X_1^n = \sum_{n=1}^N X_2^n$. In the notation of the proof of Lemma 2.3, let $\mathbf{Y^1} \in B(\mathbf{X_1})$ and set

$$\mathbf{Y_2} := \mathbf{Y_1} + (\mathbf{X_1} - \mathbf{X_2}) \in \mathcal{C} \, .$$

Then $\mathbf{X_1} + \mathbf{Y_1} = \mathbf{X_2} + \mathbf{Y_2}$, and thus $\mathbf{Y_2} \in B(\mathbf{X_2})$ because $\sum Y_1^n = \sum Y_2^n$ which implies $\mathcal{A}^{\theta(\sum Y_1^n)} = \mathcal{A}^{\theta(\sum Y_2^n)}$. Since $\pi(\mathbf{Y_1}) = \pi(\mathbf{Y_2})$ this implies $\rho(\mathbf{X_1}) \geq \rho(\mathbf{X_2})$. Interchanging the roles of $\mathbf{X_1}$ and $\mathbf{X_2}$ yields $\rho(\mathbf{X_1}) = \rho(\mathbf{X_2})$, and the map $\widetilde{\rho} : \mathcal{L}^0(\mathbb{R}) \to \overline{\mathbb{R}}$ given by

$$\widetilde{\rho}(X) := \rho(\mathbf{X}) \, ,$$

where $\mathbf{X} \in \mathcal{L}^0(\mathbb{R}^{\mathbb{N}})$ is such that $X = \sum_{n=1}^N X^n$ is well-defined. For $X_1, X_2 \in \mathcal{L}^0(\mathbb{R})$ define

$$\mathbf{X_i} := (X_i, 0, ..., 0) \in \mathcal{L}^0(\mathbb{R}^{\mathbb{N}}) \ , \quad i = 1, 2 \, .$$

Then

$$\begin{aligned} \widetilde{\rho}(\lambda X_1 + (1-\lambda) X_2) = \rho(\lambda \mathbf{X_1} + (1-\lambda)\mathbf{X_2}) &\leq \max\{\rho(\mathbf{X_1}), \rho(\mathbf{X_2})\} \\ &= \max\{\widetilde{\rho}(X_1), \widetilde{\rho}(X_2)\} \, . \end{aligned}$$

Further, if $X_1 \leq X_2$ then $\mathbf{X_1} \leq \mathbf{X_2}$ and

$$\widetilde{\rho}(X_1) = \rho(\mathbf{X_1}) \geq \rho(\mathbf{X_2}) = \widetilde{\rho}(X_2) \, .$$

So $\widetilde{\rho} : \mathcal{L}^0(\mathbb{R}) \to \overline{\mathbb{R}}$ is a quasi-convex risk measure and $\rho(\mathbf{X}) = \widetilde{\rho}(\sum_{n=1}^N X^n)$. ■

We conclude this Section by two examples that compare the risk measurement by "injecting after aggregation"as in (1.5) versus the risk measurement by "injecting before aggregation"as in (1.9) for different sets $\mathcal{C} \subset \mathcal{C}_{\mathbb{R}}$ in the situation of the worst case and the expected shortfall acceptance sets, respectively.

4.1 Example: Worst Case Acceptance Set

In this example we measure systemic risk by considering aggregated risk factors defined in terms of the aggregation rule

$$\Lambda(\mathbf{X}) := \sum_{i=1}^{N} -(X_i)^-.$$

Further, we consider the acceptance set $\mathbb{A}^W$ associated to the worst case risk measure, that is a system $\mathbf{X}$ is acceptable (or safe) if $\sum_{i=1}^{N} -(X_i)^- \in \mathbb{A}^W$ where $\mathbb{A}^W := \mathcal{L}^0_+(\mathbb{R})$, and we denote by $\rho_W : \mathcal{L}^0(\mathbb{R}) \to \overline{\mathbb{R}}$ the univariate worst case risk measure defined by

$$\rho_W(X) := \inf\left\{m \in \mathbb{R} \mid X + m \in \mathbb{A}^W\right\}.$$

The possible sets $\mathcal{C}$ are on one hand the deterministic allocations $\mathcal{C} = \mathbb{R}^N$ and on the other hand the family of constrained scenario-dependent cash allocations of the form

$$\mathcal{C}_\gamma := \{\mathbf{Y} \in \mathcal{C}_{\mathbb{R}} \mid Y_i \geq \gamma_i\,, i = 1, ...N\}\,,$$

where $\gamma := (\gamma_1, ..., \gamma_N)$, $\gamma_i \in [-\infty, 0]$. Note that for $\gamma := (-\infty, ..., -\infty)$ this family of subsets includes $\mathcal{C}_\infty = \mathcal{C}_{\mathbb{R}}$. Finally, we let the valuation be

$$\pi(\mathbf{Y}) := \sum_{i=1}^{n} Y_i.$$

The objective of the following proposition is to analyze and relate the systemic risk measurement by "injecting cash after aggregation":

$$\rho^{ag}(\mathbf{X}) := \inf\left\{y \in \mathbb{R} \mid \Lambda(\mathbf{X}) + y \in \mathbb{A}^W\right\} = \rho_W(\sum_{i=1}^{N} -(X^i)^-)\,,$$

to the systemic risk measurement by "injecting cash before aggregation", both in the case of deterministic cash allocations:

$$\rho^{\mathbb{R}^N}(\mathbf{X}) := \inf\left\{\pi(\mathbf{Y}) |\ \mathbf{Y} \in \mathbb{R}^N\,, \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}^W\right\},$$

as well as in the case of scenario-dependent cash allocations:

$$\rho^{\gamma}(\mathbf{X}) := \inf\left\{\pi(\mathbf{Y}) |\ \mathbf{Y} \in \mathcal{C}_\gamma\,, \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}^W\right\}.$$

Proposition 4.3 *It holds that*

$$\rho^{\mathbb{R}^N}(\mathbf{X}) = \sum_{i=1}^{N} \rho_W(X^i) \geq \rho^{ag}(\mathbf{X})$$

$$\rho^{\gamma}(\mathbf{X}) = \rho_W\left(\sum_{i=1}^{N}(X^i \mathbb{I}_{\{X^i \leq -\gamma_i\}} - \gamma_i \mathbb{I}_{\{X^i \geq -\gamma_i\}})\right) \leq \rho^{ag}(\mathbf{X})\,.$$

In particular, for $\gamma = 0 := (0, ..., 0)$ *we get* $\rho^0(\mathbf{X}) = \rho^{ag}(\mathbf{X})$*, and for* $\gamma = -\infty := (-\infty, ..., -\infty)$ *we get* $\rho^{-\infty}(\mathbf{X}) = \rho_W(\sum_{i=1}^{N} X^i)$*.*

Before we prove the proposition we make some comments on the results. We see that if we interpret the risk measure as capital requirement (which in this situation also is possible for ρ^{ag} since the aggregation $\Lambda(\mathbf{X})$ can be interpreted as a monetary amount), the capital requirement when "injecting before aggregation"with deterministic allocations is higher than the one when "injecting after aggregation". When allowing for "injecting before aggregation"with scenario-dependent cash allocations, the gained flexibility in allocating the cash leads to decreasing capital requirements. For fully flexible allocations the minimum amount $\rho^{-\infty}(\mathbf{X}) = \rho_W(\sum_{i=1}^{N} X^i)$ is obtained, which corresponds to the representation given in Proposition 4.2 in terms of the sum as aggregation rule. Obviously, here the relations between ρ^{ag}, $\rho^{\mathbb{R}^N}$, and ρ^{γ} depend on the choice of the acceptance set in conjunction with the aggregation function as is illustrated in the next example.

Further, from the proof below it follows that in the case $\mathcal{C} = \mathbb{R}^N$ there exists a unique allocation $\mathbf{Y}^* \in \mathbb{R}^N$ for a given $\mathbf{X} \in \mathcal{L}^0(\mathbb{R}^N)$ such that $\rho^{\mathbb{R}^N}(\mathbf{X}) = \pi(\mathbf{Y}^*)$, which implies an unambiguous ranking of the systemic riskiness of the institutions. On the other hand, in the case $\mathcal{C} = \mathcal{C}_\gamma$ there generically exist infinitely many scenario-dependent allocations $\mathbf{Y}^* \in \mathcal{C}_\gamma$ for a given $\mathbf{X} \in \mathcal{L}^0(\mathbb{R}^N)$ for which the infimum of the risk measure $\rho^{\gamma}(\mathbf{X}) = \pi(\mathbf{Y}^*)$ is obtained. In that case one needs to discuss further how to pick an allocation and to establish a ranking of systemic riskiness of the institutions.

Proof. Note that for $\mathbf{X} \in \mathcal{L}^0(\mathbb{R}^N)$ it holds that $\Lambda(\mathbf{X}) \in \mathbb{A}^W$ iff $X^i \in \mathbb{A}^W$, $i = 1, ..., N$. Thus we can rewrite

$$\rho^{\mathbb{R}^N}(\mathbf{X}) := \inf\left\{\sum_{i=1}^{N} Y^i | \mathbf{Y} \in \mathbb{R}^N, \mathbf{X} + \mathbf{Y} \in (\mathbb{A}^W)^N\right\},$$

and obviously get

$$\rho^{\mathbb{R}^N}(\mathbf{X}) = \sum_{i=1}^{N} -\text{ess.inf}(X^i) = \sum_{i=1}^{N} \rho_W(X^i),$$

and for $\mathbf{X} \in \mathcal{L}^0(\mathbb{R}^N)$ the allocation $\hat{\mathbf{Y}} := (\text{ess.inf}(X^1), ..., \text{ess.inf}(X^N))$ is the unique $\hat{\mathbf{Y}} \in \mathbb{R}^N$ such that $\rho^{\mathbb{R}^N}(\mathbf{X}) = \pi(\hat{\mathbf{Y}})$.
For ρ^γ we analogously rewrite

$$\rho^\gamma(\mathbf{X}) := \inf \left\{ \sum_{i=1}^N Y^i | \mathbf{Y} \in \mathcal{C}_\gamma \, , \mathbf{X} + \mathbf{Y} \in (\mathbb{A}^W)^N \right\}.$$

Now consider first the optimization problem

$$\widetilde{\rho}(\mathbf{X}) := \inf \left\{ \text{ess.sup}(\sum_{i=1}^N Y^i) | \mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^N), Y^i \geq \gamma_i \, , \mathbf{X} + \mathbf{Y} \in (\mathbb{A}^W)^N \right\}. \tag{4.5}$$

Then clearly $\widetilde{\rho} \leq \rho^\gamma$ and $\mathbf{Y}^* := -(X^i \mathbb{I}_{\{X^i \leq -\gamma_i\}} - \gamma_i \mathbb{I}_{\{X^i \geq -\gamma_i\}})_{i=1,...N}$ is an optimal solution of (4.5). Now define

$$\widetilde{\mathbf{Y}} := \mathbf{Y}^* + (\text{ess.sup}(\sum_{i=1}^N Y_i^*) - \sum_{i=1}^N Y_i^*, 0, ..., 0).$$

Then $\widetilde{\mathbf{Y}} \in \mathcal{C}_\gamma$ and $\rho^\gamma(\mathbf{X}) \leq \pi(\widetilde{\mathbf{Y}}) = \text{ess.sup}(\sum_{i=1}^N Y_i^*) = \widetilde{\rho}(\mathbf{X}) \leq \rho^\gamma(\mathbf{X})$, and thus

$$\begin{aligned} \rho^\gamma(\mathbf{X}) = \sum_{i=1}^N \widetilde{Y}_i &= \text{ess.sup} \left(\sum_{i=1}^N Y_i^* \right) \\ &= \text{ess.sup} \left(\sum_{i=1}^N -(X^i \mathbb{I}_{\{X^i \leq -\gamma_i\}} - \gamma_i \mathbb{I}_{\{X^i \geq -\gamma_i\}}) \right). \end{aligned}$$

Finally we remark that generically for a given $\mathbf{X} \in \mathcal{L}^0(\mathbb{R}^N)$ the above allocation $\widetilde{\mathbf{Y}} \in \mathcal{C}_\gamma$ is not unique such that $\rho^\gamma(\mathbf{X}) = \pi(\widetilde{\mathbf{Y}})$. In fact, any allocation of the form

$$\mathbf{Y}^* + (Z_1, ..., Z_N)$$

with $(Z_1, ..., Z_N) \in \mathcal{L}^0(\mathbb{R}^N)$ such that $\sum_{i=1}^N Z_i = \text{ess.sup}(\sum_{i=1}^N Y_i^*) - \sum_{i=1}^N Y_i^*$ will satisfy the desired property. ■

4.2 Example: Expected Shortfall Acceptance Set

Now consider the acceptance set associated to the "Expected Shortfall" risk measure ρ_{ES} (at some given quantile level):

$$\mathbb{A}^{ES} := \{X \in \mathcal{L}^0(\mathbb{R}) \mid \rho_{ES}(X) \leq 0\}.$$

See e.g. Föllmer and Schied [2004] for the definition of ρ_{ES}. Everything else is assumed to be as in Example 4.1. Then

$$\rho^{ag}(\mathbf{X}) = \rho_{ES}(\sum_{i=1}^{N} -(X^i)^-)\,.$$

For $\rho^{\mathbb{R}^N}$ and ρ^γ, however, $\mathbb{A}^{ES}$ gives the same result as $\mathbb{A}^W$, i.e.

$$\rho^{\mathbb{R}^N}(\mathbf{X}) = \sum_{i=1}^{N} \rho_W(X^i) \geq \rho^{ag}(\mathbf{X}) \tag{4.6}$$

$$\rho^{\gamma}(\mathbf{X}) = \rho_W\left(\sum_{i=1}^{N}(X^i \mathbb{I}_{\{X^i \leq -\gamma_i\}} - \gamma_i \mathbb{I}_{\{X^i \geq -\gamma_i\}})\right)\,. \tag{4.7}$$

Indeed, by the definition of ρ_{ES} it immediatly follows that $\sum_{i=1}^{N} -(X^i)^- \in \mathbb{A}^{ES}$ if and only if $X^i \in \mathbb{A}^W$, $i = 1, ..., N$, and (4.6) and (4.7) is then obtained from Proposition 4.3. So opposite to the situation in Example 4.1, here the risk measure when "injecting before aggregation"even with scenario-dependent allocations might be higher than the one when "injecting after aggregation". Indeed, we easily see that we always have $\rho^0 \geq \rho^{ag}$, and generically even $\rho^{-\infty} \geq \rho^{ag}$ holds. This illustrates that these kind of relations highly depends on the interplay between aggregation and acceptance set.

5 Gaussian Systems

In this Section we assume a Gaussian financial system, i.e. we let $\mathbf{X} = (X^1, \cdots, X^N)$ be an N-dimensional Gaussian random vector with covariance matrix Q, where $[Q]_{ii} := \sigma_i^2$, $i = 1, \cdots, N$, and $[Q]_{ij} := \rho_{i,j}$ for $i \neq j$, $i, j = 1, \cdots, N$, and mean vector $\mu := (\mu_1, \cdots, \mu_N)$, i.e. $\mathbf{X} \sim N(\mu, Q)$. The systemic risk measure we consider is given by

$$\rho(\mathbf{X}) := \inf\left\{\sum_{i=1}^{N} Y^i \mid \mathbf{Y} \in \mathcal{C} \subseteq \mathcal{C}_{\mathbb{R}}\,,\ \Lambda(\mathbf{X}+\mathbf{Y}) \in \mathbb{A}_\gamma\right\}, \tag{5.1}$$

where the set $\mathcal{C}_{\mathbb{R}}$ of scenario-dependent cash allocations is defined in (4.1), the aggregation rule is given by $\Lambda(\mathbf{X}) := \sum_{i=1}^{N} -(X^i - d_i)^-$ for $d_i \in \mathbb{R}$, and the acceptance set is

$$\mathbb{A}_\gamma := \left\{Z \in \mathcal{L}^0(\mathbb{R}) \mid \mathbb{E}\,[Z] \geq -\gamma\right\} \tag{5.2}$$

for some $\gamma \in \mathbb{R}^+$. Here, d_i in the aggregation rule denotes some critical liquidity level of institution i, $i = 1, ...N$, and the risk measure is concerned

with the expected total shortfall below these levels in the system. In Subsection 5.1 we compute the allocation and the systemic risk measure in case of deterministic cash allocations $\mathcal{C} := \mathbb{R}^{\mathbb{N}}$, and in Subsection 5.2 we allow for more flexible scenario-dependent allocations of the form

$$\mathcal{C} := \left\{ \mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^n) \mid \mathbf{Y} = \mathbf{m} + \alpha I_D,\, \mathbf{m}, \alpha \in \mathbb{R}^N, \sum_{i=1}^N \alpha_i = 0 \right\} \subseteq \mathcal{C}_{\mathbb{R}}, \quad (5.3)$$

where I_D is the indicator function of the event $D := \left\{ \sum_{i=1}^N X^i \leq d \right\}$ for some $d \in \mathbb{R}$. Note that the condition $\sum_{i=1}^N \alpha_i = 0$ implies that $\sum_{i=1}^N Y^i$ is constant a.s. Cash allocations in (5.3) can be interpreted as the flexibility to let the allocation depend on whether the system at time T is in trouble or not, represented by the events that $\sum_{i=1}^N X^i$ is less or greater than some critical level d, respectively. In Subsection 5.3 we then apply the results to a Gaussian system that is interconnected by the flow of capital between the institutions through a system of interacting Ornstein-Uhlenbeck diffusions.

5.1 Deterministic Cash Allocations

We now consider the case $\mathcal{C} = \mathbb{R}^{\mathbb{N}}$ and we are interested in computing the systemic risk measure

$$\rho(\mathbf{X}) := \inf \left\{ \sum_{i=1}^N m_i \mid \mathbf{m} = (m_1, \cdots, m_N) \in \mathbb{R}^{\mathbb{N}},\ \Lambda(\mathbf{X} + \mathbf{m}) \in \mathbb{A}_\gamma \right\}, \quad (5.4)$$

where for notational clarity we write $\mathbf{m}$ instead of $\mathbf{Y}$ for deterministic cash allocations. We thus need to minimize the objective function $\sum_{i=1}^N m_i$ over $\mathbb{R}^{\mathbb{N}}$ under the constrained $\Lambda(\mathbf{X}+\mathbf{m}) \in \mathbb{A}_\gamma$, which clearly is equivalent to the constraint

$$\sum_{i=1}^N \mathbb{E}\left[(X^i + m_i - d_i)^- \right] = \gamma\,. \quad (5.5)$$

This constrained optimization problem can be solved with the associated Lagrangian

$$L(m_1, ..., m_N, \lambda) := \sum_{i=1}^N m_i + \lambda(\sum_{i=1}^N \psi_i(m_i) - \gamma) \quad (5.6)$$

where $\psi_i(m_i) := \mathbb{E}\left[(X^i + m_i - d_i)^- \right]$. Since $X^i \sim N(\mu_i, \sigma_i^2)$, one obtains for $i = 1, ..., N$ that

$$\psi_i(m_i) = \frac{\sigma_i}{\sqrt{2\pi}} \exp\left[-\frac{(d_i - \mu_i - m_i)^2}{2\sigma_i^2} \right] - (m_i + \mu_i - d_i)\Phi(\frac{d_i - \mu_i - m_i}{\sigma_i}), \quad (5.7)$$

where $\Phi(x) = \int_{+\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$. By direct computation this leads to

$$\frac{\partial L(m_1, ..., m_N, \lambda)}{\partial m_i} = 1 + \lambda \Phi(\frac{d_i - \mu_i - m_i}{\sigma_i}). \tag{5.8}$$

By solving the Lagrangian system we then obtain the critical point $\mathbf{m}^* = (m_1^*, \cdots, m_N^*)$ given by

$$m_i^* = d_i - \mu_i - \sigma_i R,$$

where R solves the equation

$$P(R) := R\Phi(R) + \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{R^2}{2}\right] = \frac{\gamma}{\sum_{i=1}^{N} \sigma_i}. \tag{5.9}$$

It is easily verified that $\mathbf{m}^*$ is indeed a global minimum and thus the optimal cash allocation associated with the risk measure (5.4). The unique optimal cash allocation $\mathbf{m}^*$ now also induces a ranking of the institutions according to systemic riskiness, and we can discuss the dependence of this ranking with respect to μ_i and σ_i:

1. $\frac{\partial m_i}{\partial \mu_i} = -1$: the systemic riskiness decreases with increasing mean.

2. $\frac{\partial m_i}{\partial \sigma_i} > 0$: the systemic riskiness increases with increasing volatility. In order to show $\frac{\partial m_i}{\partial \sigma_i} > 0$ we first note that R is a solution of (5.9) if and only if R is negative. Indeed, for $R \geq 0$ the left-hand side of (5.9) is always strictly positive, the right-hand side is negative. Thus

$$\frac{\partial m_i}{\partial \sigma_i} = -R - \sigma_i \frac{\partial R}{\partial \sigma_i}. \tag{5.10}$$

By differentiating (5.9) we obtain

$$\frac{\partial P}{\partial \sigma_i} = \frac{\partial P}{\partial R} \frac{\partial R}{\partial \sigma_i} = -\frac{\gamma}{(\sum_{k=1}^{N} \sigma_k)^2}. \tag{5.11}$$

Since $\frac{\partial P}{\partial R} = \Phi(R)$, we can compute $\frac{\partial R}{\partial \sigma_i}$ and substitute it in (5.10):

$$\begin{aligned}
\frac{\partial m_i}{\partial \sigma_i} &= -R + \frac{\sigma_i \gamma}{(\sum_{k=1}^{N} \sigma_k)^2} \frac{1}{\Phi(R)} \\
&= -R + \frac{\sigma_i (\sum_{k=1}^{N} \sigma_k) P(R)}{(\sum_{k=1}^{N} \sigma_k)^2 \Phi(R)} \\
&= -R + \frac{\sigma_i (R\Phi(R) + \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{R^2}{2}\right])}{(\sum_{k=1}^{N} \sigma_k) \Phi(R)} \\
&= (\frac{\sigma_i}{\sum_{k=1}^{N} \sigma_k} - 1)R + \frac{\sigma_i}{\sum_{k=1}^{N} \sigma_k} \frac{1}{\sqrt{2\pi}\Phi(R)} \exp\left[-\frac{R^2}{2}\right].
\end{aligned}$$

Since R must be negative, this implies $\frac{\partial m_i}{\partial \sigma_i} > 0$.

5.2 A Class of Scenario-Dependent Allocations

We now allow for different allocations of the total capital at disposal depending on which state the system is in. More precisely, we differentiate between the two states that $D := \{S \leq d\}$ and $D^c = \{S > d\}$ for some level $d \in \mathbb{R}$ and $S := \sum_{i=1}^N X^i$, and consider allocations $\mathcal{C}$ given in (5.3). The systemic risk measure now becomes

$$\rho(\mathbf{X}) := \inf \left\{ \sum_{i=1}^N m_i | \ \mathbf{m} + \alpha I_D \in \mathcal{C} \, , \, \Lambda(\mathbf{X} + \mathbf{m} + \alpha I_D) \in \mathbb{A}_\gamma \right\}.$$

To compute the risk measure in this case we now need to minimize the objective function $\sum_{i=1}^N m_i$ over $(\mathbf{m}, \alpha) \in \mathbb{R}^{2N}$ under the constraints

$$\sum_{i=1}^N \alpha_i = 0 \quad \text{and} \quad \sum_{i=1}^N \mathbb{E}\left[(X^i + m_i + \alpha_i I_D - d_i)^-\right] = \gamma \, .$$

In analogy to the Section 5.2 we apply the method of Lagrange multipliers to minimize the function

$$\begin{aligned} &\phi(m_1, \cdots, m_N, \alpha_1, \cdots, \alpha_{N-1}, \lambda) = \\ &\sum_{i=1}^N m_i + \lambda \left(\Psi(m_1, \cdots, m_N, \alpha_1, \cdots, \alpha_{N-1}) - \gamma \right), \end{aligned} \tag{5.12}$$

where

$$\begin{aligned} &\Psi(m_1, \cdots, m_N, \alpha_1, \cdots, \alpha_{N-1}) := \\ &\sum_{i=1}^{N-1} \mathbb{E}\left[(X^i + m_i + \alpha_i I_D - d_i)^-\right] + \mathbb{E}\left[(X^N + m_N - \sum_{j=1}^{N-1} \alpha_j I_D - d_N)^- \right], \end{aligned}$$

as follows.

1. *By computing the derivatives with respect to* α_i, $i = 1, \cdots, N-1$: $\frac{\partial \phi}{\partial \alpha_i} = 0$ if and only if

$$F_{i,S}(d_i - m_i - \alpha_i, d) = F_{N,S}(d_N - m_N + \sum_{j=1}^{N-1} \alpha_j, d) \tag{5.13}$$

for $i = 1, \cdots, N-1$, where $F_{i,S}$ and $F_{N,S}$ are the joint distribution functions of (X^i, S) and (X^N, S) respectively.

2. *By computing the derivatives with respect to m_i, for $i = 1, \cdots, N-1$:* $\frac{\partial \phi}{\partial m_i} = 0$ if and only if

$$\Phi(\frac{d_i - \mu_i - m_i}{\sigma_i}) + F_{i,S}(d_i - m_i, d) = \Phi(\frac{d_N - \mu_n - m_N}{\sigma_n}) + F_{N,S}(d_N - m_N, d), \tag{5.14}$$

for $i = 1, \cdots, N-1$.

3. *By computing the derivatives with respect to λ:* $\frac{\partial \phi}{\partial \lambda} = 0$ if and only if $\Psi(m_1, \cdots, m_N, \alpha_1, \cdots, \alpha_{N-1}) = \gamma$, where

$$\begin{aligned}
&\Psi(m_1, \cdots, m_N, \alpha_1, \cdots, \alpha_{N-1}) = \sum_{i=1}^{N} \psi_i(m_i) \\
&+ \sum_{i=1}^{N-1} \Big[(m_i - d_i)F_{N,S}(d_i - m_i, d) - (m_i + \alpha_i - d_i)F_{i,S}(d_i - m_i - \alpha_i, d) \\
&\quad + \int_{d_i - m_i - \alpha_i}^{d_i - m_i} \int_{-\infty}^{d} x F_{i,S}(x, y) dy dx \Big] + (m_N - d_N)F_{N,S}(d_N - m_N, d) \\
&\quad - (m_N - \sum_{j=1}^{N-1} \alpha_j - d_N)F_{N,S}(d_N - m_N + \sum_{j=1}^{N-1} \alpha_j, d) \\
&\qquad + \int_{d_N - m_N + \sum_{j=1}^{N-1} \alpha_j}^{d_N - m_N} \int_{-\infty}^{d} x F_{N,S}(x, y) dy dx,
\end{aligned}$$

where ψ_i, $i = 1, \cdots, N$, are defined in (5.7).

From (5.12) and (5.13) we immediately obtain that if the X^i, $i = 1, \cdots, N$, are identically distributed, then the optimal solution is obtained for $\alpha_i = 0$, $i = 1, \cdots, N$, and corresponds to the one obtained explicitly in Section 5.1 for deterministic injections.

We now present numerical illustrations of our results in the simple case with two banks.

In Table 1 we set the means $\mu_i = 0$ for $i = 1, 2$, the standard deviations $\sigma_1 = 1$, $\sigma_2 = 3$, the acceptance level $\gamma = 0.7$ and the critical level $d = 2$. The last 2 columns show the sensitivities with respect to the correlation for deterministic allocation (case $\alpha = 0$, computed in Section 5.1) and for scenario-dependent allocation, respectively. We observe that for highly positively correlated banks the scenario-dependent allocation does not change the total capital requirement $m_1 + m_2$. Indeed, as expected, if the banks are moving together, one may have to subsidize both of them. However,

$\rho_{1,2} \downarrow$		Deterministic	Random
	m_1	0.5772	0.1597
	m_2	1.7316	1.7230
-0.8	α	0	2.8704
	$\rho = m_1 + m_2$	2.3088	1.8827
	m_1	0.5772	0.2908
	m_2	1.7316	1.7776
-0.5	α	0	2.3161
	$\rho = m_1 + m_2$	2.3088	2.0683
	m_1	0.5772	0.4490
	m_2	1.7316	1.7796
0	α	0	1.7208
	$\rho = m_1 + m_2$	2.3088	2.2286
	m_1	0.5772	0.5463
	m_2	1.7316	1.7461
0.5	α	0	1.3389
	$\rho = m_1 + m_2$	2.3088	2.2924
	m_1	0.5772	0.5737
	m_2	1.7316	1.7314
0.8	α	0	0.7905
	$\rho = m_1 + m_2$	2.3088	2.3053

Table 1: Sensitivity with respect to correlation.

when they are negatively correlated, one benefits from scenario-dependent allocation since the total allocation $m_1 + m_2$ is lower in that case.

In Table 2 we set the means $\mu_i = 0$ for $i = 1, 2$, the correlation $\rho = -0.5$, the standard deviation $\sigma_1 = 1$, the acceptance level $\gamma = 0.7$ and the critical level $d = 2$ and we show sensitivity with respect to the standard deviation σ_2 of the second bank. We observe that for equal marginals ($\sigma_1 = \sigma_2 = 1$) random allocation does not change the total capital requirement, as already stated in Section 5.2. As σ_2 increases, the systemic risk measure increases and the allocation increases with increasing standard deviation in agreement with the sensitivity analysis presented in Section 5.1 for the deterministic case. Also we observe that scenario-dependent allocation allows for smaller total capital requirement $m_1 + m_2$.

$\sigma_2 \downarrow$		Deterministic	Random
1	m_1	0.1008	0.1008
	m_2	0.1031	0.1031
	α	0	0.0002
	$\rho = m_1 + m_2$	0.2039	0.2039
5	m_1	0.8168	0.3167
	m_2	4.0816	4.1295
	α	0	3.5987
	$\rho = m_1 + m_2$	4.8984	4.4462
10	m_1	1.1417	0.4631
	m_2	11.3964	11.4333
	α	0	6.9909
	$\rho = m_1 + m_2$	12.5381	11.8963

Table 2: Sensitivity with respect to standard deviation.

5.3 Application to Models of Borrowing and Lending

We consider the Gaussian vector $X_t = (X_t^i, i = 1, \cdots, n)$ generated by the following dynamics

$$dX_t^i = \left[\sum_{j=1}^{N} p_{i,j}(X_t^j - X_t^i)\right] dt + \sigma^i \left(\rho_i dW_t^0 + \sqrt{1-\rho_i^2} dW_t^i\right), \; i = 1, \cdots, N, \tag{5.15}$$

where $\left(W_t^0, W_t^i, i = 1, \cdots, N\right)$ are independent standard Brownian motions and W_t^0 is a common noise. The lending-borrowing preferences $p_{i,j}$ are nonnegative and symmetric: $p_{i,j} = p_{j,i}$. This model is studied in detail in Carmona et al. [2015] in the mean-field context where $p_{i,j} = p/N$ with $p \geq 0$. It is shown that in that case the dynamics (5.15) emerges as a Nash equilibrium of a specific stochastic game.

The total "capitalization"is given by:

$$\begin{aligned}\sum_{i=1}^{N} X_t^i &= \sum_{i=1}^{N} x_0^i + \left(\sum_{i=1}^{N} \sigma^i \rho_i\right) W_t^0 + \sum_{i=1}^{N} \sigma^i \sqrt{1-\rho_i^2} W_t^i \\ &\stackrel{\mathcal{D}}{=} \sum_{i=1}^{N} x_0^i + \alpha B_t,\end{aligned}$$

where B_t is a standard Brownian motion and

$$\alpha^2 = \left(\sum_{i=1}^{N} \sigma^i \rho_i\right)^2 + \sum_{i=1}^{N} (\sigma^i)^2 (1-\rho_i^2).$$

We are interested in the system at a given time $t > 0$ and in quantities such as the liquidity available at time t defined by $\sum_{i=1}^{N}(X_t^i - d_i)^+$, or the shortfall $-\sum_{i=1}^{N}(X_t^i - d_i)^-$.

The class $\mathcal{C}$ of random vectors $\mathbf{Y}$ is for instance chosen as $\mathbf{Y} = \mathbf{w}Y$ where w^i's are weights, and $Y = Y_t$ is defined by

$$Y_t = y_0 + s\left(\rho_0 W_t^0 + \sqrt{1-\rho_0^2}W_t\right),$$

and W_t is a Brownian motion independent of W_t^0. This reflects that the money which can be allocated to banks can be correlated to the common factor driving the system. In fact the choice of W_t is general ranging from a linear combination of the W^i's to being independent of the W^i's. This situation corresponds to using securities (in this case bonds) as allocation at the future time t. To keep the example simple we consider the case with $s = 0$, that is $Y_t = y_0$, and constant injection as we did in Section 5.1.

5.3.1 Homogeneous Network

Here we consider the fully homogeneous case where $x_0^i = x_0$, $p_{i,j} = p/N$, $\sigma^i = \sigma$, $\rho_i = \rho$, $d_i = d$, so that the model becomes

$$dX_t^i = \left[\frac{p}{N}\sum_{j=1}^{N}(X_t^j - X_t^i)\right]dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t^i\right), \quad i = 1, \cdots, N,$$

or

$$dX_t^i = p\left[\bar{X}_t - X_t^i\right]dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t^i\right), \quad i = 1, \cdots, N, \quad (5.16)$$

where

$$\bar{X}_t = \frac{1}{N}\sum_{j=1}^{N} X_t^j.$$

In order to apply the results from Section 5.1, we will need to compute the distribution of a single X_t^i. The joint distribution of the X_t^i's being obviously Gaussian, X_t^i is Gaussian with mean μ_i and variance σ_i^2 in the notation of Section 5.1. A straightforward computation from (5.16) gives:

$$\mu_i = \mathbb{E}(X_t^i) = x_0,$$

and

$$\sigma_i^2 = \sigma^2(1-\rho^2)(1-\frac{1}{N})\left(\frac{1-e^{-2pt}}{2p}\right) + \sigma^2\left(\rho^2 + \frac{1-\rho^2}{N}\right)t. \quad (5.17)$$

Note that, even though we only consider marginal distributions in the systemic risk measure proposed in Section 5.1, these marginal distributions depend on the coupled dynamics of the X^i, in particular on the parameters p and ρ. For instance, one sees that increasing p, that is increasing liquidity, would decrease σ_i^2 (from $\sigma^2 t$ for $p = 0$ to $\sigma^2 \left(\rho^2 + \frac{1-\rho^2}{N}\right) t$ for $p = \infty$), and therefore, would decrease systemic risk according with our findings in Section 5.1.

5.3.2 Central Clearing Network

Here we consider a centralized model where bank 1 (for instance) plays a clearing role and is related to each of the other banks which are not directly related to each other. That is $p_{i,j} = p$ if $i = 1$ or $j = 1$, and $p_{i,j} = 0$ if $i \neq 1$ and $j \neq 1$; $x_0^i = x_0$ if $i \neq 1$; $\sigma^i = \sigma$ if $i \neq 1$; $\rho_i = \rho$ if $i \neq 1$; $d_i = d$ if $i \neq 1$. The model becomes

$$
\begin{aligned}
dX_t^1 &= p \sum_{j=2}^{N} (X_t^j - X_t^1) dt + \sigma_c \left(\rho_c dW_t^0 + \sqrt{1-\rho_c^2} dW_t^1 \right), \\
dX_t^i &= p(X_t^1 - X_t^i) dt + \sigma \left(\rho dW_t^0 + \sqrt{1-\rho^2} dW_t^i \right), \quad i = 2, \cdots, N,
\end{aligned}
$$

with initial conditions $X_0^1 = x_0^1$ and $X_0^i = x_0$ for $i = 2, \cdots, N$. The joint distribution of the X_t^i's is again Gaussian. Choosing $x_0^1 = x_0 = c_0/N$, we get

$$
\mu_i = \mathbb{E}(X_t^i) = \mathbb{E}(X_t^1) = c_0/N.
$$

We turn now to the computation of the variances. Applying Itô's formula we get:

$$
\begin{aligned}
&d\mathbb{E}[(X_t^i)^2] = 2p \left(\mathbb{E}(X_t^i X_t^1) - \mathbb{E}[(X_t^i)^2] \right) dt + \sigma^2 dt, \quad i = 2, \cdots, N, \\
&d\mathbb{E}[(X_t^1)^2] = 2p \sum_{j=2}^{N} \left(\mathbb{E}(X_t^1 X_t^j) - \mathbb{E}[(X_t^1)^2] \right) dt + \sigma_1^2 dt, \\
&d\mathbb{E}(X_t^i X_t^1) = p \sum_{j=2}^{N} \left(\mathbb{E}(X_t^i X_t^j) - \mathbb{E}(X_t^i X_t^1) \right) dt \\
&+ p \left(\mathbb{E}[(X_t^1)^2] - \mathbb{E}(X_t^i X_t^1) \right) dt + \sigma \sigma_c \rho \rho_c dt, \quad i = 2, \cdots, N, \\
&d\mathbb{E}(X_t^i X_t^j) = p \left(\mathbb{E}(X_t^1 X_t^j) - \mathbb{E}(X_t^i X_t^j) \right) dt + p \left(\mathbb{E}(X_t^1 X_t^i) - \mathbb{E}(X_t^i X_t^j) \right) dt \\
&+ \sigma^2 \rho^2 dt, \quad i \geq 2, j \geq 2, i \neq j.
\end{aligned}
$$

By symmetry among the X^i's for $i \geq 2$, we deduce that $\mathbb{E}[(X_t^i)^2], \mathbb{E}(X_t^i X_t^1)$, and $\mathbb{E}(X_t^i X_t^j)$ do not depend on $i \geq 2, j \geq 2$. Accordingly, we define:

$$\begin{aligned}
&\mathbb{E}[(X_t^i)^2] = v(t), \quad i \geq 2,\\
&\mathbb{E}(X_t^i X_t^1) = w(t), \quad i \geq 2,\\
&\mathbb{E}(X_t^i X_t^j) = \chi(t), \quad i \geq 2, j \geq 2, i \neq j,\\
&\mathbb{E}[(X_t^1)^2] = v_1(t).
\end{aligned}$$

These functions satisfies the differential system $Y' = pAY + B$ with:

$$Y = \begin{pmatrix} v \\ v_1 \\ w \\ \chi \end{pmatrix}, A = \begin{bmatrix} -2 & 0 & 2 & 0 \\ 0 & -2(N-1) & 2(N-1) & 0 \\ 1 & 1 & -N & N-2 \\ 0 & 0 & 2 & -2 \end{bmatrix}, B = \begin{pmatrix} \sigma^2 \\ \sigma_c^2 \\ \sigma\sigma_c\rho\rho_c \\ \sigma^2\rho^2 \end{pmatrix}.$$

We are interested in $v(t)$ and $v_1(t)$. Note that by subtracting x_0^2 to these four functions it is enough to solve the system with zero initial conditions. A straightforward but tedious computation shows that:

$$\begin{aligned}
&\sigma_1^2 = v_1(t) = \mathrm{var}(X_t^1) = \sigma^2\rho^2 t\\
&+ \frac{1}{N}\left[(\sigma^2 + 2\sigma\sigma_c\rho\rho_c - 3\sigma^2\rho^2)t + \frac{2}{p}(\sigma_c^2 - \sigma\sigma_c\rho\rho_c)\right] + \mathcal{O}(\frac{1}{N^2}),
\end{aligned}$$

and

$$\begin{aligned}
&\sigma_i^2 = v(t) = \mathrm{var}(X_t^i) = \sigma^2(1-\rho^2)\left(\frac{1-e^{-2pt}}{2p}\right) + \sigma^2\rho^2 t\\
&+ \frac{1}{N}\left[(\sigma^2 + 2\sigma\sigma_c\rho\rho_c - 3\sigma^2\rho^2)t - \sigma^2(1-\rho^2)\left(\frac{1-e^{-2pt}}{2p}\right)\right] + \mathcal{O}(\frac{1}{N^2}),
\end{aligned}$$

to be compared with the exact formula (5.17) in order to compare the systemic risk for a fully connected homogenous network with a central clearing network. At order one in $1/N$, the variance is the same but they may differ at order $1/N$. Writing $\sigma^2 + 2\sigma\sigma_c\rho\rho_c - 3\sigma^2\rho^2 = \sigma^2(1-\rho^2) + 2\sigma\rho(\sigma_c\rho_c - \sigma\rho)$, we see that the sign of $\sigma_c\rho_c - \sigma\rho$ determines which network is most stable, that is the one with smaller variance according to the conclusion in Section 5.1.

5.3.3 Heterogeneous Networks

In practical situations, the network will be heterogeneous described by a system like our simplified Gaussian model (5.15). The joint distribution will be fully characterized by the means μ_i's and by the covariance matrix

$Q = [cov(X_t^i, X_t^j)]$ which will depend on the parameters of the model, in particular the preferences $p_{i,j}$ and the individual σ^i. In that case, for given coefficients and initial conditions, one will be able to numerically compute the marginal means and variances needed in our systemic risk measures. Doing so, one will obtain the optimal allocation $m = (m_i)_{i=1,\cdots,N}$ and a ranking of the banks with respect to their systemic risk contributions.

5.3.4 An Example with Random Injections

In order to illustrate the results from Section 5.2, we consider a system of three banks. If it is fully connected and homogenous as in Section 5.3.1, then in the context of random injections in Section 5.2, by symmetry, the optimal m_i's will be equal, the α_i's will also be equal and therefore $\alpha_i = 0$ because of the constraint $\sum \alpha_i = 0$. Consequently, the injection will simply be constant as studied in Section 5.3.1.

Now we consider the case a heterogenous network with symmetric preferences such that $p_{2,3} = p/2$ and $p_{1,2} = p_{1,3} = 0$, equal starting points $x_0^i = x_0$, equal volatility $\sigma^i = \sigma$, and correlation to common noise ρ_1, $\rho_2 = \rho_3 = \rho$. The vector $X = (X^1, X^2, X^3)$ satisfies

$$\begin{aligned}
dX_t^1 &= \sigma\left(\rho_1 dW_t^0 + \sqrt{1-\rho_1^2}dW_t^1\right),\\
dX_t^2 &= \frac{p}{2}\left(X_t^3 - X_t^2\right)dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t^2\right),\\
dX_t^3 &= \frac{p}{2}\left(X_t^2 - X_t^3\right)dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}\right)dW_t^2.
\end{aligned}$$

The bank X^1 is uncoupled with the symmetric network (X^2, X^3). A straightforward computation shows that the Gaussian pair $(X_t^1, X_t^2 + X_t^3)$ (where we have aggregated X^2 and X^3) admits the covariance matrix

$$Q = \sigma^2 t \begin{pmatrix} 1 & 2\rho\rho_1 \\ 2\rho\rho_1 & 2(1+\rho^2) \end{pmatrix}.$$

In the following numerical illustration, we take $x_0^i = 0, i = 1, 2, 3$, $\sigma^2 t = 1$, $\rho = .8$ and we vary $\rho_1 \in \{-.5, -.2, 0, .2, .5\}$. In the notation of Section 5.2, this translates into $\sigma_1^2 = \sigma^2 t = 1$, $\sigma_2^2 = 2\sigma^2 t(1+\rho^2) = 3.28$ and varying the covariance $2\rho\rho_1 \in \{-.8, -.32, 0, .32, .8\}$. The results are displayed in Table 3 where we show the values of m_1, m_2, α, and the value of the systemic risk measure $m_1 + m_2$. In the deterministic column, these values do not change since they depend only on the marginal distributions and $\alpha = 0$ since in that case allocations are deterministic. Then, they can be compared with the values in the case with random allocations where we see that the gain is more pronounced for negative correlation.

$2\rho\rho_1 \downarrow$		Deterministic	Random
	m_1	0.3486	0.2671
	m_2	0.6313	0.6347
-0.8	α	0	2.1413
	$m_1 + m_2$	0.9799	0.9018
	m_1	0.3486	0.2799
	m_2	0.6313	0.6577
-0.32	α	0	1.1161
	$m_1 + m_2$	0.9799	0.9376
	m_1	0.3486	0.3062
	m_2	0.6313	0.6530
0	α	0	0.8416
	$m_1 + m_2$	0.9799	0.9592
	m_1	0.3486	0.3271
	m_2	0.6313	0.6414
0.32	α	0	0.6813
	$m_1 + m_2$	0.9799	0.9685
	m_1	0.3486	0.3436
	m_2	0.6313	0.6294
0.8	α	0	.6597
	$m_1 + m_2$	0.9799	0.9750

Table 3: Sensitivity with respect to correlation to common noise.

6 Example: Systems on a Finite Probability Space

We now consider a financial system $\mathbf{X} = (X^1, \cdots, X^N)$ that is defined on a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\Omega = \{\omega_1, \cdots, \omega_M\}$, $\mathcal{F} = 2^\Omega$, $\mathbb{P}(\omega_i) = p_i \in (0,1)$, $i = 1, \cdots, M$. The systemic risk measure we are interested in here is given by

$$\rho(\mathbf{X}) := \inf \left\{ \sum_{i=1}^N Y^i \mid \mathbf{Y} = (Y^1, \cdots, Y^N) \in \mathcal{C}^h, \Lambda(\mathbf{X} + \mathbf{Y}) \in \mathbb{A}_\gamma \right\}, \quad (6.1)$$

where as in Section 5 the acceptance set is $\mathbb{A}_\gamma = \left\{ Z \in \mathcal{L}^0(\mathbb{R}) | \ \mathbb{E}[Z] \geq -\gamma \right\}$ for $\gamma > 0$ and the admissible allocations $\mathcal{C}^h$ are introduced below. The aggregation is defined by

$$\Lambda(x_1, \cdots, x_N) := \sum_{i=1}^N - \exp\left(-\alpha_i x_i\right) \quad (6.2)$$

for $\alpha_i > 0$, $i = 1, \cdots, N$. Compared to the aggregation in Section 5, the aggregation in (6.2) is more risk averse with respect to bigger losses but also takes benefits of gains into account.

Due to the finite probability space the computation of the optimal allocation associated to the risk measure (6.1) reduces to solving a finite-dimensional system of equations even for most general scenario-dependent allocation. More precisely, let $h := (h_1, .., h_k)$ with $0 < h_1 < h_2 < \cdots < h_{k-1} < h_k = N$ represent some partition of $\{1, .., N\}$ for a given $k \in \{1, \cdots, N\}$. We then introduce the following family of allocations:

$$\mathcal{C}^h = \Big\{ \mathbf{Y} \in \mathcal{L}^0(\mathbb{R}^N) \mid \exists\ d = (d_1, \cdots, d_k) \in \mathbb{R}^k \text{ such that } \sum_{i=1}^{h_1} Y^i(w_j) = d_1,$$
$$\sum_{i=h_1+1}^{h_2} Y^i(w_j) = d_2, \cdots, \sum_{i=h_{k-1}+1}^{N} Y^i(w_j) = d_k, \text{ for } \ j = 1, \cdots, M \Big\} \subseteq \mathcal{C}_\mathbb{R}. \quad (6.3)$$

This corresponds to the situation when the regulator is constrained in the way that she cannot distribute cash freely among all financial institutions but only within k subgroups that are induced by the partition h. In other words, the risk measure is the sum of k minimal cash amounts $d_1, ..., d_k$ determined today, that at time T can be freely allocated within the k subgroups in order to make the system safe. Note that this family spans from deterministic allocations $\mathcal{C} = \mathbb{R}^N$ for $k = N$ to $\mathcal{C}_\mathbb{R}$ for $k = 1$.

For a given partition h of subgroups one can now explicitly compute a unique optimal allocation $\mathbf{Y}^*$ and the corresponding systemic risk $\rho(\mathbf{X}) = \sum_{i=1}^N Y^{i,*}$ in (6.1) by solving the corresponding Lagrangian system. For

better readability of the text we here state the explicit expressions for the following subfamily of allocations

$$\mathcal{C}^r = \Big\{ \mathbf{Y} \in \mathcal{L}^0(\mathbb{R})^N \mid \exists\ (c_N, c_{N-1}, \cdots, c_{N-r}) \in \mathbb{R}^{r+1} \text{ such that } \sum_{i=1}^{N} Y^i(w_j) = c_N,$$
$$\sum_{i=1}^{N-1} Y^i(w_j) = c_{N-1}, \cdots, \sum_{i=1}^{N-r} Y^i(w_j) = c_{N-r}, \text{ for } \ j = 1, \cdots, M \Big\} \tag{6.4}$$

for $0 \leq r \leq N-1$. This class corresponds to r subgroups of size one and one remaining bigger subgroup of size $N-r$, and the two extreme cases are recovered for $r = 0$ and $r = N-1$. Note that the summation over the Y^i's of the subgroups in (6.4) has been re-parametrized compared to (6.3) for the sake of more accessible expressions below.

The following are the optimal solutions when computing the systemic risk measure (6.1) with the set $\mathcal{C} = \mathcal{C}^r$ of type (6.4). The proof is deferred to the appendix. For notational simplicity we denote by $y_j^k := Y^{k,*}(\omega_j)$ for $k = 1, \cdots, N$, $j = 1, \cdots, M$ the optimal allocation. The optimal $c_{N-r}, \cdots, c_N$ are given by

$$c_{N-r} = -\beta_{N-r} \log\Big(\frac{\gamma}{\alpha_1 \beta_N d_{N-r}}\Big), \tag{6.5}$$

where

$$\beta_{N-r} = \sum_{i=1}^{N-r} \frac{1}{\alpha_i} \quad \text{and} \quad \beta_N = \sum_{i=1}^{N} \frac{1}{\alpha_i},$$
$$d_{N-r} = \sum_{j=1}^{M} p_j \exp\left[-\frac{1}{\beta_{N-r}} \sum_{i=1}^{N-r} X^i(w_j) - \frac{1}{\beta_{N-r}} \sum_{i=1}^{N-r} \frac{1}{\alpha_i} \log\Big(\frac{\alpha_1}{\alpha_i}\Big) \right],$$

and by

$$\begin{aligned} c_k &= c_{k-1} - \frac{1}{\alpha_k} \log\Big(\frac{\gamma}{\alpha_k \beta_N K_k}\Big) \\ &= c_{N-r} - \sum_{j=N-r+1}^{k} \frac{1}{\alpha_j} \log\Big(\frac{\gamma}{\alpha_j \beta_N K_j}\Big) \\ &= -\beta_{N-r} \log\Big(\frac{\gamma}{\alpha_1 \beta_N d_{N-r}}\Big) - \sum_{j=N-r+1}^{k} \frac{1}{\alpha_j} \log\Big(\frac{\gamma}{\alpha_j \beta_N K_j}\Big) \end{aligned} \tag{6.6}$$

for $k = N-r+1, \cdots, N$, $r \geq 1$, with

$$K_k = \sum_{j=1}^{M} p_j \exp\Big(-\alpha_k X^k(w_j)\Big).$$

In particular the optimal c_N provides the value of the systemic risk measure, i.e.

$$\rho(\mathbf{X}) = -\beta_{N-r}\log(\frac{\gamma}{\alpha_1\beta_N d_{N-r}}) - \sum_{j=N-r+1}^{N}\frac{1}{\alpha_j}\log(\frac{\gamma}{\alpha_j\beta_N K_j}). \tag{6.7}$$

The optimal allocations are given by

$$y_j^1 = \frac{1}{\alpha_1\beta_{N-r}}\sum_{i=1}^{N-r}X^i(w_j) - X^1(w_j) + \frac{1}{\alpha_1\beta_{N-r}}\sum_{i=1}^{N-r}\frac{1}{\alpha_i}\log(\frac{\alpha_1}{\alpha_i}) + \frac{1}{\alpha_1\beta_{N-r}}c_{N-r} \tag{6.8}$$

for $j = 1, \cdots, M$, by

$$\begin{aligned} y_j^k &= \frac{1}{\alpha_k}\left[\alpha_1 X^1(w_j) - \alpha_k X^k(w_j) - \log(\frac{\alpha_1}{\alpha_k}) + \alpha_1 y_j^1\right] && (6.9)\\ &= \frac{1}{\alpha_k\beta_{N-r}}\sum_{i=1}^{N-r}X^i(w_j) - X^k(w_j) - \frac{1}{\alpha_k}\log(\frac{\alpha_1}{\alpha_k}) \\ &\quad + \frac{1}{\alpha_k\beta_{N-r}}\sum_{i=1}^{N-r}\frac{1}{\alpha_i}\log(\frac{\alpha_1}{\alpha_i}) + \frac{1}{\alpha_k\beta_{N-r}}c_{N-r} && (6.10)\end{aligned}$$

for all $k = 2, \cdots, N-r-1$ and $j \in 1, \cdots, M$, and by

$$y_j^k = c_k - c_{k-1} = -\frac{1}{\alpha_k}\log(\frac{\gamma}{\alpha_k\beta_N K_k}) \tag{6.11}$$

for all $k = N-r, \cdots, N$ and $j \in 1, \cdots, M$.

Remark 6.1 *One could extend the above setting further by adding the possibility to limit cross-subsidization in the allocations. This can be done by introducing another constraint into the family* (6.3) *of cash allocations:*

$$\begin{aligned}\mathcal{C}^{h,b} = \Big\{\mathbf{Y} \in \mathcal{L}^0(\mathbb{R})^N | \quad & Y^i \geq b_i, i = 1, \cdots, N\,; \\ & \sum_{i=1}^{h_1}Y^i(w_j) = d_1, \sum_{i=h_1+1}^{h_2}Y^i(w_j) = d_2, \cdots, \sum_{i=h_{k-1}+1}^{N}Y^i(w_j) = d_k, \\ & \textit{for } \; j = 1, \cdots, M, \; \textit{and } d_t \in \mathbb{R} \textit{ for } t = 0, \cdots, k\Big\} \subseteq \mathcal{C}_{\mathbb{R}},\end{aligned}$$

where $(b_1, ..., b_N) \in \mathbb{R}^N$. *For example, putting* $b := (0, ..., 0)$ *excludes cash withdrawals from institutions and in this sense doesn't allow for any cross-subsidization. The systemic risk measure and corresponding optimal allocations solution can now be computed by resorting to the Karush Kuhn Tucker conditions (Boyd and Vandenberghe [2009]), see the computations in Pastore [2014].*

Systemic risk measure	Case
-26.36	$r = 0$
-0.56	$r = 2$, $\{X_1, X_3\}$
4.44	$r = 2$, $\{X_2, X_3\}$
63.71	$r = 2$, $\{X_2, X_4\}$
68.36	$r = 2$, $\{X_1, X_4\}$
72.96	$r = 2$, $\{X_3, X_4\}$
74.48	$r = 2$, $\{X_1, X_2\}$
79.02	$r = 3$

Table 4: Systemic risk measure.

Example 6.2 *We conclude this section with a numerical example. We consider a system of four banks represented by the random variables* X^1, X^2, X^3 *and* X^4 *on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, where* $\Omega = (\omega_1, \omega_2, \omega_3, \omega_4)$*,* $\mathcal{F} = 2^\Omega$ *and* $\mathbb{P}(\omega_1) = 0.64$*,* $\mathbb{P}(\omega_2) = \mathbb{P}(\omega_3) = 0.16$ *and* $\mathbb{P}(\omega_4) = 0.04$*. We assume that* X^4 *is independent of* X^1, X^2, X^3*, that* X^2 *is comonotone with* X^1 *and that* X^3 *is countermonotone with* X^1*. Furthermore* $X^1(w_1) = X^1(w_3) = 100, X^1(w_2) = X^1(w_4) = -50$*,* $X^2(w_1) = X^2(w_3) = 50, X^2(w_2) = X^2(w_4) = -25$*,* $X^3(w_1) = X^3(w_3) = -25, X^3(w_2) = X^3(w_4) = 50$ *and* $X^4(w_1) = X^4(w_2) = 50, X^4(w_3) = X^4(w_4) = -25$*. We set* $\alpha_i = 0.3$ *for* $i = 1, \cdots, 4$ *and* $\gamma = 50$ *and consider the set* $\mathcal{C}^r$ *defined in* (6.4)*.*

In this setting, Tables 5 and 6 reproduce the deterministic allocations and optimal scenario-dependent allocations for $r = 2$ *given by* (6.8)*,* (6.9) *and* (6.11) *as well as the systemic risk measure* ρ *given by* (6.7)*. In Table 4 we provide the systemic risk measures for all* $r = 0, 1, 2, 3$*. From Table 4 we note that the maximum and minimum value of* ρ *are obtained respectively in the deterministic (*$r = 3$*) and the fully unconstrained scenario-dependent (*$r = 0$*) cases. Whenever one groups (*X_1 *and* X_3*) or (*X_2 *and* X_3*),* ρ *is substantially reduced (*-0.56 *or* 4.44*), compared to the deterministic case (*79.02*), as these couples of vectors are counter monotone. Whenever one groups* X_4 *with any of the* X_1*,* X_2*,* X_3*, there is little difference (*$68.36, 63.71, 72.96$*) with respect to the deterministic case (*79.02*), as* X_4 *is independent from the others. Grouping* X_1 *and* X_2 *has very little effect (*74.48*) compared to the deterministic case (*79.02*), as* X_1 *and* X_2 *are comonotone.*

Groups: $\{X_1\},\{X_2\},\{X_3\},\{X_4)$	Deterministic
Y_1	36.18
Y_2	15.82
Y_3	15.82
Y_4	11.20
Systemic risk $= \sum Y_i$	**79.02**

Table 5: Case $r = 3$.

A Appendix

A.1 Gaussian Case with Random Injections

We provide here the computations necessary to minimize the function (5.12). We first consider

$$\begin{aligned}
&\mathbb{E}\left[(X^i + Y^i - d_i)^-\right] = \mathbb{E}\left[(X^i + m_i + \alpha_i I_D - d_i)^-\right] \\
&= \mathbb{E}\left[(X^i + m_i + \alpha_i - d_i)^- I_D\right] + \mathbb{E}\left[(X^i + m_i - d_i)^- I_{A^c}\right] \\
&= \mathbb{E}\left[\left\{(X^i + m_i + \alpha_i - d_i)^- - (X^i + m_i - d_i)^-\right\} I_D\right] + \mathbb{E}\left[(X^i + m_i - d_i)^-\right]
\end{aligned} \tag{A.1}$$

for $i = 1, \cdots, N$. To compute (A.1), we distinguish between the cases $\alpha_i > 0$ and $\alpha_i < 0$. Note that by the definition of $\mathcal{C}$, we cannot a priori argue on the sign of α. For $\alpha_i > 0$, we have that $\left\{X^i \le d_i - m_i\right\} = \left\{X^i \le d_i - m_i - \alpha_i\right\} \cup \left\{d_i - m_i - \alpha_i < X^i \le d_i - m_i\right\}$. Here we set $A_1 := \left\{X^i \le d_i - m_i - \alpha_i\right\}$ and $A_2 := \left\{d_i - m_i - \alpha_i < X^i \le d_i - m_i\right\}$. Then

$$(X^i + m_i + \alpha_i - d_i)^- - (X^i + m_i - d_i)^- = -\alpha_i I_{A_1} + (X^i + m_i - d_i) I_{A_2},$$

and

$$\begin{aligned}
&\mathbb{E}\left[(X^i + Y^i - d_i)^-\right] \\
&\quad = -\alpha_i \mathbb{E}\left[I_{A_1} I_D\right] + \mathbb{E}\left[(X^i + m_i - d_i) I_{A_2} I_D\right] + \mathbb{E}\left[(X^i + m_i - d_i)^-\right] \\
&\quad = (m_i - d_i) F_{i,S}(d_i - m_i, d) - (m_i + \alpha_i - d_i) F_{i,S}(d_i - m_i - \alpha_i, d) \\
&\quad + \int_{d_i - m_i - \alpha_i}^{d_i - m_i} \int_{-\infty}^{d} x f_{i,S}(x, y) dy dx + \mathbb{E}\left[(X^i + m_i - d_i)^-\right],
\end{aligned}$$

where $F_{i,S}$ and $f_{i,S}$ are the joint distribution function and the density of (X^i, S), respectively. Recall that in our setting $(X^i, S) \sim N_2(\bar{\mu}^i, \bar{Q}^i)$ with mean vector $\bar{\mu}^i = (\mu^i, \sum_{j=1}^n \mu_j)$ and covariance matrix

$$\bar{Q}^i = \begin{pmatrix} \sigma_i^2 & \sigma_i^2 + \sum_{j \neq i} \rho_{i,j} \\ \sigma_i^2 + \sum_{j \neq i} \rho_{i,j} & \sum_{j=1}^n \sigma_j^2 + \sum_{j,k=1}^n \rho_{j,k} \end{pmatrix}.$$

Groups: $\{X_1, X_2\}, \{X_3\}, \{X_4\}$	Random	Deterministic
Y_1	$Y_1(\omega_1) = Y_1(\omega_3) = 11.27$ $Y_1(\omega_2) = Y_1(\omega_4) = 36.23$ $\mathbb{E}[Y_1] = 6.23$	
Y_2	$Y_2(\omega_1) = Y_2(\omega_3) = 48.73$ $Y_2(\omega_2) = Y_2(\omega_4) = 11.23$ $\mathbb{E}[Y_1] = 41.23$	$Y_1 + Y_2 = 47.46$
Y_3		15.82
Y_4		11.20
Systemic risk $= \sum Y_i$		**74.48**
Groups: $\{X_1, X_3\}, \{X_2\}, \{X_4\}$		
Y_1	$Y_1(\omega_1) = Y_1(\omega_3) = -76.29$ $Y_1(\omega_2) = Y_1(\omega_4) = 36.21$ $\mathbb{E}[Y_1] = -53.79$	
Y_3	$Y_3(\omega_1) = Y_3(\omega_3) = 48.71$ $Y_3(\omega_2) = Y_3(\omega_4) = -63.79$ $\mathbb{E}[Y_3] = 26.21$	$Y_1 + Y_3 = -27.58$
Y_2		15.82
Y_4		11.20
Systemic risk $= \sum Y_i$		**-0.56**
Groups: $\{X_1, X_4\}, \{X_2\}, \{X_3\}$		
Y_1	$Y_1(\omega_1) = -6.64, Y_1(\omega_2) = 68.36$ $Y_1(\omega_3) = -44.14, Y_1(\omega_4) = 30.86$ $\mathbb{E}[Y_1] = 0.86$	
Y_4	$Y_4(\omega_1) = 43.36, Y_4(\omega_2) = -31.64$ $Y_4(\omega_3) = 80.86, Y_4(\omega_4) = 5.86$ $\mathbb{E}[Y_4] = 35.86$	$Y_1 + Y_4 = 36.72$
Y_2		15.82
Y_3		15.82
Systemic risk $= \sum Y_i$		**68.36**
Groups: $\{X_2, X_3\}, \{X_1\}, \{X_4\}$		
Y_2	$Y_2(\omega_1) = Y_2(\omega_3) = -58.97$ $Y_2(\omega_2) = Y_2(\omega_4) = 16.03$ $\mathbb{E}[Y_2] = -43.97$	
Y_3	$Y_3(\omega_1) = Y_3(\omega_3) = 16.03$ $Y(\omega_2) = Y_3(\omega_4) = -58.97$ $\mathbb{E}[Y_3] = 1.03$	$Y_2 + Y_3 = -42.94$
Y_1		36.18
Y_4		11.20
Systemic risk $= \sum Y_i$		**4.44**
Groups: $\{X_2, X_4\}, \{X_1\}, \{X_3\}$		
Y_2	$Y_2(\omega_1) = 5.86, Y_2(\omega_2) = 43.36$ $Y_2(\omega_3) = -31.64, Y_2(\omega_4) = 5.86$ $\mathbb{E}[Y_2] = 5.86$	
Y_4	$Y_4(\omega_1) = 5.85, Y_4(\omega_2) = -31.65$ $Y(\omega_3) = 43.35, Y_4(\omega_4) = 5.85$ $\mathbb{E}[Y_4] = 5.85$	$Y_2 + Y_4 = 11.71$
Y_1		36.18
Y_3		15.82
Systemic risk $= \sum Y_i$		**63.71**
Groups: $\{X_3, X_4\}, \{X_1\}, \{X_2\}$		
Y_3	$Y_3(\omega_1) = 47.98, Y_3(\omega_2) = 10.48$ $Y_3(\omega_3) = 10.48, Y_3(\omega_4) = -27.02$ $\mathbb{E}[Y_2] = 32.98$	
Y_4	$Y_4(\omega_1) = -27.02, Y_4(\omega_2) = 10.48$ $Y(\omega_3) = 10.48, Y_4(\omega_4) = 47.98$ $\mathbb{E}[Y_4] = -12.02$	$Y_3 + Y_4 = 20.96$
Y_1		36.18
Y_2		15.82
Systemic risk $= \sum Y_i$		**72.96**

Table 6: Case $r = 2$.

Analogous computations hold in the case $\alpha_i < 0$. Summing up, we obtain that

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^{N}(X^i + Y^i - d_i)^-\right] &= \sum_{i=1}^{N}\mathbb{E}\left[(X^i + Y^i - d_i)^-\right] \\
&= \sum_{i=1}^{N}\mathbb{E}\left[(X^i + m_i - d_i)^-\right] \\
&\quad + \sum_{i=1}^{N} I_{\alpha_i \geq 0}\left[(m_i - d_i)F_{i,S}(d_i - m_i, d) - (m_i + \alpha_i - d_i)F_{i,S}(d_i - m_i - \alpha_i, d)\right. \\
&\qquad \left. + \int_{d_i - m_i - \alpha_i}^{d_i - m_i}\int_{-\infty}^{d} x f_{i,S}(x,y)dydx\right] \\
&\quad + \sum_{i=1}^{N} I_{\alpha_i < 0}\left[(m_i - d_i)F_{i,S}(d_i - m_i, d) - (m_i + \alpha_i - d_i)F_{i,S}(d_i - m_i - \alpha_i, d)\right. \\
&\qquad \left. + \int_{d_i - m_i - \alpha_i}^{d_i - m_i}\int_{-\infty}^{d} x f_{i,S}(x,y)dydx\right] \\
&= \sum_{i=1}^{N}\mathbb{E}\left[(X^i + m_i - d_i)^-\right] \\
&\quad + \sum_{i=1}^{N}\left[(m_i - d_i)F_{i,S}(d_i - m_i, d) - (m_i + \alpha_i - d_i)F_{i,S}(d_i - m_i - \alpha_i, d)\right. \\
&\qquad \left. + \int_{d_i - m_i - \alpha_i}^{d_i - m_i}\int_{-\infty}^{d} x f_{i,S}(x,y)dydx\right] \\
&= \sum_{i=1}^{N}\mathbb{E}\left[(X^i + m_i - d_i)^-\right] \\
&\quad + \sum_{i=1}^{N-1}\left[(m_i - d_i)F_{i,S}(d_i - m_i, d) - (m_i + \alpha_i - d_i)F_{i,S}(d_i - m_i - \alpha_i, d)\right. \\
&\qquad \left. + \int_{d_i - m_i - \alpha_i}^{d_i - m_i}\int_{-\infty}^{d} x f_{i,S}(x,y)dydx\right] \\
&\quad + (m_N - d_N)F_{N,S}(d_N - m_N, d) - (m_N - \sum_{j=1}^{N-1}\alpha_j - d_N)F_{N,S}(d_N - m_N \\
&\qquad + \sum_{j=1}^{N-1}\alpha_j, d) + \int_{d_N - m_N + \sum_{j=1}^{N-1}\alpha_j}^{d_N - m_N}\int_{-\infty}^{d} x f_{N,S}(x,y)dydx,
\end{aligned}$$

where in the last equality we have used the constraint $\sum_{j=1}^{N}\alpha_j = 0$. We now denote by μ_i, σ_i the mean and the quadratic variation of X^i, $i = 1, \cdots, N$,

and $\Phi(x) = \int_{+\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$. Set $\bar{f}_{i,S}(x,y) = \int_{-\infty}^{y} f_{i,S}(x,s)ds$.

1. By computing the derivatives with respect to α_i, $i = 1, \cdots, N-1$, we obtain $\frac{\partial \phi}{\partial \alpha_i} = 0$ if and only if

$$
\begin{aligned}
0 = \lambda \Big(& (m_i + \alpha_i - d_i) \bar{f}_{i,S}(d_i - m_i - \alpha_i, d) - F_{i,S}(d_i - m_i - \alpha_i, d) \\
& + (d_i - m_i - \alpha_i) \int_{-\infty}^{d} f_{i,S}(d_i - m_i - \alpha_i, y) dy \\
& + F_{N,S}(d_N - m_N + \sum_{j=1}^{N-1} \alpha_j, d) \\
& - (m_N - \sum_{j=1}^{N-1} \alpha_j - d_N) \bar{f}_{N,S}(d_N - m_N + \sum_{j=1}^{N-1} \alpha_j, d) \\
& + (m_N - \sum_{j=1}^{N-1} \alpha_j - d_N) \int_{-\infty}^{d} f_{N,S}(d_N - m_N + \sum_{j=1}^{N-1} \alpha_j, y) dy \Big) \\
= \lambda & \left(F_{N,S}(d_N - m_N + \sum_{j=1}^{N-1} \alpha_j, d) - F_{i,S}(d_i - m_i - \alpha_i, d) \right).
\end{aligned}
$$

   We then obtain that the equation above has a solution if $\lambda = 0$ or when

$$
F_{i,S}(d_i - m_i - \alpha_i, d) = F_{N,S}(d_N - m_N + \sum_{j=1}^{N-1} \alpha_j, d) \tag{A.2}
$$

   for $i = 1, \cdots, N-1$.

2. By computing the derivatives with respect to m_i, for $i = 1, \cdots, N$, we obtain $\frac{\partial \phi}{\partial m_i} = 0$ if and only if

$$
\begin{aligned}
0 = 1 + \lambda \Big(& \Phi(\frac{d_i - \mu_i - m_i}{\sigma_i}) - (m_i - d_i) \bar{f}_{i,S}(d_i - m_i, d) + F_{i,S}(d_i - m_i, d) \\
& - F_{i,S}(d_i - m_i - \alpha_i, d) + (m_i + \alpha_i - d_i) \bar{f}_{i,S}(d_i - m_i - \alpha_i, d) \\
& + (d_i - m_i - \alpha_i) \int_{-\infty}^{d} f_{i,S}(d_i - m_i - \alpha_i, y) - (d_i - m_i) \int_{-\infty}^{d} f_{i,S}(d_i - m_i, y) dy \Big) \\
= 1 + \lambda & \left(\Phi(\frac{d_i - \mu_i - m_i}{\sigma_i}) + F_{i,S}(d_i - m_i, d) - F_{N,S}(d_N - m_N + \sum_{j=1}^{N-1} \alpha_j, d) \right),
\end{aligned}
$$

   where we have used (5.8), (A.2) and the notation above. In particular

$$
\lambda = - \left(\Phi(\frac{d_N - \mu_N - m_N}{\sigma_N}) + F_{N,S}(d_N - m_N, d) - F_{N,S}(d_N - m_N + \sum_{j=1}^{N-1} \alpha_j, d) \right)^{-1}, \tag{A.3}
$$

if the denominator is different from zero. By (A.3) we then obtain

$$\begin{aligned}&\Phi(\frac{d_i-\mu_i-m_i}{\sigma_i})+F_{i,S}(d_i-m_i,d)\\&=\Phi(\frac{d_N-\mu_N-m_N}{\sigma_N})+F_{N,S}(d_N-m_N,d),\end{aligned}\tag{A.4}$$

for $i=1,\cdots,N-1$.

A.2 Example on a Finite Probability Space

We give the proof of the optimal solutions (6.5)-(6.11) in Section 6.

Proof. We restrict the proof to the case $r=0$. The general case can be obtained following the same steps. For further details on the proof in the general case, we refer to Pastore [2014]. Note that the following computations apply in any other case when the derivatives of U_i are invertible for all $i=1,\cdots,N$. We can rewrite the definition of ρ in this particular setting as follows:

$$\rho(\mathbf{X}):=\inf\left\{c\in\mathbb{R}|\ \sum_{i=1}^{N}Y^i=c,\mathbb{E}\left[\sum_{i=1}^{N}\exp\left(-\alpha_i(X^i+Y^i)\right)\right]\leq\gamma\right\}.\tag{A.5}$$

Note that for $r=0$ we now have $y_j^N:=c-\sum_{i=1}^{N-1}y_j^i$ for $j=1,\cdots,M$. We compute ρ by using the method of Lagrange multipliers to minimize the function

$$\begin{aligned}&\phi(c,y_1^1,\cdots,y_M^1,\cdots,y_1^{N-1},\cdots,y_M^{N-1},\lambda)\\&=c+\lambda\left(\sum_{j=1}^{M}p_j\left[\sum_{k=1}^{N-1}\exp\left(-\alpha_k(X^k(\omega_j)+y_j^k)\right)\right.\right.\\&\qquad\left.\left.+\exp\left(-\alpha_N(X^N(\omega_j)+c-\sum_{i=1}^{N-1}y_j^i)\right)\right]-\gamma\right).\end{aligned}$$

We have:

1. *By computing the derivatives with respect to* y_j^k, $k=1,\cdots,N-1$, $j=1,\cdots,M$: $\frac{\partial\phi}{\partial y_j^k}=0$ if and only if for all fixed $j=1,\cdots,M$

$$\alpha_k\exp\left(-\alpha_k(X^k(\omega_j)+y_j^k)\right)=\alpha_N\exp\left(-\alpha_N(X^N(\omega_j)+c-\sum_{i=1}^{N-1}y_j^i)\right).\tag{A.6}$$

This also implies that for all fixed $j=1,\cdots,M$

$$\alpha_k\exp\left(-\alpha_k(X^k(\omega_j)+y_j^k)\right)=\alpha_1\exp\left(-\alpha_1(X^1(\omega_j)+y_j^1)\right)\tag{A.7}$$

for all $k = 1, \cdots, N-1$, i.e.

$$y_j^k = \frac{1}{\alpha_k}\left[\alpha_1 X^1(\omega_j) - \alpha_k X^k(\omega_j) - \log\frac{\alpha_1}{\alpha_k} + \alpha_1 y_j^1\right]. \tag{A.8}$$

Furthermore, by (A.6) we obtain that

$$y_j^1 = \frac{1}{\alpha_1\beta_N}\sum_{i=1}^{N} X^i(\omega_j) - X^1(\omega_j) + \frac{1}{\alpha_1\beta_N}\sum_{i=1}^{N}\frac{1}{\alpha_i}\log\frac{\alpha_1}{\alpha_i} + \frac{1}{\alpha_1\beta_N}c, \tag{A.9}$$

where $\beta_N = \sum_{k=1}^{N}\frac{1}{\alpha_k}$ as before.

2. *By computing the derivatives with respect to c*: $\frac{\partial\phi}{\partial c} = 0$ if and only if

$$0 = 1 - \alpha_N\lambda\sum_{j=1}^{M} p_j \exp\left(-\alpha_N(X^N(\omega_j) + c - \sum_{i=1}^{N-1} y_j^i)\right). \tag{A.10}$$

We can insert (A.6) in (A.10) and obtain

$$0 = 1 - \alpha_k\lambda\sum_{j=1}^{M} p_j \exp\left(-\alpha_k(X^k(\omega_j) + y_j^k)\right),$$

i.e.

$$\sum_{j=1}^{M} p_j \exp\left(-\alpha_k(X^k(\omega_j) + y_j^k)\right) = \frac{1}{\alpha_k\lambda} \tag{A.11}$$

for all $k = 1, \cdots, N-1$.

3. *By computing the derivatives with respect to λ*: $\frac{\partial\phi}{\partial\lambda} = 0$ if and only if

$$\begin{aligned}\gamma &= \mathbb{E}\sum_{i=1}^{N}\exp\left(-\alpha_i(X^i + Y^i)\right) \\ &= \sum_{j=1}^{M} p_j\left[\sum_{k=1}^{N-1}\exp\left(-\alpha_k(X^k(\omega_j) + y_j^k)\right) + \exp\left(-\alpha_N(X^N(\omega_j) + c - \sum_{i=1}^{N-1} y_j^i)\right)\right].\end{aligned} \tag{A.12}$$

We now substitute (A.10) and (A.11) in (A.12) and obtain:

$$\begin{aligned}\gamma &= \sum_{j=1}^{M} p_j \left(\sum_{k=1}^{N-1} \exp\Big(-\alpha_k (X^k(\omega_j) + y_j^k)\Big) \right. \\ &\qquad \left. + \exp\left(-\alpha_N (X^N(\omega_j) + c - \sum_{i=1}^{N-1} y_j^i)\right)\right) \\ &= \sum_{k=1}^{N-1} \sum_{j=1}^{M} p_j \exp\Big(-\alpha_k (X^k(\omega_j) + y_j^k)\Big) \\ &\qquad + \sum_{j=1}^{M} p_j \exp\left(-\alpha_N (X^N(\omega_j) + c - \sum_{i=1}^{N-1} y_j^i)\right) \\ &= \sum_{k=1}^{N-1} \frac{1}{\alpha_k \lambda} + \frac{1}{\alpha_N \lambda} \quad = \quad \frac{1}{\lambda} \sum_{k=1}^{N} \frac{1}{\alpha_k}. \end{aligned} \tag{A.13}$$

Hence

$$\lambda = \frac{1}{\gamma} \sum_{k=1}^{N} \frac{1}{\alpha_k} = \frac{\beta_N}{\gamma}. \tag{A.14}$$

We now compute c by inserting (A.8), (A.9) and (A.14) in (A.11) for $k = 1$:

$$e^{-\frac{c}{\beta_N}} \sum_{j=1}^{M} p_j \exp\left(-\frac{1}{\beta_N} (\sum_{i=1}^{N} X^i(\omega_j) + \sum_{i=1}^{N} \frac{1}{\alpha_i} \log \frac{\alpha_1}{\alpha_i})\right) = \frac{\gamma}{\alpha_1 \beta_N}. \tag{A.15}$$

Hence the systemic risk measure, i.e. the optimal c, is given by

$$\rho(\mathbf{X}) = c^* = -\beta_N \log \left[\frac{\gamma}{\alpha_1 \beta_N d_N}\right], \tag{A.16}$$

where $d_N = \sum_{j=1}^{M} p_j \exp\Big(-\frac{1}{\beta_N} \sum_{i=1}^{N} X^i(\omega_j) - \frac{1}{\beta_N} \sum_{i=1}^{N} \frac{1}{\alpha_i} \log \frac{\alpha_1}{\alpha_i}\Big)$.
By substituting the optimal value for c in (A.8) and (A.9) we also obtain the optimal allocations

$$\begin{aligned} y_j^k &= \frac{1}{\alpha_k \beta_N} \sum_{i=1}^{N} X^i(\omega_j) - X^k(\omega_j) - \frac{1}{\alpha_k} \log(\frac{X^j \alpha_1}{\alpha_k}) \\ &\qquad + \frac{1}{\alpha_k \beta_N} \sum_{i=1}^{N} \frac{1}{\alpha_i} \log \frac{\alpha_1}{\alpha_i} - \frac{1}{\alpha_k} \log \left[\frac{\gamma}{\alpha_k \beta_N d}\right], \end{aligned}$$

for $j = 1, \cdots, M$ and $k = 1, \cdots, N$. ■

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