Asymptotics of a Two-Scale Stochastic Volatility Model

J.P. Fouque* G. Papanicolaou† K.R. Sircar‡

To Jacques-Louis Lions on the occasion of his seventieth birthday

Abstract

We present an asymptotic analysis of derivative prices arising from a stochastic volatility model of the underlying asset price that incorporates a separation between the short (tick-by-tick) time-scale of fluctuation of the price and the longer (less rapid) time-scale of volatility fluctuations. The model includes leverage or skew effects (a negative correlation between price and volatility shocks), and a nonzero market price of volatility risk. The results can be used to estimate the latter parameter, which is not observable, from at-the-money European option prices. Detailed simulations and estimation of parameters are presented in [6].

1 Introduction

Stochastic volatility models have become popular for derivative pricing and hedging in the last ten years as the existence of a nonflat implied volatility surface (or term-structure) has been noticed and become more pronounced, especially since the 1987 crash. This phenomenon, which is well-documented in, for example, [9, 12], stands in empirical contradiction to the consistent use of a classical Black-Scholes (constant volatility) approach to pricing options and similar securities. However, it is clearly desirable to maintain as many of the features as possible that have contributed to this model's popularity and longevity, and the natural extension pursued in the literature and in practice has been to modify the specification of volatility in the stochastic dynamics of the underlying asset price model.

One approach, termed the implied deterministic volatility (IDV) approach [5, Chapter 8], is to suppose volatility is a deterministic function of the asset price X_t: volatility $= \sigma(t, X_t)$, so that the stochastic differential equation modeling the asset price becomes

$$dX_t = \mu X_t dt + \sigma(t, X_t) X_t dW_t.$$

The function $C(t,x)$ giving the no-arbitrage price of a European derivative security at time t when the asset price $X_t = x$ then satisfies the generalized Black-Scholes PDE

$$C_t + \frac{1}{2}\sigma^2(t,x)x^2 C_{xx} + r(xC_x - C) = 0,$$

with r the constant risk free interest rate and with terminal condition appropriate for the contract. This has the nice feature that the market is complete which, in this context, means that the

*CNRS-CMAP, Ecole Polytechnique, 91128 Palaiseau Cedex France, *fouque@paris.polytechnique.fr*. This work was done while visiting the Department of Mathematics, Stanford University.

†Department of Mathematics, Stanford University, Stanford CA 94305, *papanico@math.stanford.edu*

‡Department of Mathematics, University of Michigan, Ann Arbor MI 48109, *sircar@math.lsa.umich.edu*

derivative's risk can (theoretically) be perfectly hedged by the underlying, and there is no volatility risk premium to be estimated.

However, while attempts to infer a local volatility surface from market data by tree methods [13] or relative-entropy minimization [2] or interpolation [15] have yielded interesting qualitative properties of the (risk-neutral) probability distribution used by the market to price derivatives (such as excess skew and leptokurtosis in comparison to the lognormal distribution), this approach has not produced a stable surface that can be used consistently and with confidence over time.

For this reason, we concentrate on the "pure" stochastic volatility approach of [8] reviewed in [16, Section 1.2] or [3], in which volatility σ_t is modeled as an Itô process driven by a Brownian motion that has a component independent of the Brownian motion W_t driving the asset price. There has been a lot of analysis of specific Itô models in the literature [14, 17, 7] by numerical and analytical methods, many of which have ignored skew effects and/or the volatility risk premium for tractability. Our goal [6] is to estimate these parameters from market data and to test their stability over time and thus the potential usefulness of stochastic volatility models for hedging derivatives. What is (to our knowledge) new here in comparison with previous empirical work on stochastic volatility models is our keeping of these two factors, use of high-frequency (intraday) data, and an asymptotic simplification of option prices predicted by the model that allows for easy estimation of the volatility risk premium from at-the-money market option prices.

The latter exploits the separation of time-scales first introduced (in this context) in [16]. It is often observed that while volatility might fluctuate considerably over the many months comprising the lifetime of an options contract, it does not do so as rapidly as the stock price itself. That is, there are periods when the volatility is high, followed by periods when it is low. Within these periods, there might be much fluctuation of the stock price (as usual), but the volatility can be considered *relatively* constant until its next "major" fluctuation. The "minor" volatility fluctuations within these periods are relatively insignificant, especially as far as option prices, which come from an average of a functional of possible paths of the volatility, are concerned.

Many authors, for example [1], have proposed nonparametric estimation of the pricing measure for derivatives. The analysis in [16] is independent of specific modeling of the volatility process, but results in bands for option prices that describe potential volatility risk in relation to its historical autocorrelation decay structure, while obviating the need to estimate the risk premium. However, the market in at-the-money European options is liquid and its historical data can be used to estimate this premium[1]. For this reason, we shall attempt this with a model that is highly parametric, but complex enough to reflect an important number of observed volatility features:

1. volatility is positive;
2. volatility is rapidly mean-reverting (see for example [10]);
3. volatility shocks are negatively correlated with asset price shocks. That is, when volatility goes up, stock prices tend to go down and *vice-versa*. This is often referred to as leverage [4], and it at least partially accounts for a skewed distribution for the asset price that lognormal or zero-correlation stochastic volatility models do not exhibit. The skew is documented in empirical studies of historical stock data.

2 Model

The model we choose is that volatility is the exponential of a mean-reverting Ornstein-Uhlenbeck (OU) process (or, equivalently, $\log \sigma_t$ is mean-reverting OU). With a suitable initial distribution,

[1] This was suggested to us by Darrell Duffie.

the volatility process is stationary and ergodic which allows us to use averaging principles to approximate the option price, separating the minor and major fluctuations. This model has been considered in [14] and it is related to EGARCH models which, as shown in [11], are weak approximations to the continuous-time diffusion. Another model that is stationary and can be similarly implemented and analyzed is when σ_t is a mean-reverting Feller (or Cox-Ingersoll-Ross) process [3].

The final ingredient is to model the two time scales described previously. To this end, we introduce a small parameter $\varepsilon > 0$ describing the discrepancy between the scales, and model the volatility as $\sigma_t^\varepsilon = \sigma_{t/\varepsilon}$, where σ_t is exponential OU. Thus the volatility is the σ_t process "speeded-up" to reflect that there are many major fluctuations over the life of the options contract (this time scale is $\mathcal{O}(1)$ in the usual time-unit of years), but not as many as there are minor Itô fluctuations.

We define $Y_t := \log \sigma_t$ and suppose it satisfies

$$dY_t = \alpha(m - Y_t)dt + \beta d\hat{Z}_t$$

for constants $\alpha > 0, \beta > 0, m$ and $\hat{Z}_t$ a Brownian motion. Then $Y_t^\varepsilon := \log \sigma_t^\varepsilon$ is described by

$$dY_t^\varepsilon = \frac{\alpha}{\varepsilon}(m - Y_t^\varepsilon)dt + \frac{\beta}{\sqrt{\varepsilon}}d\hat{Z}_t$$

where now α and β have been replaced by α/ε and $\beta/\sqrt{\varepsilon}$ to model rapid mean reversion and overall variance of order one. Finally, to incorporate the correlation (skew) effect $d\langle W, \hat{Z}\rangle_t = \rho dt$, we write $\hat{Z}_t = \rho W_t + \sqrt{1-\rho^2}Z_t$, where W and Z are independent Brownian motions, to arrive at the final stochastic volatility model for the stock price X_t^ε

$$\begin{aligned} dX_t^\varepsilon &= \mu X_t^\varepsilon dt + e^{Y_t^\varepsilon} X_t^\varepsilon dW_t && (1) \\ dY_t^\varepsilon &= \frac{\alpha}{\varepsilon}(m - Y_t^\varepsilon)dt + \frac{\beta}{\sqrt{\varepsilon}}\left(\rho dW_t + \sqrt{1-\rho^2}dZ_t\right) && (2) \end{aligned}$$

Then, by the usual no-arbitrage argument, as detailed for example in [16, Section 1.3], the European call option price $C^\varepsilon(t,x,y)$ satisfies

$$\begin{aligned} C_t^\varepsilon + \frac{1}{2}e^{2y}x^2 C_{xx}^\varepsilon + \frac{\rho\beta x e^y}{\sqrt{\varepsilon}}C_{xy}^\varepsilon + \frac{\beta^2}{2\varepsilon}C_{yy}^\varepsilon + r\left(xC_x^\varepsilon - C^\varepsilon\right) & \\ + \left(\frac{\alpha}{\varepsilon}(m-y) - \frac{\lambda\beta}{\sqrt{\varepsilon}}\right)C_y^\varepsilon &= 0 \qquad (3) \\ C^\varepsilon(T,x,y) &= (x-K)^+ \end{aligned}$$

where λ is the market price of volatility risk which we assume constant. If $C^\varepsilon(t,x,y)$ satisfies this equation then from Itô's formula $C^\varepsilon = C^\varepsilon(t, X_t^\varepsilon, Y_t^\varepsilon)$ satisfies the stochastic differential equation

$$dC^\varepsilon = [rC^\varepsilon + (\mu - r)X^\varepsilon C_x^\varepsilon + \lambda\frac{\beta}{\sqrt{\varepsilon}}C_y^\varepsilon]dt + e^{Y_t^\varepsilon}X_t^\varepsilon C_x^\varepsilon dW_t + \frac{\beta}{\sqrt{\varepsilon}}C_y^\varepsilon d\hat{Z}_t \qquad (4)$$

From this expression we see that an infinitesimal change in the volatility risk $\beta/\sqrt{\varepsilon}$ changes the infinitesimal rate of return of the option by λ times the change in volatility risk. This is why λ is called the market price of volatility risk.

3 Asymptotic Analysis

Now, as $\varepsilon \downarrow 0$, the distinction between the time scales disappears and the major fluctuations occur infinitely often. In this limit, volatility can be approximated by a constant as far as averages of functionals of its path are concerned (that is, weakly), and we return to the classical Black-Scholes setting. What is of interest is the next term in the asymptotic approximation of $C^\varepsilon(t,x,y)$ valid for small ε, that describes the influence of ρ, λ and the randomness ($\beta > 0$) of the volatility.

To obtain this, let us write (3) as $\mathcal{L}^\varepsilon C^\varepsilon = 0$, where

$$\mathcal{L}^\varepsilon := \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2,$$

and

$$\begin{aligned}
\mathcal{L}_0 &:= \frac{1}{2}\beta^2\frac{\partial^2}{\partial y^2} + \alpha(m-y)\frac{\partial}{\partial y}, \\
\mathcal{L}_1 &:= \rho\beta x e^y\frac{\partial^2}{\partial x \partial y} - \lambda\beta\frac{\partial}{\partial y}, \\
\mathcal{L}_2 &:= \frac{\partial}{\partial t} + \frac{1}{2}e^{2y}x^2\frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right).
\end{aligned}$$

Then, constructing an expansion

$$C^\varepsilon(t,x,y) = C_0(t,x,y) + \sqrt{\varepsilon}C_1(t,x,y) + \varepsilon C_2(t,x,y) + \cdots,$$

we find, comparing powers of $\varepsilon << 1$,

$$\mathcal{L}_0 C_0 = 0$$

at the $\mathcal{O}(\varepsilon^{-1})$ level. Since $\mathcal{L}_0$ involves only y-derivatives and is the generator of the OU process Y_t, its null space is spanned by any nontrivial constant function, and it must be that C_0 does not depend on y: $C_0 = C_0(t,x)$.

At the next order, $\mathcal{O}(\varepsilon^{-1/2})$, we have

$$\mathcal{L}_1 C_0 + \mathcal{L}_0 C_1 = 0, \tag{5}$$

and since $\mathcal{L}_1$ takes y-derivatives, $\mathcal{L}_1 C_0 = 0$. By the same reasoning, (5) implies that $C_1 = C_1(t,x)$. Thus, up till $\mathcal{O}(\varepsilon)$, the option price does not depend on the current volatility.

Comparing $\mathcal{O}(1)$ terms,

$$\mathcal{L}_0 C_2 + \mathcal{L}_2 C_0 = 0.$$

Given $C_0(t,x)$, this is a Poisson equation for $C_2(t,x,y)$ and there will be no solution unless $\mathcal{L}_2 C_0$ is in the orthogonal complement of the null space of $\mathcal{L}_0^*$ (Fredholm Alternative). This is equivalent to saying that $\mathcal{L}_2 C_0$ has mean zero with respect to the invariant measure of the OU process. We denote this

$$\langle \mathcal{L}_2 C_0 \rangle_{ou} = 0,$$

where $\langle\cdot\rangle_{ou}$ denotes the expectation with respect to this invariant measure which is $\mathcal{N}(m,\nu^2)$, where $\nu^2 := \beta^2/2\alpha$:

$$\langle f \rangle_{ou} = \frac{1}{\sqrt{2\pi\nu^2}}\int_{-\infty}^{\infty} e^{-(y-m)^2/2\nu^2} f(y)dy.$$

Since C_0 is independent of y and $\mathcal{L}_2$ only depends on y through the e^{2y} coefficient, $\langle \mathcal{L}_2 C_0\rangle_{ou} = \langle \mathcal{L}_2\rangle_{ou} C_0$, and

$$\langle \mathcal{L}_2 \rangle_{ou} = \mathcal{L}_{BS}(\hat{\sigma}) := \frac{\partial}{\partial t} + \frac{1}{2}\hat{\sigma}^2 x^2\frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right),$$

where $\hat{\sigma}^2 := \langle e^{2y}\rangle_{ou} = e^{2m+2\nu^2}$.

Thus $C_0(t,x) = C_{BS}(t,x;\hat{\sigma})$, and the first term in the expansion is the Black-Scholes pricing formula with the averaged volatility constant $\hat{\sigma}$. The λ and ρ have thus far played no role, and we proceed to find the next term in the approximation, $C_1(t,x)$.

Comparing terms of $\mathcal{O}(\varepsilon)$, we find

$$\mathcal{L}_0 C_3 = -\left(\mathcal{L}_1 C_2 + \mathcal{L}_2 C_1\right), \tag{6}$$

which we look at as a Poisson equation for $C_3(t,x,y)$. Just as the Fredholm solvability condition for C_2 determined the equation for C_0, the solvability for (6) will give us the equation for $C_1(t,x)$. Substituting for $C_2(t,x,y)$ with

$$C_2 = -\mathcal{L}_0^{-1}\left(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle_{ou}\right)C_0,$$

this condition is

$$\left\langle \mathcal{L}_2 C_1 - \mathcal{L}_1\mathcal{L}_0^{-1}\left(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle_{ou}\right)C_0\right\rangle_{ou} = 0,$$

where

$$\langle\mathcal{L}_2 C_1\rangle_{ou} = \langle\mathcal{L}_2\rangle_{ou} C_1 = \mathcal{L}_{BS}(\hat{\sigma})C_1$$

since C_1 does not depend on y.

Defining

$$\mathcal{A} := \left\langle \mathcal{L}_1\mathcal{L}_0^{-1}\left(\mathcal{L}_2 - \langle\mathcal{L}_2\rangle_{ou}\right)\right\rangle_{ou},$$

the equation determining C_1 is

$$\mathcal{L}_{BS}(\hat{\sigma})C_1 = \mathcal{A}C_0,$$

as C_0 does not depend on y.

Again, using that $\mathcal{L}_0$ acts only on y-dependent functions, we can write

$$\mathcal{A} = \left\langle \left(\rho\beta x e^y \frac{\partial^2}{\partial x\partial y} - \lambda\beta\frac{\partial}{\partial y}\right)\left(\frac{1}{2}\phi(y)x^2\frac{\partial^2}{\partial x^2}\right)\right\rangle_{ou},$$

where

$$\mathcal{L}_0\phi(y) = \frac{1}{2}\beta^2\phi''(y) + \alpha(m-y)\phi'(y) = e^{2y} - \langle e^{2y}\rangle_{ou},$$

and so

$$\mathcal{A} = Ax^3\frac{\partial^3}{\partial x^3} + Bx^2\frac{\partial^2}{\partial x^2},$$

with

$$\begin{aligned} A &:= \frac{1}{2}\rho\beta\langle e^y\phi'\rangle_{ou} \\ B &:= \rho\beta\langle e^y\phi'\rangle_{ou} - \frac{\lambda\beta}{2}\langle\phi'\rangle_{ou}. \end{aligned}$$

Thus we must solve

$$\begin{aligned} \mathcal{L}_{BS}(\hat{\sigma})C_1 &= Ax^3\frac{\partial^3 C_{BS}}{\partial x^3}(\hat{\sigma}) + Bx^2\frac{\partial^2 C_{BS}}{\partial x^2}(\hat{\sigma}) \\ &= \frac{xe^{-d_1^2/2}}{\hat{\sigma}\sqrt{2\pi\,(T-t)}}\left(B - A\left[1 + \frac{d_1}{\hat{\sigma}\sqrt{T-t}}\right]\right), \end{aligned}$$

where

$$d_1 = \frac{\log(x/K) + (r + \frac{1}{2}\hat{\sigma}^2)(T-t)}{\hat{\sigma}\sqrt{T-t}},$$

and where we have used the explicit expression for the Black-Scholes price $C_{BS}(\hat{\sigma})$. Using the Green's function for the Black-Scholes PDE, as given for example in [16, Appendix C], and logarithmic transformations to reduce to Gaussian integrals, we find that

$$C_1 = \frac{xe^{-d_1^2/2}}{\hat{\sigma}\sqrt{2\pi}} \left(A\frac{d_1}{\hat{\sigma}} + (A-B)\sqrt{T-t} \right).$$

Finally, we compute

$$\begin{aligned} \langle e^y \phi' \rangle_{ou} &= -\frac{2e^{3m}}{\beta^2} \left(e^{9\nu^2/2} - e^{5\nu^2/2} \right) \\ \langle \phi' \rangle_{ou} &= -\frac{2\hat{\sigma}^2}{\alpha}, \end{aligned}$$

and so

$$C_1 = \frac{xe^{-d_1^2/2}}{\hat{\sigma}\sqrt{2\pi}} \left(\theta\rho \left(-\frac{d_1}{\hat{\sigma}} + \sqrt{T-t} \right) - \frac{\lambda\beta}{\alpha}\hat{\sigma}^2\sqrt{T-t} \right), \tag{7}$$

where $\theta := e^{3m}(e^{9\nu^2/2} - e^{5\nu^2/2})/\beta$ is a positive constant. Note that to order $\sqrt{\varepsilon}$, C^ε is decreasing in λ.

We can now calculate the implied volatility I^ε defined by $C^\varepsilon = C_{BS}(I^\varepsilon)$. Constructing an expansion $I^\varepsilon = \hat{\sigma} + \sqrt{\varepsilon}I_1 + \cdots$, we find that

$$\begin{aligned} I_1 &= C_1(t,x) \left[\frac{\partial C_{BS}(t,x;\hat{\sigma})}{\partial \hat{\sigma}} \right]^{-1} \\ &= -\theta\frac{\rho}{\hat{\sigma}} \left(\frac{d_1}{\hat{\sigma}\sqrt{T-t}} - 1 \right) - \frac{\lambda\beta\hat{\sigma}}{\alpha}, \end{aligned} \tag{8}$$

which shows that

$$I^\varepsilon = \hat{\sigma} + (\frac{\alpha}{\varepsilon})^{-1/2} \left\{ \rho \frac{e^{3\nu^2/2} - e^{-\nu^2/2}}{\sqrt{2}\nu(T-t)} \left[\log\left(\frac{K}{x}\right) - (r - \frac{\hat{\sigma}^2}{2})(T-t) \right] - \sqrt{2}\nu\lambda\hat{\sigma} \right\} + \mathcal{O}((\frac{\alpha}{\varepsilon})^{-1}), \tag{9}$$

where we have used the expression for θ above and where $\hat{\sigma} = e^{m+\nu^2}$ and $\nu^2 = \beta^2/2\alpha$. We have also expressed the expansion for I^ε in terms of the inverse of the fast mean reversion rate $(\frac{\alpha}{\varepsilon})^{-1}$. For $\rho < 0$, which is the usual case, this gives a decreasing implied volatility curve when plotted against strike price K, that is, a decreasing smirk.

The analysis gives rise to an explicit formula describing the geometry of the implied volatility surface across strike prices and expiration dates. In particular, the relationship to the risk premium parameter λ in (9) considerably simplifies the procedure for its estimation, which otherwise would be a computationally-intensive inverse problem for the PDE (3).

4 Conclusions

We have shown that an incomplete market asset modeled by a fast mean reverting stochastic volatility leads to an asymptotic formula for options pricing and associated implied volatility (9). This formula involves in a direct way the otherwise unobservable market price of volatility risk λ, which can then be estimated by fitting it to observed smirks (observed implied volatility as a function of strike price K). The other parameters in the model, the mean and variance of the log volatility m and ν^2 and the fast mean reversion rate α/ε, can be estimated from historical asset price data. The remaining parameter of the model, the skew ρ, can in principle also be estimated from historical asset price data but it is better in practice to estimate it by fitting formula (9) to option pricing data, as is done for the market price of volatility risk λ. This is done in [6].

References

[1] Y. Aït-Sahalia and A. Lo. Nonparametric estimation of state-price densities implicit in financial asset prices. Technical report, Graduate School of Business, University of Chicago, November 1996.

[2] M. Avellaneda, C. Friedman, R. Holmes, and D. Samperi. Calibrating volatility surfaces via relative-entropy minimization. *Appl. Math. Finance*, 4(1):37–64, 1997.

[3] C. Ball and A. Roma. Stochastic Volatility Option Pricing. *J. Financial and Quantitative Analysis*, 29(4):589–607, December 1994.

[4] F. Black. Studies in stock price volatility changes. In *Proceedings of the 1976 Business Meeting of the Business and Economic Statistics Section*, pages 177–181. Amer. Statist. Assoc., 1976.

[5] D. Duffie. *Dynamic Asset Pricing Theory, 2nd ed.* Princeton University Press, 1996.

[6] J.P. Fouque, G. Papanicolaou, and K.R. Sircar. A Two-Scale Stochastic Volatility Model: Asymptotic Analysis and Estimation. *Preprint*, 1998.

[7] S. Heston. A closed-form solution for options with Stochastic Volatility with applications to bond and currency options. *Review of Financial Studies*, 6(2):327–343, 1993.

[8] J. Hull and A. White. The Pricing of Options on Assets with Stochastic Volatilities. *J. Finance*, XLII(2):281–300, June 1987.

[9] J. Jackwerth and M. Rubinstein. Recovering probability distributions from contemporaneous security prices. *Early Draft*, October 1995.

[10] A. Melino and S.M. Turnbull. Pricing Foreign Currency Options with Stochastic Volatility. *J. of Econometrics*, 45:239–265, 1990.

[11] D.B. Nelson. Arch models as diffusion approximations. *J. Econometrics*, 45(1-2):7–38, July/Aug 1990.

[12] M. Rubinstein. Nonparametric Tests of Alternative Option Pricing Models. *J. Finance*, XL(2):455–480, June 1985.

[13] M. Rubinstein. Implied binomial trees. *J. Finance*, LXIX(3):771–818, July 1994.

[14] L. Scott. Option Pricing when the Variance changes randomly: Theory, Estimation, and an Application. *J. Financial and Quantitative Analysis*, 22(4):419–438, December 1987.

[15] D. Shimko. Bounds of probability. *RISK*, 6(4):33–37, April 1993.

[16] K.R. Sircar and G.C. Papanicolaou. Stochastic volatility, smile and asymptotics. *Submitted to Applied Mathematical Finance*, April 1997.

[17] E. Stein and J. Stein. Stock Price Distributions with Stochastic Volatility: An Analytic Approach. *Review of Financial Studies*, 4(4):727–752, 1991.