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WAVES IN RANDOM MEDIA AND TWO-PARAMETER DIFFUSIONS

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Abstract. We consider a one-dimensional linear wave equation with a small random dissipative field and with a boundary condition corresponding to the so-called Goursat problem. In [**3**], we have studied the case of a mean zero dissipative field and we have obtained the convergence in distribution of the solution, properly rescaled, to a two-parameter diffusion. In this paper we study the noncentered case and we show that the fluctuation field converges also to a two-parameter diffusion. These results are examples of two-parameter diffusions arising naturally from the propagation of waves in random media.

1. Introduction

We study a one-dimensional wave in a small ergodic dissipation field with boundary conditions corresponding to the so-called Goursat problem. Denoting by x the space variable, τ the time variable and $u(x,\tau)$ the wave function, the equation can be written:

$$\frac{\partial^2 u}{\partial \tau^2} - \frac{\partial^2 u}{\partial x^2} = \epsilon D(x,\tau)u, \quad \tau \geq |x| \tag{1.1}$$

$$u(-\tau,\tau) = u(\tau,\tau) = 1 \tag{1.2}$$

where ϵ is small, $0 < \epsilon << 1$, and $D(x,\tau)$ is a space-time ergodic dissipation field. u is given and equalled to a constant along the characteristic lines and the equation is to be solved inside of the light cone $\tau \geq |x|$.

Rotating the axis by the change of variables $(s = \tau + x, t = \tau - x)$, denoting $u(x,\tau)$ by $Y(s,t)$ and $D(x,\tau)$ by $F(s,t)$, our problem becomes:

$$\frac{\partial^2 Y}{\partial s \partial t} = \epsilon F(s,t)Y, \quad st \geq 0 \tag{1.3}$$

$$Y(s,0) = Y(0,t) = 1 \tag{1.4}$$

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This paper is in final form and no version of it will be submitted for publication elsewhere or, in the integral form:

$$Y(s,t) = 1 + \epsilon \int_0^s \int_0^t F(u,v)Y(u,v)dudv \tag{1.5}$$

For simplicity we consider a very particular random field F; we assume:

$$F(s,t) = \sum_{k=1}^{\infty}\sum_{l=1}^{\infty} F_{k,l} 1_{[k-1,k)}(s) 1_{[l-1,l)}(t) \tag{1.6}$$

where $(F_{k,l})_{k\geq 1, l\geq 1}$ is an i.i.d. sequence of bounded random variables; M, m and σ^2 will respectively denote a bound for $|F|$, $\mathbb{E}(F)$ and $\mathrm{Var}(F)$.

From equation (1.5), it is clear that $Y(s,t)$ (or $u(x,\tau)$) converges to the constant 1. In order to observe the effect of the randomness we perform a space-time rescaling; the first order will be given by $(s \to s/\sqrt{\epsilon}, t \to t/\sqrt{\epsilon})$. With the notations $Y^{\epsilon}(s,t) = Y(s/\sqrt{\epsilon}, t/\sqrt{\epsilon})$ and $F_{\epsilon}(s,t) = F(s/\sqrt{\epsilon}, t/\sqrt{\epsilon})$, one get the following rescaled integral equation:

$$Y^{\epsilon}(s,t) = 1 + \int_0^s \int_0^t F_{\epsilon}(u,v)Y^{\epsilon}(u,v)dudv \tag{1.7}$$

Denoting the mean value of F by m, $Y^{\epsilon}(s,t)$ converges to the deterministic function $y(s,t)$, unique solution to the following equation:

$$y(s,t) = 1 + m\int_0^s \int_0^t y(u,v)dudv \tag{1.8}$$

which can be represented as $y(s,t) = f(mst)$ where f is the function given by the series $f(\rho) = \sum_{n=0}^{\infty} \frac{\rho^n}{(n!)^2}$.

In the following section we study the non-centered case $m \neq 0$; we prove that the fluctuation field, defined by $Z^{\epsilon} = \frac{1}{\sqrt{\epsilon}}(Y^{\epsilon} - y)$, converges in distribution to a two-parameter diffusion.

In the last section we present, without details, the main result obtained in **[3]** for the centered case $m = 0$.

2. The fluctuation field

Definition 2.1. *The* fluctuation field Z^{ϵ} *is defined for every* $s \geq 0$ *and* $t \geq 0$ *by:*

$$Z^{\epsilon}(s,t) = \frac{1}{\sqrt{\epsilon}}\{Y^{\epsilon}(s,t) - y(s,t)\} \tag{2.1}$$

where Y^{ϵ} *is given by (1.7) and* y *is given by (1.8).*

With the notation $W_\epsilon(u,v) = \frac{1}{\sqrt{\epsilon}}\{F_\epsilon(u,v) - m\}$, we get the following equation for Z^ϵ:

$$\begin{aligned} Z^\epsilon(s,t) = m\int_0^s\int_0^t Z^\epsilon(u,v)dudv \quad &+ \quad \int_0^s\int_0^t y(u,v)W_\epsilon(u,v)dudv \\ &+ \quad \sqrt{\epsilon}\int_0^s\int_0^t Z^\epsilon(u,v)W_\epsilon(u,v)dudv \end{aligned} \tag{2.2}$$

We have the following result for the second term on the righthand side of (2.2):

Lemma 2.2. *The two-parameter continuous process*

$$\left(\int_0^s\int_0^t y(u,v)W_\epsilon(u,v)dudv\right)_{s\geq 0, t\geq 0}$$

converges in distribution, as $\epsilon \searrow 0$, to the continuous gaussian process

$$\left(\sigma\int_0^s\int_0^t y(u,v)dW(u,v)\right)_{s\geq 0, t\geq 0}$$

where W is a standard brownian sheet.

Proof. Let us recall a sufficient condition, obtained in [**1**], for the tightness of the laws on $C([0,+\infty)^2, \mathbb{R})$ of a family of continuous two-parameter processes (X^ϵ): for every $S > 0$ and $T > 0$, there exists a constant $C(S,T)$ independent of ϵ such that:

$$\mathbb{E}\{|\triangle_{[z,z']}(X^\epsilon)|^4\} \leq C(S,T)|\triangle_{[z,z']}|^2 \tag{2.3}$$

where $z = (s,t)$ and $z' = (s',t')$ such that $0 \leq s \leq s' \leq S$ and $0 \leq t \leq t' \leq T$, $|\triangle_{[z,z']}| = (s'-s)(t'-t)$ and $\triangle_{[z,z']}(X^\epsilon)$ denotes the increment of X^ϵ over the rectangle $[z,z']$, namely:

$$\triangle_{[z,z']}(X^\epsilon) = X^\epsilon(s',t') - X^\epsilon(s,t') - X^\epsilon(s',t) + X^\epsilon(s,t)$$

This estimate is easily obtained for $\int_0^s\int_0^t y(u,v)W_\epsilon(u,v)dudv$ by decomposing the double integral over small squares of size $\sqrt{\epsilon}\times\sqrt{\epsilon}$ and by using the independence of the sequence $(F_{k,l})$.
A simple generalization of the central limit theorem gives the identification of the limit: the increments over disjoint rectangles are independent and the law of an increment is gaussian with mean zero and variance $\sigma^2\int_s^{s'}\int_t^{t'}|y(u,v)|^2dudv$.

The next step is to control the fourth moment of Z^ϵ:

Lemma 2.3. *For every $S > 0$ and $T > 0$, there exists a constant $C(S,T)$ independent of ϵ such that:*

$$\sup_{0\leq s\leq S, 0\leq t\leq T} \mathbb{E}\{|Z^\epsilon(s,t)|^4\} \leq C(S,T) \tag{2.4}$$

PROOF. For $s \leq S$, $t \leq T$ and constants $C_i(S,T)$, independent of ϵ, which may change from line to line, we get from equation (2.2):

$$
\begin{aligned}
|Z^\epsilon(s,t)|^4 &\leq C_1(S,T)\int_0^s\int_0^t |Z^\epsilon(u,v)|^4 dudv \\
&+ C_2\left|\int_0^s\int_0^t y(u,v)W_\epsilon(u,v)dudv\right|^4 \\
&+ C_3(S,T)\int_0^s\int_0^t |Z^\epsilon(u,v)|^4|\sqrt{\epsilon}W_\epsilon(u,v)|^4 dudv
\end{aligned}
$$

Taking the expectation, using the tightness estimate in the proof of lemma 2.2 for the second term and using the fact that $|\sqrt{\epsilon}W_\epsilon| \leq 2M$ in the third term, we obtain:

$$
\mathbb{E}\{|Z^\epsilon(s,t)|^4\} \leq C_1(S,T) + C_2(S,T)\int_0^s\int_0^t \mathbb{E}\{|Z^\epsilon(u,v)|^4\}dudv \tag{2.5}
$$

The proof is completed by an application of a two-parameter version of Gronwall's lemma.

PROPOSITION 2.4. *The laws of* $(Z^\epsilon(s,t))_{s\geq 0, t\geq 0}$ *form a tight family on the space* $C([0,+\infty)^2, \mathbb{R})$.

PROOF. We apply the sufficient condition (2.3) to (Z^ϵ): from (2.2) we get:

$$
\begin{aligned}
\triangle_{[z,z']}(Z^\epsilon) = m\int_s^{s'}\int_t^{t'} Z^\epsilon(u,v)dudv \quad &+ \int_s^{s'}\int_t^{t'} y(u,v)W_\epsilon(u,v)dudv \\
&+ \sqrt{\epsilon}\int_s^{s'}\int_t^{t'} Z^\epsilon(u,v)W_\epsilon(u,v)dudv
\end{aligned}
\tag{2.6}
$$

Using lemma 2.3 and $|\sqrt{\epsilon}W_\epsilon| \leq 2M$, we deduce:

$$
\mathbb{E}\{|\triangle_{[z,z']}(Z^\epsilon)|^4\} \leq C_1(S,T)|\triangle_{[z,z']}|^4 + C_2\mathbb{E}\{\left|\int_s^{s'}\int_t^{t'} y(u,v)W_\epsilon(u,v)dudv\right|^4\}
$$

The inequality $|\triangle_{[z,z']}|^4 \leq C(S,T)|\triangle_{[z,z']}|^2$ and the tightness estimate for $\int_0^s\int_0^t y(u,v)W_\epsilon(u,v)dudv$ give the sufficient condition 2.3 for (Z^ϵ).

The following lemma enables us to handle the last term in (2.2).

LEMMA 2.5. *For every* $s \geq 0$ *and* $t \geq 0$*:*

$$
\lim_{\epsilon\searrow 0} \mathbb{E}\{\left|\sqrt{\epsilon}\int_0^s\int_0^t Z^\epsilon(u,v)W_\epsilon(u,v)dudv\right|^2\} = 0
$$

PROOF. Let us introduce the following notations:

$$
\underline{u} = \sqrt{\epsilon}[u/\sqrt{\epsilon}]\ ,\ \underline{v} = \sqrt{\epsilon}[v/\sqrt{\epsilon}]
$$

where [.] denotes the integer part.
One can decompose $Z^\epsilon(u,v)$ into:

$$Z^\epsilon(u,v) = Z^\epsilon(\underline{u},\underline{v}) + \triangle_{[(\underline{u},\underline{v}),(u,v)]}(Z^\epsilon) + \triangle_{[(\underline{u},0),(u,\underline{v})]}(Z^\epsilon) + \triangle_{[(0,\underline{v}),(\underline{u},v)]}(Z^\epsilon)$$

From lemma 2.3 we have a uniform control of $\mathbb{E}\{|Z^\epsilon(u,v)|^2\}$ by a constant depending only upon S and T.
By (2.2) one can see that $Z^\epsilon(\underline{u},\underline{v})$, $\triangle_{[(\underline{u},0),(u,\underline{v})]}(Z^\epsilon)$ and $\triangle_{[(0,\underline{v}),(\underline{u},v)]}(Z^\epsilon)$ are independent of $W_\epsilon(u,v)$ as soon as u or v is not a multiple of $\sqrt{\epsilon}$.
Using these two properties, a simple computation gives:

$$\lim_{\epsilon\searrow 0}\mathbb{E}\left\{\left|\sqrt{\epsilon}\int_0^s\int_0^t \Psi(u,v)W_\epsilon(u,v)dudv\right|^2\right\} = 0$$

where $\Psi(u,v)$ is $Z^\epsilon(\underline{u},\underline{v})$ or $\triangle_{[(\underline{u},0),(u,\underline{v})]}(Z^\epsilon)$ or $\triangle_{[(0,\underline{v}),(\underline{u},v)]}(Z^\epsilon)$.
For $\Psi(u,v) = \triangle_{[(\underline{u},\underline{v}),(u,v)]}(Z^\epsilon)$ we get the same result by a direct estimate of $\mathbb{E}\{|\Psi(u,v)|^2\}$ and the bound $\sqrt{\epsilon}W_\epsilon(u,v) \leq 2M$. Putting these four estimates together one obtain the lemma.

We are now in position to give the main result of this section.

THEOREM 2.6. *The fluctuation field $(Z^\epsilon(s,t))$ converges in distribution, as $\epsilon\searrow 0$, to the unique solution $Z(s,t)$ of the following two-parameter stochastic differential equation:*

$$Z(s,t) = m\int_0^s\int_0^t Z(u,v)dudv + \sigma\int_0^s\int_0^t y(u,v)dW(u,v) \tag{2.7}$$

where y is the deterministic solution of (1.8) and W is a standard brownian sheet.

PROOF. The family (Z^ϵ) being tight by proposition 2.4, let us denote by (Z^{ϵ_n}) a converging subsequence and by Z its limit;

$$(Z(s,t))_{s\geq 0,t\geq 0} \longrightarrow \left(Z(s,t) - m\int_0^s\int_0^t Z(u,v)dudv\right)_{s\geq 0,t\geq 0}$$

being a continuous map on $C([0,+\infty)^2,\mathbb{R})$, one get that:

$$Z^{\epsilon_n}(s,t) - m\int_0^s\int_0^t Z^{\epsilon_n}(u,v)dudv \xrightarrow{\mathcal{D}} Z(s,t) - m\int_0^s\int_0^t Z(u,v)dudv$$

On the other hand, by lemmas 2.2 and 2.5:

$$\int_0^s\int_0^t \{y(u,v) + \sqrt{\epsilon_n}Z^{\epsilon_n}(u,v)\}W_{\epsilon_n}(u,v)dudv \xrightarrow{\mathcal{D}} \sigma\int_0^s\int_0^t y(u,v)dW(u,v)$$

Therefore Z is a solution to equation (2.7); a simple strong uniqueness result for this equation (see for instance [**4**]) completes the proof.

We end this section by the following remark: the corresponding one-parameter problem, unrelated to the physical wave propagation problem, can be written as follows:

$$Y(t) = 1 + \epsilon \int_0^t F(u)Y(u)du$$

With the rescaling $Y^\epsilon(t) = Y(t/\epsilon)$ and $F_\epsilon(t) = F(t/\epsilon)$, one can write the equation corresponding to (1.7):

$$Y^\epsilon(t) = 1 + \int_0^t F_\epsilon(u)Y^\epsilon(u)du$$

whose solution is $Y^\epsilon(t) = \exp(\int_0^t F_\epsilon(u)du)$ which converges to $y(t) = \exp(mt)$. The fluctuation field is then given explicitely by:

$$Z^\epsilon(t) = \frac{1}{\sqrt{\epsilon}}(\exp(\int_0^t F_\epsilon(u)du) - \exp(mt))$$

which converges in distribution to the Orstein-Uhlenbeck process, B being a standard brownian motion:

$$Z(t) = \sigma \int_0^t \exp(m(t-u))dB(u)$$

solution of the stochastic differential equation:

$$Z(t) = m \int_0^t Z(u)du + \sigma \int_0^t y(u)dB(u)$$

3. The centered case

In the centered case, $m = \mathbb{E}(F) = 0$, $y(s,t)$ is constant which means that the scaling used in (1.7) is not the appropriate one; we have to look at the solution on a longer scale in space and time in order to observe the effect of the randomness; the second order scaling will be given by: $(s \to s/\epsilon, t \to t/\epsilon)$. With the notations $X^\epsilon(s,t) = X(s/\epsilon, t/\epsilon)$ and $F^\epsilon(s,t) = F(s/\epsilon, t/\epsilon)$, one get the following rescaled equation:

$$X^\epsilon(s,t) = 1 + \frac{1}{\epsilon}\int_0^s \int_0^t F^\epsilon(u,v)X^\epsilon(u,v)dudv \tag{3.1}$$

This problem is much more involved than the one studied in the previous section; this is due to the fact that the integral in (3.1) does not go to 0 and is not a continuous functional of the approximate brownian sheet $\frac{1}{\epsilon}\int_0^s\int_0^t F^\epsilon(u,v)dudv$. The following result is the main result of [**3**]; we give it here without proof.

Theorem 3.1. *(Theorem 1.1 [**3**]) (X^ϵ) converges in distribution, as $\epsilon \searrow 0$, to the two-parameter diffusion X, unique solution of the following stochastic differential equation:*

$$X(s,t) = 1 + \sigma \int_0^s \int_0^t X(u,v)dW(u,v) + \frac{\sigma^2}{4}\int_0^s\int_0^t X(u,v)dudv \tag{3.2}$$

where $W(s,t)$ is a standard brownian sheet and the second integral being the Stratonovich correction to the first Ito's stochastic integral.

Our final remark is that, also in this case, the corresponding one-parameter problem is very easy since the equation for $X^\epsilon(t) = X(t/\epsilon^2)$ and $F^\epsilon(t) = F(t/\epsilon^2)$ becomes:

$$X^\epsilon(t) = 1 + \frac{1}{\epsilon}\int_0^t F^\epsilon(u)X^\epsilon(u)du$$

which is solved explicitely:

$$X^\epsilon(t) = \exp\left(\frac{1}{\epsilon}\int_0^t F^\epsilon(u)du\right)$$

and converges to $X(t) = \exp(\sigma B(t))$ solution of:

$$X(t) = 1 + \sigma\int_0^t X(u)dB(u) + \frac{\sigma^2}{2}\int_0^t X(u)du = \sigma\int_0^t X(u)\circ dB(u)$$

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