Wave Propagation in Randomly Layered Media

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Abstract

An acoustical pulse propagating in a randomly layered slab and scattered by the random inhomogeneities generates a reflected signal and a transmitted pulse followed by its coda. In the regime of separation of scales (correlation length of the random inhomogeneities much smaller than typical wavelengths contained in the pulse much smaller than the large scale of the medium comparable to the size of the slab) an asymptotic analysis of the governing stochastic equations leads to the determination of the asymptotic probability distributions for the reflected and transmitted signals. Moreover it is possible to relate the local power spectral densities of the reflected signal to the large scale variations of the medium as shown by Asch, Kohler, Papanicolaou, Postel and White [1]; this is done through a system of hyperbolic transport equations and leads to a solution to the inverse problem which consists in recovering these large scale variations from the reflected signals. A time reversal method is applied to obtain consistent estimators for these local power spectral densities.

1 The One-Dimensional Case

We consider an acoustic wave travelling in a one-dimensional random medium located in the slab $0 \le x \le L$ and satisfying the linear conservation laws:

$$\left\{ \begin{array}{ll} \rho(x)\frac{\partial u}{\partial t}(x,t) + \frac{\partial p}{\partial x}(x,t) & = 0 \\ \frac{1}{K(x)}\frac{\partial p}{\partial t}(x,t) + \frac{\partial u}{\partial x}(x,t) & = 0 \end{array} \right.$$

where $u(x,t)$ and $p(x,t)$ are, respectively, the velocity and the pressure of the wave, whereas $\rho(x)$ and $K(x)$ are the density and the bulk modulus of the medium; the random fluctuations are modelled as follows:

$$\left\{ \begin{array}{ll} \rho(x) & = \rho_0(x)(1+\eta(\frac{x}{\varepsilon^2})) \\ \frac{1}{K(x)} & = \frac{1}{K_0(x)}(1+\nu(\frac{x}{\varepsilon^2})) \end{array} \right.$$

Here ρ_0 and K_0, varying on the scale of the slab, represent the deterministic parameters or macroscopic properties of the medium, $\eta(\frac{x}{\varepsilon^2})$ and $\nu(\frac{x}{\varepsilon^2})$ are varying on the small scale ε^2 (the correlation length of the random medium) and represent the stationary centered random fluctuations of the medium. The random fluctuations in the medium need not to be weak.

Defining the deterministic acoustic impedance by $I_0(x) = (\rho_0(x)K_0(x))^{1/2}$ and the deterministic acoustic velocity by $c_0(x) = (K_0(x)/\rho_0(x))^{1/2}$, we introduce the right going wave $A(x,t)$ and the left going wave $B(x,t)$ defined by:

$$\left\{ \begin{array}{ll} A & = I_0^{-1/2}p + I_0^{1/2}u \\ B & = -I_0^{-1/2}p + I_0^{1/2}u \end{array} \right.$$

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They satisfy an hyperbolic system of first order equations, with random coefficients, for which we can write the following boundary conditions corresponding to a pulse entering the slab from the left at time $t = 0$ and no wave entering the slab from the right:

$$\begin{cases} A(0,t) & = \varepsilon^{-\gamma} f(\frac{t}{\varepsilon}) \\ B(L,t) & = 0 \end{cases}$$

where f is the pulse shape, $f(t/\varepsilon)$ containing wavelengths of order ε (larger than the correlation length ε^2 and smaller than the size of the slab), and $\varepsilon^{-\gamma}$ is the amplitude of the incoming pulse, the energy entering the slab being of order $\varepsilon^{1-2\gamma}$ ($\gamma = 1/2$ corresponds to a fixed energy and $\gamma = 0$ corresponds to fixed amplitudes).
The transmitted and reflected waves are respectively given by $A(L,t)$ and $B(0,t)$.

1.1 The Transmitted Pulse

The analysis of the transmitted pulse (coherent transmitted front) in the one-dimensional case has been done in [12] and [6]. Introducing the deterministic travel time defined by

$$\tau_0(x) = \int_0^x \frac{dy}{c_0(y)}$$

the transmitted pulse, viewed in the scale ε of the initial pulse, is given by:

$$A(L, \tau_0(L) + \varepsilon\sigma)$$

where σ is the time in the ε-scale.
In the regime of constant amplitudes ($\gamma = 0$), the probability distribution of this transmitted pulse converges, as ε goes to 0, to the probability distribution of the initial pulse spreaded by a convolution with a Gaussian density and shifted by a Gaussian random variable. Both the variances of the Gaussian density and the Gaussian shift depend upon the size of the slab (distance of propagation) and the covariance of the random medium.
This result is a prolongation of the O'Doherty-Anstey theory [13] which asserts that, for weak fluctuations, which in fact we did not assume, the travelling pulse retains its shape up to a low spreading (convolution by a Gaussian density) and is deterministic when observed by an observer travelling at the same random velocity as the wave ($c(x) = (K(x)/\rho(x))^{1/2}$), while it is stochastic (shift by a Gaussian random variable) when the observer's velocity is the deterministic velocity of the wave ($c_0(x)$).

1.2 The Reflected Signal

The reflected signal $B(0,t)$ obtained by a pulse scattered by a randomly layered medium in the regime of separation of scales described above has been studied by G. Papanicolaou and his coauthors. The one-dimensional case is treated in a series of papers [4], [3], [2] and [14], the last paper being devoted to the one-dimensional inverse problem.
The first observation of this study is that, apparently, the information about the large scale variations of the medium is lost in the noise present in the reflected signal. This noise is due to the multiple scattering of the pulse as it propagates into the medium. One of the main ideas of this theory is that the local properties of this non-stationary random signal contain information. More precisely, locally around a time t_0 and in the scale ε of the initial pulse (large compared to the correlation length ε^2 of the inhomogeneities but small compared to the distance of propagation), this signal, given by $B(0, t_0 + \varepsilon\sigma)$, is

asymptotically (as ε goes to 0) stationary, centered and normally distributed. The spectral density of the asymptotic stationary Gaussian process is related to the determistic part of the coefficients of the medium ($\rho_0(x)$ and $K_0(x)$) through an hyperbolic system of transport equations (the so-called W-equations). This is obtained by an invariant embedding method and an asymptotic analysis (in the white noise approximation regime) of the differential equations with random coefficients describing the wave field. Tools such as convergence of stochastic processes or stochastic calculus with respect to Brownian motions are needed in this study. In the case of a uniform background (no large scale variations of the medium), this spectral density can be computed explicitely; this is very important for comparison to various simulations [1].

1.2.1 The Inverse Problem. The inverse problem consists in the reconstruction of the large scale variations of the medium appearing as coefficients in the underlying transport equations. The practical solution to this problem requires good statistical estimators for the local covariances of the reflected signals, namely the mathematical expectation of $B(0,t_0)B(0,t_0+\varepsilon\sigma)$. This has been done by using a windowed Fourier transform [1]. An estimate based on a wavelet transform [7] has been proposed as well as an estimate for the small parameter ε representing the separation of scales i.e. the ratio of the size of the inhomogeneities with the wavelengths or ratio of the wavelenths with the distance of propagation.

1.2.2 A Time Reversal Method. Consistent estimators for these local covariances have been obtained by a time reversal method in [8]. This method is the mathematical analysis in the one-dimensional situation of time-reversal mirrors. A time-reversal mirror is a device, elaborated at the ESPCI-Paris by M. Fink and his team [9] in the context of ultrasounds, enabling them to memorize an acoustical signal and to resend a part of it back into the same medium after a time-reversal. Using this method, it has been shown [8] that one obtains a new reflected signal which becomes asymptotically deterministic and whose Fourier transform is nothing more than the spectral density. The medium itself is doing the computation of the local covariance in the time domain and therefore produces the best estimator one can wish.

2 The 3-D Layered Case

The results presented above in the 1-D case have been generalized to the 3-D layered case. The medium is layered which means that its properties vary only in one direction (the depth into the medium by analogy with Geophysics). We assume that only the bulk modulus contains random fluctuations on the smaller scale.

The transmitted front has been studied in [5]: in the same regime of separation of scales, the asymptotic probability distribution of the pressure field at the bottom of the slab generated by a pulse at the surface, is obtained; the main difference with the one-dimensional case being a convolution with the derivative of a Gaussian density instead of a Gaussian density itself, only one Gaussian variable being needed to described the distribution of the field at various offsets. The proof of this result requires a non trivial simultaneous use of diffusion-approximation and stationary phase theorems (see [5]). A multimode situation is studied in [11].

The reflected signal as well as the inverse problem in the 3-D layered case have been studied in the review paper [1]. A generalization to elastic waves can be found in the recent paper [10] and the case where the medium is also slowly varying in the transverse directions is studied in [15].

The generalization of the time reversal method to the 3-D layered case is a work in progress.

References

[1] M.Asch, W. Kohler, G. Papanicolaou, M. Postel and B. White, *Frequency content of randomly scattered signals*, SIAM Review, 33 (1991), pp. 519–626.

[2] M. Asch, G. Papanicolaou, M. Postel, P. Sheng and B. White, *Frequency content of randomly scattered signals, Part I*, Wave Motion, 12 (1990), pp. 429–450.

[3] R. Burridge, G. Papanicolaou, P. Sheng and B. White, *Probing a random medium with a pulse*, SIAM J. Appl. Math., 49 (1989), pp. 582–607.

[4] R. Burridge, G. Papanicolaou and B. White, *Statistics for pulse reflexion from a randomly layered medium*, SIAM J. Appl. Math., 47 (1987), pp. 146–168.

[5] J. Chillan and J. P. Fouque, *Pressure fields generated by acoustical pulses propagating in randomly layered media*, SIAM J. Appl. Math., (1998), to appear.

[6] J. F. Clouet and J. P. Fouque, *Spreading of a pulse travelling in random media*, Annals of Applied Probability, 4 (1994), pp. 1083–1097.

[7] J. F. Clouet, J. P. Fouque and M. Postel, *Spectral analysis of ramdomly scattered signals using the wavelet transform*, Wave Motion, 22 (1995), pp. 145–170.

[8] J. F. Clouet and J. P. Fouque, *A time-reversal method for an acoustical pulse propagating in randomly layered media*, Wave Motion, 25 (1997), pp. 361–368.

[9] M. Fink, *Time reversal mirrors*, J. Phys. D: Appl. Phys., 26 (1993), pp. 1333–1350.

[10] W. Kohler, G. Papanicolaou and B. White, *Localization and Mode Conversion for Elastic Waves in Randomly Layered Media*, Wave Motion, 23 (1996), pp. 1–22 and 181–201.

[11] P. Lewicki, R. Burridge and M. De Hoop, *Beyond effective medium theory: pulse stabilization for multimode wave propagation in high-contrast layered media*, SIAM J. Appl. Math., 56 (1996), pp . 256–276.

[12] P. Lewicki, R. Burridge and G. Papanicolaou, *Pulse stabilisation in a strongly heterogeneous layered medium*, Wave Motion, 20 (1994), pp. 177–195.

[13] R. F. O'Doherty and N. A. Anstey, *Refections on amplitudes*, Geophysical Prospecting, 19 (1971), pp. 430–458.

[14] G. Papanicolaou, M. Postel, P. Sheng and B. White, *Frequency content of randomly scattered signals, Part II*, Wave Motion, 12 (1990), pp. 527–549.

[15] K. Solna, *Stable spreading of acoustic pulses due to laminated microstructure*, Ph.D. dissertation, Stanford University (1997).