ACOUSTIC PULSES PROPAGATING IN RANDOMLY LAYERED MEDIA

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Abstract. We present mathematical results which have been obtained in the last fifteen years for waves propagating in randomly layered media in the framework of separation of scales introduced by G. Papanicolaou and his coauthors. Basic facts on diffusion-approximation and on the use of stochastic calculus will be recalled. We show how a multi-frequency asymptotic analysis gives a precise description of the probability distribution of the transmitted front generated by an incoming pulse. An inverse problem consists in reconstructing the macroscopic variations of the medium from the noisy reflected signal. We present the solution proposed by G. Papanicolaou and his coauthors and show how it depends on statistical estimates for the local power spectral densities of the non stationary reflected signal. We conclude by another approach to this estimation problem which uses a time reversal method introduced in the context of ultrasounds by M. Fink and his group.

1. Introduction

Waves propagating in layered media can be studied by using the propagator technique which consists in decomposing the wave into right and left going waves traveling in the direction where the coefficients of the medium are varying. By using a Fourier transform in the other directions it is then possible to reduce the problem to an initial value problem in the frequency domain. Assuming that the medium is random with a correlation length much smaller than the typical distances of propagation, pulses containing intermediate wavelengths can be used to probe the medium. This is done by an asymptotic analysis as these scales separate and by using diffusion-approximation results. This approach has been initiated by G. Papanicolaou and his coauthors. We refer to the review paper [1] for a detailed presentation of the method. The goal of this contribution is to present in a condensed way the results obtained in these recent years using this technique.

In Section 2 we present the simplest case of a one-dimensional random medium. We recall the basic facts needed to describe Markovian coefficients, we introduce the different scales and we write carefully the boundary conditions. Simulations borrowed from [1] show the transmitted pulse and the refected signal. Integral representations of these quantities of interest are given.

Section 3 is devoted to the study of the transmitted front. The basic diffusion-approximation result needed in this study is presented and applied to deduce the shape of the front. Generalizations to the three-dimensional layered case are indicated at the end of the section.

The reflected signal is studied in Section 4. We review the asymptotic analysis developed in [1], the limiting spectral densities and the system of transport equations used in the inverse problem.

In Section 5 we present recent results obtained by using time reversal mirror techniques. We show that sending back into the same medium a piece of the reflected signal after time reversal will produce a refocalization at the initial source in time and in space.

2. A One-Dimensional Model

2.1. ACOUSTIC WAVE EQUATIONS

We consider a simple one-dimensional model where the space variable is denoted by x and the density and the bulk modulus of the medium are respectively $\rho(x)$ and $K(x)$. We assume that these coefficients vary only in the slab $\{0 \le x \le L\}$ and remain constant outside. The velocity and the pressure will be denoted respectively by $u(x,t)$ and $p(x,t)$. The linearized acoustic equations are:

$$\left\{ \begin{array}{ll} \rho(x)\frac{\partial u}{\partial t}(x,t) + \frac{\partial p}{\partial x}(x,t) & = 0 \\ \frac{1}{K(x)}\frac{\partial p}{\partial t}(x,t) + \frac{\partial u}{\partial x}(x,t) & = 0, \end{array} \right. \tag{1}$$

with appropriate boundary conditions which will be added later on.

2.2. RANDOM FLUCTUATIONS OF THE COEFFICIENTS

We assume that the coefficients ρ and $1/K$ are made of macroscopic and microscopic variations. The macroscopic variations will be described by the smooth positive functions $\rho_0(x)$ and $1/K_0(x)$ and the microscopic random fluctuations will be described by the stationary centered stochastic processes $\eta(x/\varepsilon^2)$ and $\nu(x/\varepsilon^2)$. The "microscopic" correlation length of the medium is represented by the small parameter ε^2 (since we will need an intermediate scale it is convenient to denote this small scale by ε^2 instead of ε). More comments on the role of the small parameter ε will be given in Section 2.4.2. Our coefficients can be written as follows:

$$\rho(x) = \begin{cases} \rho_0(x)(1+\eta(x/\varepsilon^2)) & \text{for } 0 \leq x \leq L \\ 1 & \text{for } x < 0 \text{ or } x > L, \end{cases} \tag{2}$$

and

$$\frac{1}{K(x)} = \begin{cases} \frac{1}{K_0(x)}(1+\nu(x/\varepsilon^2)) & \text{for } 0 \leq x \leq L \\ 1 & \text{for } x < 0 \text{ or } x > L. \end{cases} \tag{3}$$

We assume that $|\eta|$ and $|\nu|$ are bounded by a constant C strictly smaller than 1 so that the density ρ and the bulk modulus K are bounded and positive. Denoting by $\mathbb{E}\{\cdots\}$ the ensemble average with respect to the invariant distribution of the random fluctuations, the correlation length of the medium is:

$$\int_0^\infty \mathbb{E}\{\eta(0)\eta(x/\varepsilon^2)\}dx = \varepsilon^2\alpha_\eta, \tag{4}$$

where α_η is the integrated covariance of the random fluctuations of ρ given by:

$$\alpha_\eta = \int_0^\infty \mathbb{E}\{\eta(0)\eta(x)\}dx, \tag{5}$$

and a similar expression for α_ν.

The fluctuations η and ν are not supposed to be small which means that we are in the "high-contrast" situation.

The macroscopic impedance $I_0(x)$ and sound velocity $C_0(x)$ are defined as follows:

$$I_0(x) = \sqrt{\rho_0(x)K_0(x)}, \tag{6}$$

and

$$C_0(x) = \sqrt{\frac{K_0(x)}{\rho_0(x)}}. \tag{7}$$

Finally we shall assume that the random processes η and ν have nice decorrelation (or mixing) properties which can be described as follows: let

x and y be two points such that $0 < x < y$ and denote by A an event which depends on $(\eta(z), \nu(z))$ only for $0 \leq z \leq x$ and by B an event which depends on $(\eta(z), \nu(z))$ only for $z \geq y$.

We assume that $|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| \to 0$ exponentially fast as $|y - x| \nearrow +\infty$ and uniformly in the events A and B.

2.3. MARKOVIAN MODELS

Markovian models are models which depend on an auxiliary Markov process taking its values in a state space S. These models are fairly general and are well-suited for explicit computations. To avoid technical difficulties we shall restrict ourself to compact state spaces S and even more than that to finite spaces. In the usual context of Markov processes the variable x represents time while in our situation it is space. We may still call it time when dealing with the theory of Markov processes which we summarize now.

We denote by $(q_x)_{x \geq 0}$ a Markov process with values in S and we assume that q is homogeneous which means that the probability to be in the state s' at time y given that q is in the state s at the earlier time x depends only on $y - x$. This probability will be denoted as follows:

$$\mathbb{P}(q_y = s'/q_x = s) = P_{y-x}(s, s'). \tag{8}$$

For functions $f : S \to \mathbb{R}$, $P_x f(s) = \mathbb{E}\{f(q_x)/q_0 = s\}$ defines a *semi-group* $(P_x)_{x \geq 0}$. Its *infinitesimal generator* Q is obtained as the limit of $x^{-1}(P_x - Id)$ as $x \downarrow 0$. For an infinitesimal time Δx one can write:

$$\Delta x Q(s, s') = \mathbb{P}\{\text{jump from } s \text{ to } s' \text{ in time } \Delta x\},$$

and

$$1 + \Delta x Q(s, s) = \mathbb{P}\{\text{no jump from } s \text{ in time } \Delta x\},$$

since $Q(s, s) = -\sum_{s'=s}$. In the simplest case of two states, $S = \{s_1, s_2\}$, the infinitesimal generator Q is the 2×2 matrix

$$Q = \begin{pmatrix} -\lambda & +\lambda \\ +\lambda & -\lambda \end{pmatrix},$$

where $\lambda > 0$ is the jump rate. In that case we have for an infinitesimal time Δx:

$$\begin{aligned} \mathbb{P}\{\text{jump from } s_1 \text{ to } s_2 \text{ in time } \Delta x\} &= \\ \mathbb{P}\{\text{jump from } s_2 \text{ to } s_1 \text{ in time } \Delta x\} &= \lambda \Delta x, \end{aligned}$$

and also

$$1 - \lambda \Delta x = \mathbb{P}\{\text{no jump from } s_1 \text{ (resp. } s_2) \text{ in time } \Delta x\}.$$

The *invariant*, or *equilibrium*, distribution is the unique probability distribution $\bar{\mathbb{P}}$ on S such that if the initial distribution of q_0 is $\bar{\mathbb{P}}$ then the distribution of q_x at later times $x > 0$ remains equal to $\bar{\mathbb{P}}$. $\bar{\mathbb{P}}$ satisfies the equilibrium equation $\bar{\mathbb{P}}Q = 0$ and it can be shown that $P_x f(s)$ converges exponentially fast to $\bar{\mathbb{P}}f = \int f d\bar{\mathbb{P}}$ as $x \nearrow \infty$ for every initial state s.

In the two states case $\bar{\mathbb{P}}$ is simply $(1/2, 1/2)$. The jump times are Poisson distributed: that is the differences between consecutive jumps are independent and exponentially distributed with the probability density function $\lambda \exp(-\lambda x)$, $x > 0$, λ being the jump rate. Taking $s_1 = s$ and $s_2 = -s$ for some s such that $|s| < 1$, one can construct a fluctuation process by setting $\eta(x) = q_x$, initially distributed according to the invariant probability $\bar{\mathbb{P}} = (1/2, 1/2)$. This process can be represented as follows: let $(X_1, X_2, \ldots)$ be a sequence of independent identically distributed exponential(λ) random variables. $\eta(0)$ being uniformly distributed over $\{-s, s\}$, $\eta(x)$ remains constant until X_1, the first jump time, it then jumps to the other value and remains constant until $X_1 + X_2$, the second jump time; it jumps like that between s and $-s$ at the successive jump times $X_1 + \cdots + X_k$ as in the following Figure:

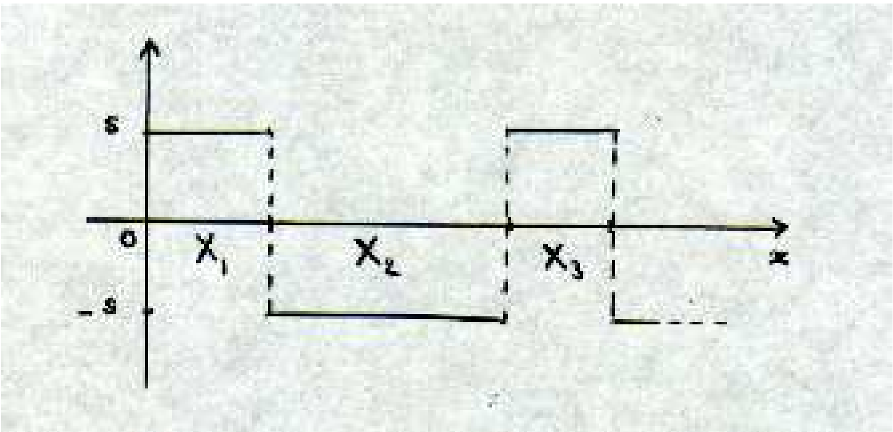

Figure 1. One realization of a two-state Markov process.

The fluctuation process $\eta(x/\varepsilon^2)$ is simply obtained by replacing the X_i's by $\varepsilon^2 X_i$. A simple exercise shows that α_η given by (5) is equal to $s^2/2\lambda$.

More generally the fluctuation processes η and ν will be obtained by setting $\eta(x/\varepsilon^2) = \eta(q_{x/\varepsilon^2})$ and $\nu(x/\varepsilon^2) = \nu(q_{x/\varepsilon^2})$ where q is a Markov process on S under its invariant distribution $\mathbb{P}$ and η and ν are two real functions defined on S such that $|\eta|, |\nu| \leq C < 1$. We assume that η and ν are centered: $\mathbb{E}\{\eta(x/\varepsilon^2)\} = \int \eta d\mathbb{P} = 0$ and $\mathbb{E}\{\nu(x/\varepsilon^2)\} = \int \nu d\mathbb{P} = 0$.

2.4. BOUNDARY CONDITIONS

2.4.1. *Right and left going waves*

In order to provide (1) with appropriate boundary conditions we introduce the right and left going waves A and B as follows:

$$\left\{ \begin{array}{ll} A & = I_0^{-\frac{1}{2}} p + I_0^{\frac{1}{2}} u \\ B & = -I_0^{-\frac{1}{2}} p + I_0^{\frac{1}{2}} u. \end{array} \right. \tag{9}$$

They satisfy the following equations:

$$\begin{aligned} \frac{\partial}{\partial x}\begin{pmatrix} A \\ B \end{pmatrix} = -\frac{1}{C_0(x)}\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} m^\varepsilon(x) & n^\varepsilon(x) \\ -n^\varepsilon(x) & -m^\varepsilon(x) \end{pmatrix}\right]\frac{\partial}{\partial t}\begin{pmatrix} A \\ B \end{pmatrix} \\ + \frac{d}{dx}\left(\log\sqrt{I_0(x)}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}, \end{aligned} \tag{10}$$

where we have defined:

$$\begin{array}{ll} m^\varepsilon(x) & = \frac{1}{2}(\eta(x/\varepsilon^2) + \nu(x/\varepsilon^2)), \\ n^\varepsilon(x) & = \frac{1}{2}(\eta(x/\varepsilon^2) - \nu(x/\varepsilon^2)). \end{array} \tag{11}$$

For instance in the particular case where only $1/K(x)$ contains random fluctuations we have $\eta = 0$ and:

$$\begin{pmatrix} m^\varepsilon(x) & n^\varepsilon(x) \\ -n^\varepsilon(x) & -m^\varepsilon(x) \end{pmatrix} = \frac{1}{2}\nu(x/\varepsilon^2)\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}. \tag{12}$$

The boundary conditions we impose on A and B correspond to a two-point boundary value problem for equation (10).

At the right end of the slab we impose a radiation condition so that nothing is coming from $+\infty$:

$$B(L,t) = 0 \quad \text{for every } t \quad \text{(radiation condition)}. \tag{13}$$

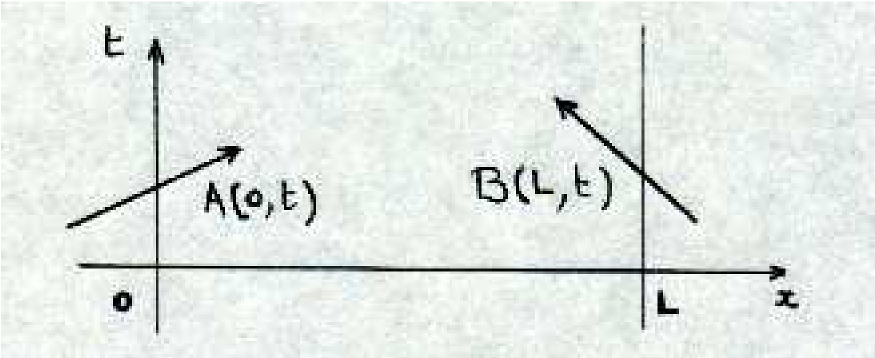

Figure 2. Waves entering the slab.

At the left end of the slab we have an incoming right going wave corresponding to a pulse entering the slab at $x = 0$. The shape of this pulse will be given by a smooth compactly supported function f. It is rescaled such that the typical wavelength is small of order ε and a factor $\varepsilon^{-\gamma}$ is introduced in order to rescale also the amplitudes:

$$A(0,t) = \varepsilon^{-\gamma} f(t/\varepsilon) \quad \text{(scaled pulse).} \tag{14}$$

The equations being linear, the multiplicative factor $\varepsilon^{-\gamma}$ plays no particular role for a constant ε. The case $\gamma = 0$ corresponds to constant amplitudes while the case $\gamma = \frac{1}{2}$ corresponds to a constant energy entering the slab since in that case $\int |A(0,t)|^2 dt = \int |f(t)|^2 dt$ which is independent of ε.

2.4.2. *Separation of scales*

We shall perform an asymptotic analysis of the problem as $\varepsilon \downarrow 0$ in the **regime of separation of scales** which can be summarized as follows:

$$\begin{pmatrix} \text{correlation length} \\ \approx \ \ \varepsilon^2 \end{pmatrix} \ll \begin{pmatrix} \text{wavelength} \\ \approx \ \ \varepsilon \end{pmatrix} \ll \begin{pmatrix} \text{distance of propagation} \\ \approx \ \ 1 \end{pmatrix}. \tag{15}$$

We would like to insist here on the role of the small parameter ε which is only a convenient mathematical tool to embed our problem in a family of similar problems (indexed by ε) for which we can perform an asymptotic analysis and use the limit as an approximation for the original problem. In practical situations the following parameters will be given:

$$\begin{cases} l: & \text{correlation length,} \\ L: & \text{distance of propagation} \approx 1, \\ \lambda: & \text{wavelength.} \end{cases}$$

where we use a unit of length of the order of the typical distances of propagation. Recall that we are in the high-contrast regime which means that the fluctuations are not supposed to be weak.

If the following condition is satisfied:

$$\frac{\lambda}{L} \approx \frac{l}{\lambda} \ll 1,$$

one can define a small parameter ε by

$$\frac{\lambda}{L} \approx \frac{l}{\lambda} \approx \varepsilon,$$

which implies $\lambda \approx \varepsilon$ and $l \approx \varepsilon^2$ since $L \approx 1$. In our model this is achieved by taking a pulse of the form $f(t/\varepsilon)$ and fluctuations of the form $\eta(x/\varepsilon^2)$. For instance, in the context of Geophysics, typical frequencies are 20Hz, the sound speed being about 2km/s, we get λ = 100m= 10^{-1}km. The correlation length being like 2-3m, our condition will be satisfied for L's of order a few kilometers and ε will be like 10^{-2} which is small.

2.4.3. *Simulations*

Figure (3) shows a simulation which we borrowed from [1], Figure 1.1a (we like to thank the authors of this review paper here). The bottom part shows one realization of the one-dimensional velocity profile $C_0(x)$ with its macroscopic variation (on the scale of kilometers in the context of Geophysics) and the random fluctuations with a small correlation length (of order a few meters) as described in (3), η, the random fluctuations in the density $\rho(x)$, being 0 in this model. The top part shows the propagation of a pulse into this medium. The width of the pulse is chosen such that we are in the regime of separation of scales (15) which is of the order of a hundred meters. One can observe the spreading of the front of the pulse which moves with the mean speed and the formation of a *coda* due to multiple scattering. There is no attenuation in this model.

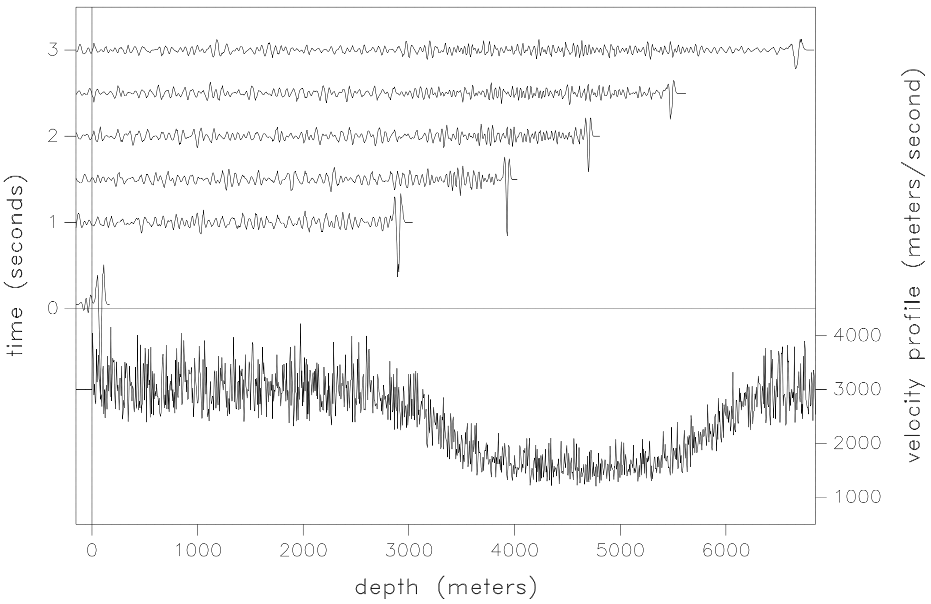

Figure 3. Velocity profile (bottom) - transmitted pulse (top)

Figure (4) (also borrowed from [1], Figure 1.1b) shows the reflected signal at the surface $x = 0$ as a function of the travel time which will be defined in the following paragraph. The bottom part shows the reflected signal when the fluctuations in the sound speed are removed (the scale is on the right). The top part shows the very fluctuating reflected signal when random fluctuations are present (the scale is on the left). One can observe the small signal to noise ratio (of order 10^{-3}) and guess that extracting information from the noisy reflected signal is not easy.

The goal of this paper is to present recent complements to the theory elaborated in [4], [3], [2], [16] and presented in the review paper [1].

2.5. QUANTITIES OF INTEREST: TRANSMISSION AND REFLECTION

2.5.1. *Transmission*

The transmitted signal is the right going wave at $x = L$, namely $A(L, t)$. Defining the *travel time* in the deterministic macroscopic medium by

$$\tau(x) = \int_0^x \frac{dy}{C_0(y)}, \tag{16}$$

our quantity of interest is the pulse front observed around the arrival time

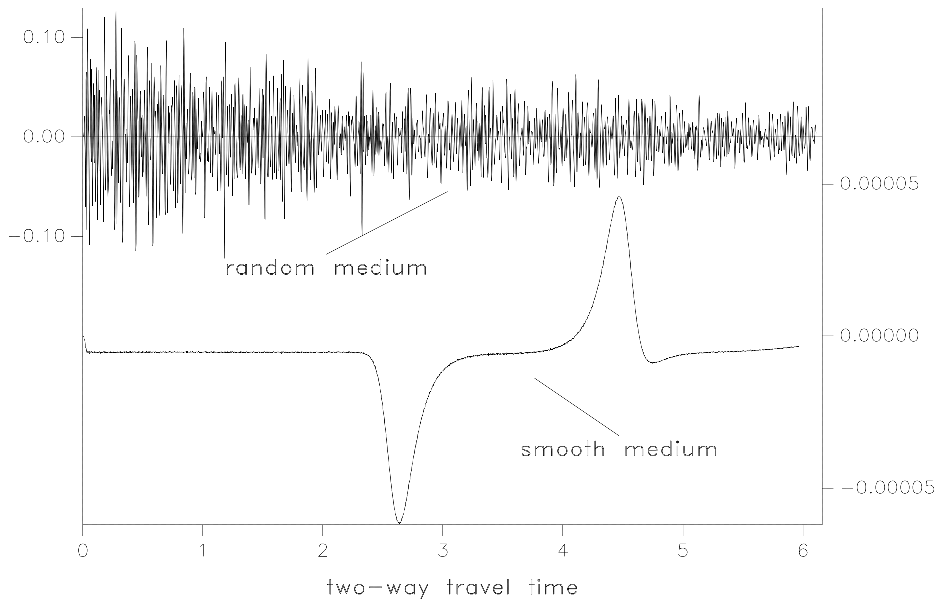

Figure 4. Reflected signal with and without noise

$\tau(L)$ in the original scale ε of the initial pulse (14). This consists in introducing a new scaled time variable σ and setting $t = \tau(L)+\varepsilon\sigma$. The quantity to be studied is

$$a^\varepsilon(x,\sigma) = A(x, \tau(x) + \varepsilon\sigma), \tag{17}$$

and the pulse front is $a^\varepsilon(L,\sigma)$.

2.5.2. *Reflection*

The reflected signal is the left going wave at $x = 0$, namely $B(0,t)$. This signal can be observed locally by opening a time window centered at time t_0 and in the scale ε. The quantity of interest is then $B(0, t_0 + \varepsilon\sigma)$. The quantity to be studied is

$$b^\varepsilon(x,\sigma) = B(x, -\tau(x) + \varepsilon\sigma), \tag{18}$$

and the reflected signal becomes $b^\varepsilon(0, \frac{t_0}{\varepsilon} + \sigma)$.

The wave functions a^ε and b^ε are A and B along the characteristics $t = \tau(x)$ and $t = -\tau(x)$ in the scale ε in time.

2.5.3. *Integral representations*

The medium being time-independent it is natural to perform a Fourier transform in time. We define $\hat{a^\varepsilon}(x,\omega) = \int e^{i\omega\sigma} a^\varepsilon(x,\sigma)d\sigma$ and similarly for

$\hat{b^\varepsilon}(x,\omega)$. Taking a Fourier transform of equations (9) and using the definitions (17) and (18) we get

$$\frac{d}{dx}\left(\begin{array}{c}\hat{a^\varepsilon}\\ \hat{b^\varepsilon}\end{array}\right) = \left(\frac{1}{\varepsilon}P^\varepsilon(x,\omega) + Q^\varepsilon(x,\omega)\right)\left(\begin{array}{c}\hat{a^\varepsilon}\\ \hat{b^\varepsilon}\end{array}\right), \tag{19}$$

where

$$P^\varepsilon(x,\omega) = \frac{i\omega}{C_0(x)}\left(\begin{array}{cc} m^\varepsilon(x) & n^\varepsilon(x)e^{-2i\omega\frac{\tau(x)}{\varepsilon}} \\ -n^\varepsilon(x)e^{2i\omega\frac{\tau(x)}{\varepsilon}} & -m^\varepsilon(x)\end{array}\right), \tag{20}$$

and

$$Q^\varepsilon(x,\omega) = \frac{d}{dx}\left(\log\sqrt{I_0(x)}\right)\left(\begin{array}{cc} 0 & e^{-2i\omega\frac{\tau(x)}{\varepsilon}} \\ e^{2i\omega\frac{\tau(x)}{\varepsilon}} & 0\end{array}\right). \tag{21}$$

The boundary conditions for (19) are simply

$$\hat{a^\varepsilon}(0,\omega) = \varepsilon^{-\gamma}\hat{f}(\omega) \ , \ \hat{b^\varepsilon}(L,\omega) = 0. \tag{22}$$

Denoting by $(\hat{a_1^\varepsilon}, \hat{b_1^\varepsilon})$ the solution of (19) where the first boundary condition is replaced by $\hat{a^\varepsilon}(0,\omega) = 1$ in (22), $\varepsilon^{-\gamma}\hat{f}(\omega)$ being a simple multiplicative factor we obtain the following integral representations for our quantities of interest:

$$A(L,\tau(L)+\varepsilon\sigma) = a^\varepsilon(L,\sigma) = \frac{\varepsilon^{-\gamma}}{2\pi}\int e^{-i\omega\sigma}\hat{f}(\omega)\hat{a_1^\varepsilon}(L,\omega)d\omega, \tag{23}$$

and

$$B(0,t_0+\varepsilon\sigma) = b^\varepsilon(0,\frac{t_0}{\varepsilon}+\sigma) = \frac{\varepsilon^{-\gamma}}{2\pi}\int e^{-i\omega(\frac{t_0}{\varepsilon}+\sigma)}\hat{f}(\omega)\hat{b_1^\varepsilon}(0,\omega)d\omega. \tag{24}$$

2.5.4. *Hyperbolicity and the role of L*

By the asumptions we have made on the coefficients of equation (1) we know that there is a finite maximum speed of propagation C_{max}. Therefore, for any given time $T > 0$, the reflected signal before T, $(B(0,t), t \leq T)$, does not depend on L as soon as $L > C_{max}T$. It is interesting to notice that nevertheless $\hat{b_1^\varepsilon}(0,\omega)$ in (24) does depend on L. Of course the transmitted front depends on L.

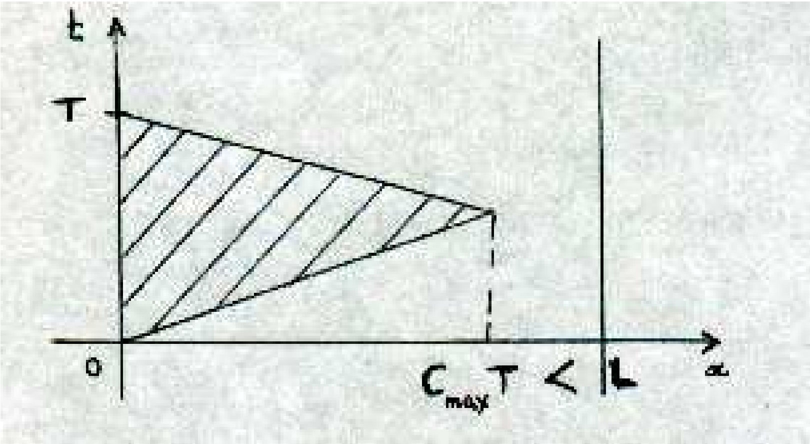

Figure 5. Hyperbolicity

3. The Transmitted Front

3.1. ONE-DIMENSIONAL MODEL

The transmitted front $A(L, \tau(L) + \varepsilon\sigma)$ has the integral representation (23) where $(\hat{a}_1^\varepsilon, \hat{b}_1^\varepsilon)$ is the solution of (19) with the bondary conditions $\hat{a}_1^\varepsilon(0,\omega) = 1$ and $\hat{b}_1^\varepsilon(L,\omega) = 0$. This is a linear two-point boundary value problem well centered and scaled in the sense that the random coefficients appearing in P^ε, such as $\eta(x/\varepsilon^2)$, are centered ($\mathbb{E}\{P^\varepsilon\} = 0$) and $\frac{1}{\varepsilon}\eta(x/\varepsilon^2)$ is approximately a white noise.

It is convenient to introduce the *propagator* for this problem. It is a complex 2×2 matrix $Y^\varepsilon(x,\omega)$ solution of (19) with the initial condition $Y^\varepsilon(0,\omega) = Id$. Noting that (α, β) is a solution of (19) implies that $(\bar{\beta}, \bar{\alpha})$ is also a solution one can show that the propagator can be written

$$Y^\varepsilon = \begin{pmatrix} \alpha & \bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}. \tag{25}$$

The trace of $\frac{1}{\varepsilon}P^\varepsilon + Q^\varepsilon$ being 0, the determinant of Y^ε is constant equal to 1. In other words $|\alpha|^2 - |\beta|^2 = 1$ for every x and ω. By the definition of the propagator and the fact that $\hat{a}_1^\varepsilon(0,\omega) = 1$ we have

$$\begin{pmatrix} \hat{a}_1^\varepsilon(x,\omega) \\ \hat{b}_1^\varepsilon(x,\omega) \end{pmatrix} = Y^\varepsilon(x,\omega) \begin{pmatrix} 1 \\ \hat{b}_1^\varepsilon(0,\omega) \end{pmatrix}, \tag{26}$$

which applied at $x = L$ gives

$$\hat{a_1^\varepsilon}(L,\omega) = \frac{1}{\bar{\alpha}(L,\omega)} \quad , \quad \hat{b_1^\varepsilon}(0,\omega) = -\frac{\beta(L,\omega)}{\bar{\alpha}(L,\omega)}. \tag{27}$$

The property $\det Y^\varepsilon = 1$ is the *conservation of energy* relation

$$|\hat{a_1^\varepsilon}(L,\omega)|^2 + |\hat{b_1^\varepsilon}(0,\omega)|^2 = 1 \tag{28}$$

which implies in particular that $\hat{a_1^\varepsilon}(L,\omega)$ appearing in (23) is uniformly bounded by 1.

3.2. CHARACTERIZATION OF THE FRONT

In the regime of constant amplitudes, $\gamma = 0$ in (23), the transmitted front admits the integral representation

$$a^\varepsilon(L,\sigma) = \frac{1}{2\pi}\int e^{-i\omega\sigma}\hat{f}(\omega)\hat{a_1^\varepsilon}(L,\omega)d\omega. \tag{29}$$

In order to characterize its limiting probability distribution as $\varepsilon \downarrow 0$ one needs to study the joint distribution of $\hat{a_1^\varepsilon}(L,\omega)$ simultaneously for all the ω's. This is a complicated infinite dimensional problem. One way to treat it is to study all the moments of the front and reduce it to an infinite number of finite dimensional problems by the following relation

$$\begin{aligned} \mathbb{E}\left\{a^\varepsilon(L,\sigma_1)^{p_1}\cdots a^\varepsilon(L,\sigma_k)^{p_k}\right\} = \left(\frac{1}{2\pi}\right)^p \int e^{-i\Sigma\sigma_j\omega_j^l}\prod\left(\hat{f}(\omega_j^l)\right) \\ \times\mathbb{E}\left\{\prod\left(\hat{a_1^\varepsilon}(L,\omega_j^l)\right)\right\}\prod(d\omega_j^l) \end{aligned} \tag{30}$$

where $p_1,\ldots,p_k$ are integers, $p = p_1 + \cdots + p_k$ and

$$\left(\omega_j^l;\ j = 1,\ldots,k;\ l := l_j = 1,\ldots,p_j\right)$$

forms a finite set of frequencies. The asymptotic limit of this moment requires to study the joint distribution of $\left\{\hat{a_1^\varepsilon}(L,\omega_1),\ldots,\hat{a_1^\varepsilon}(L,\omega_p)\right\}$ for p distinct frequencies $(\omega_1,\ldots,\omega_p)$. This amounts to the study of a system of p equations like (19) for which we define a $2p \times 2p$ propagator matrix $Y^\varepsilon(x,\omega_1,\ldots,\omega_p)$ obtained by puting the $Y^\varepsilon(x,\omega_j)$'s on the diagonal. This propagator is still the solution of an equation like (19) with

$P^\varepsilon(x, \omega_1, \ldots, \omega_p)$ (resp. $Q^\varepsilon(x, \omega_1, \ldots, \omega_p)$) obtained by puting the $P^\varepsilon(x, \omega_j)$'s (resp. $Q^\varepsilon(x, \omega_j)$'s) on the diagonal. The initial condition remains

$$Y^\varepsilon(0, \omega_1, \ldots, \omega_p) = Id$$

for the $2p \times 2p$ Identity matrix.

The limit as $\varepsilon \downarrow 0$ of the solution of equations of this type is described in the theory of "diffusion-approximation" that we summarize in the next paragraph.

3.3. A LIMIT THEOREM

3.3.1. *The main result*

Let (q_x) be a Markov process as in Section 2.3 and (Y_x^ε) defined as the unique $\mathbb{R}^d$-valued solution of the differential equation

$$\frac{dY_x^\varepsilon}{dx} = \frac{1}{\varepsilon} F\left(x, \frac{x}{\varepsilon}, q_{x/\varepsilon^2}, Y_x^\varepsilon\right) + G\left(x, \frac{x}{\varepsilon}, q_{x/\varepsilon^2}, Y_x^\varepsilon\right), \tag{31}$$

starting with the initial condition $Y_0^\varepsilon = y_0 \in \mathbb{R}^d$. F and G are smooth functions, periodic in the second variable (the fast phase $\frac{x}{\varepsilon}$), and F is centered in the sense that

$$\mathbb{E}\left\{F(x', \tau, q_x, y)\right\} = 0 \tag{32}$$

when the first, second and fourth variables are fixed and the expectation is with respect to the invariant distribution of the Markov process (q_x). Note that (19) is a linear equation of this form, the first variable x appearing in $C_0(x)$ and $I_0(x)$ and the second variable, the fast phase, appearing in the exponentials. q is the Markov process driving the fluctuations η and ν on the scale ε^2.

The **main result** of this Section asserts that the process $(Y_x^\varepsilon)_{x\geq 0}$ (which is not Markovian by itself) converges in probability distribution, as $\varepsilon \downarrow 0$, to the d-dimensional Markov diffusion process $(Y_x)_{x\geq 0}$ which admits the following infinitesimal generator:

$$\begin{aligned} L_x = \int_0^\infty \langle \mathbb{E}\left\{F(x, \tau, q_0, y) \cdot \nabla_y F(x, \tau, q_t, y) \cdot \nabla_y\right\}\rangle_\tau dt \\ + \langle \mathbb{E}\left\{G(x, \tau, q_0, y)\right\}\rangle_\tau, \end{aligned} \tag{33}$$

where $\langle \cdots \rangle_\tau$ denotes an average over a period with respect to the variable τ.

3.3.2. *A simple case*

In order to give an idea of how a formula like (33) is derived, let us consider the simple case where F only depends on q and y and $G = 0$. Equation (31) reduces to

$$\frac{dY_x^\varepsilon}{dx} = \frac{1}{\varepsilon} F\left(q_{x/\varepsilon^2}, Y_x^\varepsilon\right), \tag{34}$$

where $Y_0^\varepsilon = y_0 \in \mathbb{R}^d$ and $\mathbb{E}\left\{F(q_x, y)\right\} = 0$.

Denoting q_{x/ε^2} by q_x^ε, the pair $(q_x^\varepsilon, Y_x^\varepsilon)$ is a Markov process on $S \times \mathbb{R}^d$ which admits the infinitesimal generator

$$L^\varepsilon = \frac{1}{\varepsilon^2} Q + \frac{1}{\varepsilon} F(q, y) \cdot \nabla_y, \tag{35}$$

easily obtained by taking a limit as $x \downarrow 0$ in

$$\frac{1}{x}\left(\mathbb{E}\{f(q_x^\varepsilon, Y_x^\varepsilon)/q_0 = q, Y_0^\varepsilon = y_0\} - f(q, y)\right),$$

for any smooth function f and Q being the infinitesimal generator of the Markov process (q_x) as described in Section 2.3.

For a given real smooth function f we define a perturbed test function f^ε by

$$f^\varepsilon(q, y) = f(y) + \varepsilon f_1(q, y) + \varepsilon^2 f_2(q, y), \tag{36}$$

for functions f_1 and f_2 to be determined as follows: find an infinitesimal generator L such that $|L^\varepsilon f^\varepsilon - Lf|$ goes to 0 as $\varepsilon \downarrow 0$. When computing $L^\varepsilon f^\varepsilon$ there is no $\mathcal{O}(1/\varepsilon^2)$ term since Q in (35) acts on the variable q and f is independent of q. The condition for the $\mathcal{O}(1/\varepsilon)$ term to vanish is

$$Qf_1 + F \cdot \nabla_y f = 0, \tag{37}$$

which is a Poisson equation which admits the solution $f_1 = (-Q)^{-1}(F \cdot \nabla_y f)$ since F is centered with respect to the invariant distribution of the Markov process (q_x). This solution can also be written

$$f_1(q, y) = \int_0^\infty \mathbb{E}\{F(q_t, y) \cdot \nabla_y f/q_0 = q\} dt. \tag{38}$$

The $\mathcal{O}(1)$ term is equal to $Qf_2 + F \cdot \nabla_y f_1$. By substracting and adding $\mathbb{E}\{F \cdot \nabla_y f_1\}$, f_2 can be chosen such that

$$Qf_2 + F \cdot \nabla_y f_1 - \mathbb{E}\{F \cdot \nabla_y f_1\} = 0,$$

since this is again a Poisson equation in Q with a centered second term. It follows that the $\mathcal{O}(1)$ term is reduced to $\mathbb{E}\{F \cdot \nabla_y f_1\}$ which is the desired infinitesimal generator L which, acting on f, can also be written

$$Lf = \int_0^\infty \mathbb{E}\{F(q_0, y) \cdot \nabla_y (F(q_t, y) \cdot \nabla_y f)\} dt \tag{39}$$

which is formula (33) in this simple case. The averaging with respect to the variable τ in the general case (33) is done after the averaging with respect to q since $\tau = x/\varepsilon$ is a fast variable but slower than x/ε^2.

In the simple linear case

$$\frac{dY_x^\varepsilon}{dx} = \frac{1}{\varepsilon} \eta(x/\varepsilon^2) F(Y_x^\varepsilon),$$

the infinitesimal generator L reduces to the simple second order operator

$$L = \alpha_\eta F(y) \cdot \nabla_y (F(y) \cdot \nabla_y)$$

where α_η is defined in (5).

In general the limiting process (Y_x) is a Markov diffusion with infinitesimal generator given by (33). This process can be represented as solution to a *stochastic differential equation* involving *Brownian motions*. Ito's stochastic calculus can be used in the computation of expressions such as $\mathbb{E}\{\phi(Y_x)\}$ which will approximate $\mathbb{E}\{\phi(Y_x^\varepsilon\}$ for ε small.

3.4. APPLICATION TO THE FRONT SHAPE

The expectation appearing in (30) can be written as the expectation of a functional ϕ of the propagator:

$$\mathbb{E}\left\{\prod \left(\hat{a_1^\varepsilon}(L, \omega_j^l)\right)\right\} = \mathbb{E}\left\{\phi\left(Y_L^\varepsilon(\omega_1, \ldots, \omega_p)\right)\right\},$$

which will be approximated, for ε small, by $\mathbb{E}\left\{\phi\left(Y_L(\omega_1, \ldots, \omega_p)\right)\right\}$ as explained in the previous Section. In this case the multi-frequency limiting propagator $Y_x(\omega_1, \ldots, \omega_p)$ is obtained as $Y_x^\varepsilon(\omega_1, \ldots, \omega_p)$ in Section 3.2 by putting the $Y_x(\omega_j)$'s on the diagonal where each $Y_x(\omega_j)$ can be written as in (25). Using (27) we see that the functional ϕ is given by

$$\phi(Y(\omega_1, \ldots, \omega_p)) = \prod_{j=1}^p \left(\frac{1}{\bar{\alpha}(\omega_j)}\right).$$

The use of Ito's stochastic calculus enables us to derive explicit formulas for expectations of such functionals. We refer to [6] for details where it is shown that $a^\varepsilon(L,\sigma)$ converges in probability distribution as $\varepsilon \downarrow 0$ to $a(L,\sigma)$ which admits the following integral representation similar to (29)

$$a(L,\sigma) = \frac{1}{2\pi}\int e^{-i\omega\sigma}\hat{f}(\omega)\hat{a_1}(L,\omega)d\omega, \tag{40}$$

where

$$\hat{a_1}(L,\omega) = \exp\left(i\omega Z_L - \omega^2\alpha_n\int_0^L \frac{dy}{C_0(y)^2}\right), \tag{41}$$

and Z_L is a centered Gaussian random variable with variance $2\alpha_m\int_0^L \frac{dy}{C_0(y)^2}$. The coefficients α_n and α_m are associated by (5) to m and n given by (11). From (40) and (41) one can easily deduce

$$a(L,\omega) = (f \star G_L)(\sigma + Z_L), \tag{42}$$

where G_L is the centered Gaussian density with variance $2\alpha_n\int_0^L \frac{dy}{C_0(y)^2}$.

The result (42) tells us that the transmitted front, in the same scale as the orinal pulse, has approximately a deterministic shape obtained by a convolution of the original pulse f with a Gaussian kernel G_L but shifted by a Gaussian random variable Z_L. This *stabilization effect* , the deterministic shape, has also been obtained independently in [14]. As observed in [6], this is the only coherent energy exiting the slab in the sense that $A(L,t+\varepsilon\sigma)$ for $t \neq \tau(L)$ or $B(0,t+\varepsilon\sigma)$ for any t, go to 0 as $\varepsilon \downarrow 0$ in this regime of constant amplitudes $\gamma = 0$ in (14).

3.5. THE 3-D LAYERED CASE

We briefly summarize here the result obtained in [5]. The space is now three-dimensional and we assume that the coefficients of the acoustic equations depend only on the "depth", the vertical direction z, the transverse directions being denoted by x and y. We assume that a point source is placed at the surface $z = 0$ and that the pressure is observed at the bottom of the random slab $z = -L$ as shown in Figure (6). We suppose that only the bulk modulus contains random fluctuations so that we have the same description of the coefficients than in (2) and (3) with $0 \leq x \leq L$ replaced by $-L \leq z \leq 0$ and η being set to 0.

Using the specific Fourier transform

$$\hat{\phi}(\omega,h,k) = \int\int\int e^{iw(t-hx-ky)}\phi(t,x,y)dtdxdy \tag{43}$$

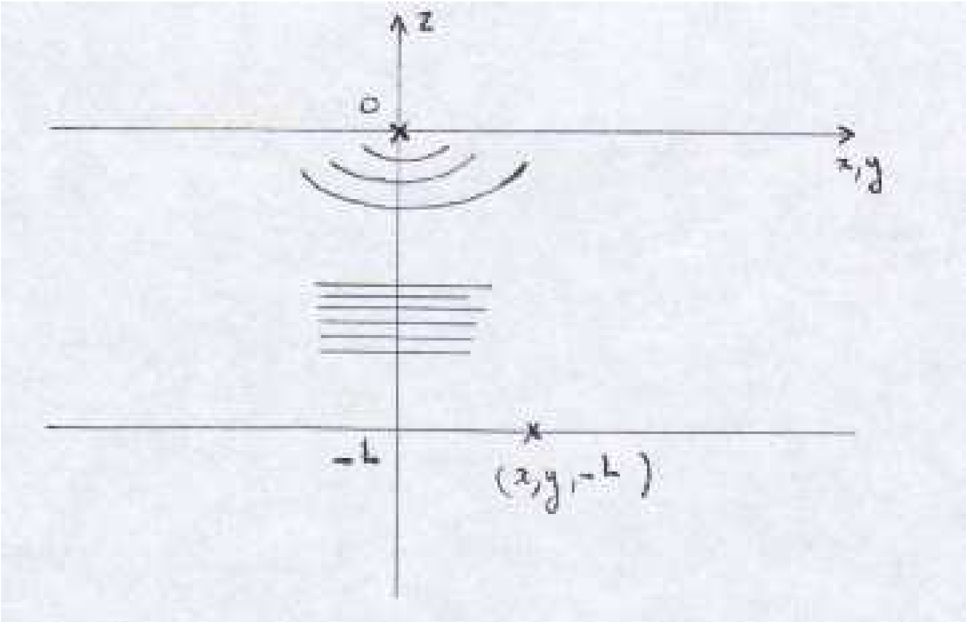

Figure 6. A point source over a 3-D randomly stratified medium.

and defining the slowness $\kappa = \sqrt{h^2 + k^2}$, the travel time

$$\tau(z, \kappa) = \int_o^z \frac{\sqrt{(1 - C_0(s)^2\kappa^2)}}{C_0(s)} ds,$$

and the impedance

$$I_0(z, \kappa) = \frac{\rho_0(z)C_0(z)}{\sqrt{1 - C_0(s)^2\kappa^2)}},$$

it is possible to obtained an integral representation similar to (23) for the transmitted pressure $p(t_0 + \varepsilon\sigma, x, y, -L)$ in the ε-scale of the initial pulse, around a time t_0, at the bottom $z = -L$ and at the offset (x, y). This integral representation in very similar to (23) but involves also a fast phase $e^{i\phi/\varepsilon}$.

In [5] it is shown how to apply simultaneously a diffusion-approximation result as in Section 3.3 and a *stationary phase* argument to derive the asymptotic transmitted pressure field.

The position $(x, y, -L)$ being given, we solve $\nabla\phi = 0$ to get the values (h_0, k_0) where the integral concentrate. Solving $\phi = 0$ for these values one can deduce the right time t_0 where the front arrives at this given position. For this value t_0 and jointly in (x, y) , the transmitted pressure $p(t_0 +$

$\varepsilon\sigma, x, y, -L)$ converges in probability distribution to

$$\beta(x, y, -L)\,(f \star G'_L(x, y))\,(\sigma + \alpha(x, y)Z_L). \tag{44}$$

The coefficients β and α are described in details in [5], β being the coefficient obtained in the geometric acoustic theory. We simply remark that, like in the one-dimensional case (42), the stabilization of the front is still there but now the convolution is with the derivative of a Gaussian kernel. It is also noticable that only one Gaussian random variable Z_L is needed to described the pressure field for all the offsets (x, y).

4. Reflected Signals

4.1. THE REFLECTION COEFFICIENT

Reformulating in the slab $[x, L]$, with the left extremity varying, the one-dimensional problem described in Section 3.1 , we define a reflection coefficient $R^\varepsilon(x, \omega)$ and a transmission coefficient $T^\varepsilon(x, \omega)$ corresponding respectively to $\hat{b}_1^\varepsilon$ and $\hat{a}_1^\varepsilon$ defined in Section 2.5.3.

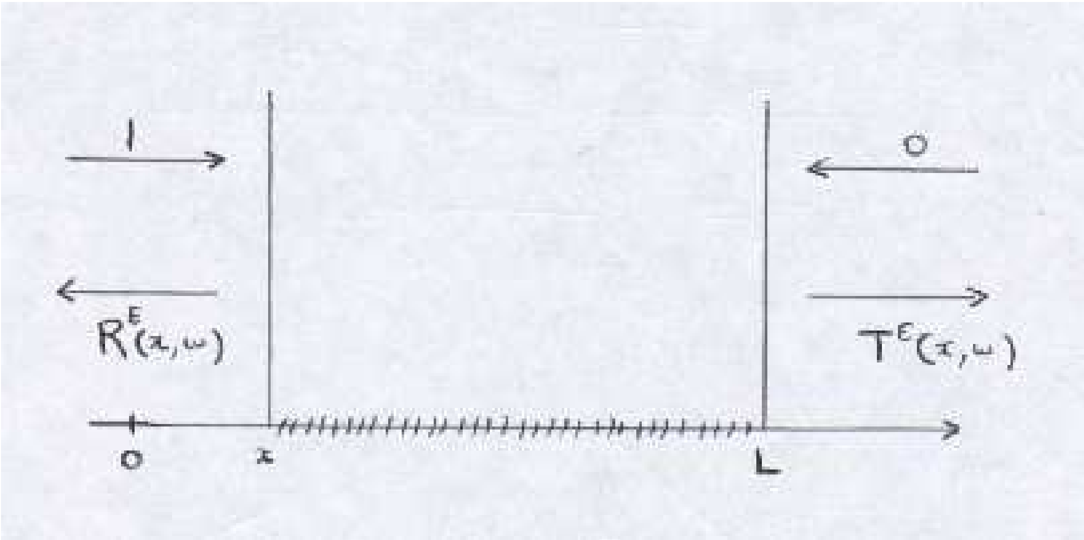

Figure 7. Relection coefficient; invariant embedding.

This technique is known as *invariant embedding*. The conservation of energy relation (28) becomes

$$|R^\varepsilon|^2 + |T^\varepsilon|^2 = 1. \tag{45}$$

It turns out that the reflection coefficient R^ε satisfies a *Riccati equation* which we write in the case $C_0(x) = 1$ and $\nu = 0$ for simplicity

$$\frac{dR^\varepsilon}{dx} = \frac{i\omega}{2\varepsilon}\eta(x/\varepsilon^2)\left(e^{2i\omega\frac{x}{\varepsilon}} + 2R^\varepsilon + (R^\varepsilon)^2 e^{-2i\omega\frac{x}{\varepsilon}}\right), \tag{46}$$

with the final condition $R^\varepsilon(L,\omega) = 0$. At $x = 0$ we have $R^\varepsilon(0,\omega) = \hat{b}_1^\varepsilon(0,\omega)$ and the integral representation for the reflected signal

$$B(0, t_0 + \varepsilon\sigma) = \varepsilon^{-\gamma}\frac{1}{2\pi}\int e^{-i\omega\sigma}e^{-i\omega\frac{t_0}{\varepsilon}}\hat{f}(\omega)R^\varepsilon(0,\omega)d\omega. \tag{47}$$

For $\gamma = \frac{1}{2}$, in the regime of constant energy, it has been shown that the stochastic signal $\{B(0, t_0 + \varepsilon\sigma)\}_{\sigma\in\mathbb{R}}$ converges in probability distribution to a stationary centered Gaussian process whose covariance is described in the following Section (we refer to [1] for this result). That is in this sense that the reflected signal $B(0,t)$ is locally stationary.

4.2. SPECTRAL DENSITY

4.2.1. *Covariance*

The covariance of the reflected signal in the ε scale, around the time t and for the pulse f is given by

$$C_{t,f}^\varepsilon(\sigma) = \mathbb{E}\left\{\overline{B^\varepsilon(0,t)}B^\varepsilon(0,t+\varepsilon\sigma)\right\}, \tag{48}$$

which, by using the integral representation (47), can be written

$$\varepsilon^{-2\gamma}\frac{1}{4\pi^2}\int\int e^{-i\omega_2\sigma}e^{i(\omega_1-\omega_2)\frac{t}{\varepsilon}}\overline{\hat{f}}(\omega_1)\hat{f}(\omega_2)\mathbb{E}\left\{\overline{R^\varepsilon(0,\omega_1)}R^\varepsilon(0,\omega_2)\right\}d\omega_1 d\omega_2. \tag{49}$$

To compensate the effect of the fast phase we concentrate this integral on nearby frequencies by the change of variables $\omega_1 - \omega_2 = \varepsilon h$ and $\omega_2 = \omega$. The covariance (48) becomes

$$\varepsilon^{-2\gamma+1}\frac{1}{4\pi^2}\int\int e^{-i\omega_2\sigma}e^{ith}\overline{\hat{f}}(\omega+\varepsilon h)\hat{f}(\omega)\mathbb{E}\left\{\overline{R^\varepsilon(0,\omega+\varepsilon h)}R^\varepsilon(0,\omega)\right\}d\omega dh. \tag{50}$$

Denoting by $\Lambda^\varepsilon(\omega, t)$ the inverse Fourier transform with respect to h of

$\mathbb{E}\left\{\overline{R^\varepsilon(0,\omega+\varepsilon h)}R^\varepsilon(0,\omega)\right\}$ and using $\overline{\hat{f}}(\omega+\varepsilon h)\hat{f}(\omega) \approx |\hat{f}(\omega)|^2$, the covariance (48) can be written

$$\varepsilon^{-2\gamma+1}\frac{1}{2\pi}\int e^{-i\omega\sigma}|\hat{f}(\omega)|^2\Lambda^\varepsilon(\omega,t)d\omega. \tag{51}$$

Choosing $\gamma = \frac{1}{2}$, it can be proved that $\Lambda^\varepsilon(\omega,t)$ converges to a limit $\Lambda(\omega,t)$ as $\varepsilon \downarrow 0$ (see [1] for details). The covariance of the limiting centered stationary Gaussian process is then equal to

$$\frac{1}{2\pi}\int e^{-i\omega\sigma}|\hat{f}(\omega)|^2\Lambda(\omega,t)d\omega. \tag{52}$$

where $|\hat{f}(\omega)|^2\Lambda(\omega,t)$ is its spectral density.

4.2.2. *The uniform background case*
In the uniform background case, $C_0(x) = 1$, the limiting spectral density can be explicitely computed and is given by

$$\Lambda(\omega,t) = \frac{\alpha\omega^2}{4(1+\omega^2\alpha t)^2} \tag{53}$$

where $\alpha = \alpha_\eta$, defined in (5), in the case $\nu = 0$ for instance.

This explicit case is important for the validation of numerical studies.

4.2.3. *The general case*
In the general case, C_0 varying, the limiting spectral density is no longer explicitely known.

From the Riccati equation (46) for R^ε we can deduce an infinite system of equations for the quantities

$$\overline{R^\varepsilon(x,\omega+\varepsilon h)}^M R^\varepsilon(x,\omega)^N, \qquad M,N \geq 0 \quad \text{integers.}$$

It is then possible to study the limit, as $\varepsilon \searrow 0$, of the expectations of these quantities. It turns out that the diagonal terms obtained for $M = N$ form, in the limit, a closed system. Denoting these limiting expectations by

$$\widetilde{W}^N(x,h) = \lim_{\varepsilon\downarrow 0}\mathbb{E}\left\{\overline{R^\varepsilon(x,\omega+\varepsilon h)}^N R^\varepsilon(x,\omega)^N\right\},$$

their Fourier transforms in h, denoted by $\{W^N(x,t), N \geq 0\}$, satisfy a system of *transport equations* described in the following Section.

4.3. TRANSPORT EQUATIONS

It is shown in [1] that the $\{W^N(x,t), N \geq 0\}$ satisfy the following system of transport equations referred as the *W-equations*:

$$\frac{\partial W^N}{\partial x} - \frac{2N}{C_0(x)}\frac{\partial W^N}{\partial t} - \frac{2\omega^2 \alpha N^2}{C_0(x)^2}\left(W^{N+1} - 2W^N + W^{N-1}\right) = 0 \qquad (54)$$

with the condition $W^N|_{x=L} = \delta(t)\delta_{N,0}$ at $x = L$.

Estimating $\Lambda(\omega, t) = W^1(0,t)$ from the observed reflected signal, an inverse problem consists in reconstructing the sound speed $C_0(x)$ which appears in the coefficients of (54). We refer to [1] and [16] for a complete treatment of the inverse problem. We simply remark that it involves a statistical estimation of the spectral density $\Lambda(\omega, t)$. A windowed Fourier transform is used in [1] and a wavelet transform is proposed in [7]. In th next Section we summarize a method using a time reversal mirror technique and developped in [8].

5. Time Reversal Mirror Techniques

A time reversal mirror is, roughly speaking, a device which is capable of receiving an acoustic signal in time, keeping it in memory, and sending it back into the medium in the reversed direction of time. In the context of ultrasounds, time-reversal mirrors have been developed and their effects have been studied experimentally by Mathias Fink and his team at the Laboratoire Ondes et Acoustique (ESPCI-Paris). We refer to [9]. We shall exploit here the possibility of sending back only a part of the reflected signal after time reversal. The idea being that the content of the reflected random signal around the arrival time t is related in some way to the macroscopic properties of the medium at a depth corresponding to a travel time $t/2$. Sending back, in the time-independent medium, this portion of the signal, the new reflected signal should contain more information about the properties of the medium at this precise depth. It will turn out that this method enables us to have direct access to quantites such as local correlations of the reflected signal without any computation, these correlations being made by the medium itself and therefore very robust.

In the next Section, we present this idea in the simple one-dimensional model containing the main property of separation of scales described in Section 2.4.2.

We recall that, from (47), the reflected signal has the integral representation

$$B(0,t) = \varepsilon^{-\gamma} \frac{1}{2\pi} \int e^{-i\omega \frac{t}{\varepsilon}} \hat{f}(\omega) R^\varepsilon(0,\omega) d\omega. \tag{55}$$

We use a scaled time-window $G_\varepsilon(t) = G(t/\varepsilon^\rho)$ where G is anything like an indicator function $1_{(-M,M)}(t)$ or a Gaussian density. The reflected signal $B(0,t)$ is "cut" by $G_\varepsilon(t)$ around the time t_0 and time-reversed to obtain the new signal $B(0,t_0 - t)G_\varepsilon(t)$. In the ε-scale this new signal is

$$g(\sigma) = B(0, t_0 - \varepsilon\sigma) G(-\frac{\sigma}{\varepsilon^\rho}), \tag{56}$$

to be sent in the same medium.

In order to obtain an integral representation for the reflection from this new pulse, we need its Fourier transform which is given by the convolution

$$\hat{g}(\omega) = \varepsilon^{\rho-\gamma-1} \int e^{i\omega' \frac{t_0}{\varepsilon}} \overline{\hat{f}}(\omega') \overline{\hat{G}}((\omega - \omega')\varepsilon^{\rho-1}) d\omega'. \tag{57}$$

Using (47) with the pulse f replaced by the new pulse g we have the integral representation for the new reflected signal denoted by $S^\varepsilon_{t_0}$:

$$S^\varepsilon_{t_0}(\sigma) = \frac{1}{2\pi} \int e^{-i\omega\sigma} e^{-i\omega \frac{t_0}{\varepsilon}} \hat{g}(\omega) R^\varepsilon(0,\omega) d\omega. \tag{58}$$

Using (57) in (58) we get

$$\begin{aligned} S^\varepsilon_{t_0}(\sigma) = \varepsilon^{\rho-\gamma-1} \tfrac{1}{2\pi} \textstyle\int\int \quad & e^{-i\omega\sigma} e^{-i(\omega' - \omega)\frac{t_0}{\varepsilon}} \\ & \overline{\hat{f}}(\omega') \overline{R^\varepsilon(0,\omega')} R^\varepsilon(0,\omega) \overline{\hat{G}}((\omega - \omega')\varepsilon^{\rho-1}) d\omega' d\omega. \end{aligned} \tag{59}$$

As in Section 4.2.1, in order to compensate the fast phase, we perform the change of variables $\omega' - \omega = \varepsilon h$, keeping ω, to obtain

$$S^\varepsilon_{t_0}(\sigma) = \frac{1}{2\pi} \int e^{-i\omega\sigma} \widehat{S^\varepsilon_{t_0}}(\omega) d\omega, \tag{60}$$

with

$$\widehat{S^\varepsilon_{t_0}}(\omega) = \varepsilon^{\rho-\gamma} \int e^{iht_0} \overline{\hat{f}}(\omega + \varepsilon h) \overline{R^\varepsilon(0, \omega + \varepsilon h)} R^\varepsilon(0,\omega) \hat{G}(\varepsilon^\rho h) dh. \tag{61}$$

We are now ready to perform an asymptotic analysis as $\varepsilon \downarrow 0$.

5.2. ASYMPTOTICS

In the regime of constant amplitudes, $\gamma = 0$, we set $\rho = 0$ in (61). This means that the window $G_\varepsilon(t)$ is independent of ε but can be chosen small compared to 1.

The first moment $\mathbb{E}\{\widehat{S^\varepsilon_{t_0}}(\omega)\}$ involves quantities such as

$$\mathbb{E}\{\overline{R^\varepsilon(0,\omega+\varepsilon h)}R^\varepsilon(0,\omega)\}.$$

Applying the asymptotic analysis described in Section 4.2.1 one can show that

$$\lim_{\varepsilon\downarrow 0}\mathbb{E}\{\widehat{S^\varepsilon_{t_0}}(\omega)\} = 2\pi\overline{\hat{f}}(\omega)\,(\Lambda(\omega,t)\star G(t))\,(t_0)$$

Using second moments one can actually prove that $\hat{S^\varepsilon_{t_0}}(\omega)$ itself converges to this same deterministic limit. We refer to [8] for more details.

To summarize, the Fourier transform of the new reflected signal $S^\varepsilon_{t_0}(\sigma)$ leads to a consistent estimator of the spectral density $\Lambda(\omega,t)$, up to a convolution by the window $G(t)$.

We conclude this Section by two remarks.

This convergence has been obtained when the time-reversed reflected signal is observed at time t_0, the same time around which we have cut the original reflected signal. Let us suppose that it is observed at time $t_0+\Delta t_0$. Then a fast phase $e^{i\omega\Delta t_0/\varepsilon}$ appears in the integral (59) and kills it so that, for ε small, no new reflected signal is observed at time $t_0+\Delta t_0$. This is observed experimentally (see [9]).

If $\hat{f}(0) \neq 0$, which means that the pulse contains low frequencies, it is possible to use the low frequency limit

$$\lim_{\omega\downarrow 0}\frac{\Lambda(\omega,t)}{\omega^2}$$

in the inverse problem for $C_0(x)$ in (54).

5.3. BACK AND FORTH

Following an idea used in the context of target selection [10], it is possible to repeat the same procedure at another time t_1. The new pulse will be $S^\varepsilon_{t_0}$ whose Fourier transform $\widehat{S^\varepsilon_{t_0}}(\omega)$ is given by (61).

We now perform the following:

1. Send $S^\varepsilon_{t_0}$ back in the medium without time-reversal.
2. Get the reflected signal.
3. Cut it around time t_1 with the new window $H_\varepsilon(t) = H(t/\varepsilon^\delta)$.
4. Time-reverse it and send it back in the medium.
5. Get the new reflected signal denoted by $S^\varepsilon_{t_0,t_1}(\sigma)$.

Applying (61) with $\varepsilon^{-\gamma}\hat{f}$, t_0 and G_ε replaced respectively by $\widehat{S^\varepsilon_{t_0}}$, t_1 and H_ε we get

$$\widehat{S^\varepsilon_{t_0,t_1}}(\omega) = \varepsilon^\delta \int e^{iht_1}\overline{\widehat{S^\varepsilon_{t_0}}}(\omega+\varepsilon h)\overline{R^\varepsilon(0,\omega+\varepsilon h)}R^\varepsilon(0,\omega)\hat{H}(\varepsilon^\delta h)dh. \quad (62)$$

Replacing $\widehat{S^\varepsilon_{t_0}}(\omega+\varepsilon h)$ given in (61) we obtain

$$\begin{aligned}\widehat{S^\varepsilon_{t_0,t_1}}(\omega) = \quad & \varepsilon^{\delta+\rho-\gamma}\int\int e^{iht_1}e^{-ih't_0}\hat{f}(\omega+\varepsilon(h+h')) \\ & \times\left\{R^\varepsilon(0,\omega)\overline{R^\varepsilon(0,\omega+\varepsilon h)}^2 R^\varepsilon(0,\omega+\varepsilon(h+h'))\right\} \\ & \times\hat{G}(\varepsilon^\rho h')\hat{H}(\varepsilon^\delta h)dhdh'.\end{aligned} \quad (63)$$

In this integral three frequencies are involved which is not good for an asymptotics. Rewriting

$$e^{iht_1}e^{-ih't_0} = e^{-i(h+h')t_0}e^{it(t_0+t_1)},$$

fixing $t = t_0+t_1$ and integrating with respect to t_0 between 0 and t (instead of $-\infty$ and $+\infty$ by causality), we use

$$\int_0^t dt_0 e^{-i(h+h')t_0}\ldots = \int \delta_0(h+h')\ldots$$

to obtain

$$\begin{aligned}\int_0^t S^\varepsilon_{t_0,t-t_0}(\sigma)dt_0 = \quad & \varepsilon^{\delta+\rho-\gamma}\frac{1}{2\pi}\int e^{-i\omega\sigma}\hat{f}(\omega) \\ & \times\int e^{iht}\left\{R^\varepsilon(0,\omega)^2\overline{R^\varepsilon(0,\omega+\varepsilon h)}^2\right\}\hat{G}(\varepsilon^\rho h)\hat{H}(\varepsilon^\delta h)dhd\omega.\end{aligned} \quad (64)$$

Using $\gamma = \delta+\rho$ (with $\rho = 0$ and $\delta = \gamma$ for instance), the Fourier transform of $\int_0^t S^\varepsilon_{t_0,t-t_0}(\sigma)dt_0$ gives a consistent estimator, as $\varepsilon \downarrow 0$, for $W^2(0,t)$ which can be used in the transport equations (54) to improve the estimation of the coefficients in the inverse problem. This procedure can be obviously carried out to higher order $W^N(0,t)$.

5.4. THE 3-D STRATIFIED CASE

With the notations introduced in Section 3.5, the scaled reflected velocity can be written

$$u^\varepsilon(t_0+\varepsilon\sigma, x_0+\varepsilon\Delta x, y_0+\varepsilon\Delta y, 0)$$

where (x_0, y_0) is the offset center of observation, t_0 is the time window center and the last component is $z = 0$. A time-window of this reflected velocity will be used as a new pulse sent into the medium after time-reversal. The new pulse in the ε scale is

$$g(\sigma, \Delta x, \Delta y) = u^\varepsilon(t_0 - \varepsilon\sigma, x_0 + \varepsilon\Delta x, y_0 + \varepsilon\Delta y, 0) G_\varepsilon(-\sigma, \Delta x, \Delta y).$$

This new source will generates a new reflected velocity at the surface $u(t + \varepsilon\sigma, x + \varepsilon\Delta x, y + \varepsilon\Delta y, 0)$ for which an integral representation can be written. This representation involves the slowness (Section 3.5) and we refer to [11] for more details on this generalization of Section 2 to the three-dimensional case, as well as the asymptotic analysis as $\varepsilon \downarrow 0$.

The main result obtained in [11] says that this new reflected velocity is asymptotically 0 (for ε small) , unless $(x, y) = (-x_0, -y_0)$ and $t = t_0$. In other words we have refocusing at the right time and the right location.

6. Conclusion

We have presented in the simplest way, we believe, the technique of diffusion-approximation applied to waves propagating in randomly layered media. This has been done in the regime of separation of scales explained in Section 2.4.2. We have shown how to use this technique to study the shape of the transmitted front (Section 3) and the reflected signal (Section 4). In the last Section 5 we have presented recent results using time reversal mirror techniques.

References

1. Asch, M., Kohler, W., Papanicolaou, G., Postel, M., and White, B. (1991) Frequency content of randomly scattered signals, *SIAM Review* **33**, 519-626.
2. Asch, M., Papanicolaou, G., Postel, M., Sheng, P., and White, B. (1990) Frequency content of randomly scattered signals, Part I, *Wave Motion* **12**, 429-450.
3. Burridge, B., Papanicolaou, G., Sheng, P., and White, B. (1989) Probing a random medium with a pulse, *SIAM J. Appl. Math.* **49**, 582-607.
4. Burridge, B., Papanicolaou, G., and White, B. (1987) Statistics for pulse reflexion from a randomly layered medium, *SIAM J. Appl. Math.* **47** (1987, 146-168.
5. Chillan, J. and Fouque, J.P. (1998) Pressure fields generated by acoustical pulses propagating in randomly layered media, *SIAM J. Appl. Math.* **58**, 1532-1546.
6. Clouet, J.F. and Fouque, J.P. (1994) Spreading of a pulse travelling in random media, *Annals of Applied Probability* **4**, 1083-1097.
7. Clouet, J.F., Fouque, J.P., and Postel, M. (1995) Spectral analysis of ramdomly scattered signals using the wavelet transform, *Wave Motion* **22**, 145-170.
8. Clouet, J.F. and Fouque, J.P. (1997) A time-reversal method for an acoustical pulse propagating in randomly layered media, *Wave Motion* **25**, 361-368.
9. Fink, M. (1993) Time reversal mirrors, *J. Phys. D: Appl. Phys.* **26**, 1333-1350.
10. Fink, M. and Prada, (1994) Eigenmodes of the time reversal operator: a solution to selective focusing in multiple-target media, *Wave Motion* **20**, 151-163.
11. Fouque, J.P. and Ndzie, J. (in preparation) Refocusing of a time-reversed acoustical pulse propagating in randomly layered media.

12. Kohler, W., Papanicolaou, G., and White, B. (1996) Localization and Mode Conversion for Elastic Waves in Randomly Layered Media, *Wave Motion* **23**, 1-22 and 181-201.
13. Lewicki, P., Burridge, R., and De Hoop, M. (1996) Beyond effective medium theory: pulse stabilization for multimode wave propagation in high-contrast layered media, *SIAM J. Appl. Math.* **56**, 256-276.
14. Lewicki, P., Burridge, R., and Papanicolaou, G. (1994) Pulse stabilisation in a strongly heterogeneous layered medium, *Wave Motion* **20**, 177-195.
15. O'Doherty, R.F. and Anstey, N.A. (1971) Refections on amplitudes, *Geophysical Prospecting* **19**, 430-458.
16. Papanicolaou, G., Postel, M., Sheng, P., and White, B. (1990) Frequency content of randomly scattered signals, Part II, *Wave Motion* **12**, 527-549.
17. Solna, K. (1997) Stable spreading of acoustic pulses due to laminated microstructure, Ph.D. dissertation, Stanford University.