A time-reversal method for an acoustical pulse propagating in randomly layered media

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Abstract

In the recent years a considerable amount of mathematical work have been devoted to the study of reflected signals obtained by the propagation of pulses in randomly layered media. We refer to [1] for an extensive survey and applications to inverse problems. The analysis is based on separation of scales between the correlation scale of the inhomogeneities present in the medium, the typical wavelengths of the pulse and the macroscopic variations of the medium. On the other hand, in the context of ultrasounds, time-reversal mirrors have been developed and their effects have been studied experimentally by Mathias Fink and his team at the Laboratoire Ondes et Acoustique (ESPCI-Paris). We refer to [2]. Our goal is to present a mathematical analysis of a time-reversal method for analysing reflected signals in the model described in [1]. We restrict our analysis to the one-dimensional case, the three-dimensional layered case being the content of a forthcoming paper. It is noticeable that we do not introduce new mathematics in the problem but simply put together an already existing mathematical theory and a new device, the time-reversal mirror.

1 Introduction

A time-reversal mirror is, roughly speaking, a device which is capable of receiving an acoustic signal in time, keeping it in memory, and sending it back into the medium in the reversed direction of time. We shall exploit here the possibility of sending back only a part of the reflected signal after time-reversal. The idea being that the content of the reflected random signal around the arrival time t is related in some way to the macroscopic properties of the medium at a depth corresponding to a travel time $t/2$. Sending back, in the time-independent medium, this portion of the signal, the new reflected signal should contain more information about the properties of the medium at this precise depth. It will turn out that this method enables us to have direct access to quantites such as local correlations of the reflected signal without any computation, these correlations being made by the medium itself and therefore

very robust.
In the next section, we present a simple one-dimensional model containing the main property of separation of scales. The third section is devoted to the mathematical analysis of our method in the particular case of uniform background (i.e. without macroscopic variations).
In the fourth section, we extend our method to macroscopically varying media and we discuss the possibility to improve estimates needed in the inverse problem.

2 The model

We are not looking for generality and we restrict ourself to the most simple model containing the main features of the problem. Let us consider the wave equation in one dimension of space:

$$\frac{1}{c(x)^2}\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=0 \tag{1}$$

where $\frac{1}{c(x)^2}=\frac{1}{c_0(x)^2}(1+\eta(\frac{x}{\varepsilon^2}))$ for $0\leq x\leq L$, $c_0(x)$ being a smooth function matching constants on $x\leq 0$ and $x\geq L$; η is a random centered fluctuation, bounded by a constant strictly less than one; we assume η to be stationary with nice mixing properties; ε^2 is a very small scale corresponding to the correlation scale of the inhomogeneities present in the medium. Introducing the pressure p, equation (1) corresponds to the acoutic equations:

$$\left\{\begin{array}{ll}\rho(x)\frac{\partial u}{\partial t}(x,t)+\frac{\partial p}{\partial x}(x,t) & =0\\ \frac{1}{K(x)}\frac{\partial p}{\partial t}(x,t)+\frac{\partial u}{\partial x}(x,t) & =0\end{array}\right. \tag{2}$$

where we have set the bulk modulus constant equal to 1 and $\rho(x)=\frac{1}{c(x)^2}$. We could have also considered a randomly varying bulk modulus and introduced a linear dissipation term like in [3]. The boundary conditions are expressed in terms of the right and left going waves:

$$\left\{\begin{array}{ll}A=p+u & A(0,t)=\frac{1}{\varepsilon^\gamma}f(\frac{t}{\varepsilon})\\ B=-p+u & B(L,t)=0\end{array}\right. \tag{3}$$

where f is the pulse shape, $f(\frac{t}{\varepsilon})$ containing wavelengths of order ε; $\varepsilon^{-\gamma}$ is the magnitude of the incoming pulse: $\gamma=0$ corresponds to amplitudes of order 1, $\gamma=\frac{1}{2}$ to energies of order 1 and $\gamma=1$ to strong energies.
The second boundary condition on B at $x=L$, expresses the fact that nothing is coming from $+\infty$. Let us remark that the hyperbolicity of equation (2) implies that the reflected signal at $x=0$, $B(0,t)$ for $t\leq T$, is independent of L provided that L has been chosen large enough.
Our quantity of interest is the reflected signal viewed in various scales in time. In the following paragraphs we recall the results, obtained by G. Papanicolaou and his coauthors, on the limiting law, as $\varepsilon\to 0$, of $B(0,t+\varepsilon\sigma)$, namely the reflected signal in a time window, centered at some macroscopic time, in the ε-scale.

3 Uniform background

We consider the case $c_0(x) = 1$ for all x. Changing variables and rescaling (3) as follows:

$$\begin{cases} a^\varepsilon(x,\sigma) & = A(x, x+\varepsilon\sigma) \\ b^\varepsilon(x,\sigma) & = B(x, -x+\varepsilon\sigma) \end{cases} \tag{4}$$

we obtain the boundary conditions:

$$\begin{cases} a^\varepsilon(0,\sigma) & = \frac{1}{\varepsilon^\gamma} f(\sigma) \\ b^\varepsilon(L,\sigma) & = 0 \end{cases} \tag{5}$$

and our quantity of interest becomes:

$$B(0, t+\varepsilon\sigma) = b^\varepsilon(0, \frac{t}{\varepsilon} + \sigma) \tag{6}$$

Denoting by $\hat{a}^\varepsilon$ and $\hat{b}^\varepsilon$ the Fourier transforms in time of a^ε and b^ε ($\hat{a}^\varepsilon(x,\omega) = \int e^{i\omega\sigma} a^\varepsilon(x,\sigma) d\sigma$ and similarly for b^ε) we obtain the equation:

$$\frac{d}{dx} \begin{bmatrix} \hat{a}^\varepsilon \\ \hat{b}^\varepsilon \end{bmatrix} = \frac{i\omega}{2\varepsilon} \eta(\frac{x}{\varepsilon^2}) \begin{bmatrix} 1 & e^{-2i\omega\frac{x}{\varepsilon}} \\ -e^{2i\omega\frac{x}{\varepsilon}} & -1 \end{bmatrix} \begin{bmatrix} \hat{a}^\varepsilon \\ \hat{b}^\varepsilon \end{bmatrix} \tag{7}$$

with boundary conditions:

$$\begin{cases} \hat{a}^\varepsilon(0,\omega) & = \frac{1}{\varepsilon^\gamma} \hat{f}(\omega) \\ \hat{b}^\varepsilon(L,\omega) & = 0 \end{cases} \tag{8}$$

We consider also similar quantities $\hat{a}_1^\varepsilon$ and $\hat{b}_1^\varepsilon$ which solve (7) with boundary conditions:

$$\begin{cases} \hat{a}_1^\varepsilon(0,\omega) & = 1 \\ \hat{b}_1^\varepsilon(L,\omega) & = 0 \end{cases} \tag{9}$$

By linearity, we get $\begin{bmatrix} \hat{a}^\varepsilon \\ \hat{b}^\varepsilon \end{bmatrix} = \frac{1}{\varepsilon^\gamma} \hat{f}(\omega) \begin{bmatrix} \hat{a}_1^\varepsilon \\ \hat{b}_1^\varepsilon \end{bmatrix}$. We have now a Fourier representation of the reflected signal (6):

$$b^\varepsilon(0, \frac{t}{\varepsilon} + \sigma) = \frac{1}{2\pi} \int e^{-i\omega\sigma} e^{-i\frac{\omega t}{\varepsilon}} \frac{1}{\varepsilon^\gamma} \hat{f}(\omega) \hat{b}_1^\varepsilon(0,\omega) d\omega \tag{10}$$

Let us remark again that, while $\hat{b}_1^\varepsilon(0,\omega)$ depends on L, $b^\varepsilon(0, \frac{t}{\varepsilon} + \sigma)$ does not for $t + \varepsilon\sigma < T$.

3.1 Regime of constant energy

In the regime of energy of order 1, $\gamma = \frac{1}{2}$, it has been shown [4], that $(B(0, t+\varepsilon\sigma))_{\sigma\in\mathbb{R}}$ converges in distribution, as $\varepsilon \to 0$, to a centered stationary gaussian process; the full mathematical proof

of this convergence involves regularization in the $t-$variable [5].
The covariance $C^{\varepsilon}_{t,f}(\sigma) = \mathbb{E}\{B(0,t)B(0,t+\varepsilon\sigma)\}$ can be computed from (10):

$$C^{\varepsilon}_{t,f}(\sigma) = \frac{1}{4\pi^2\varepsilon}\int\int e^{-i\omega_2\sigma}e^{i\frac{\omega_1 t}{\varepsilon}}e^{-i\frac{\omega_2 t}{\varepsilon}}\overline{\hat{f}}(\omega_1)\hat{f}(\omega_2)\mathbb{E}\{\overline{\hat{b}^{\varepsilon}_1}(0,\omega_1)\hat{b}^{\varepsilon}_1(0,\omega_2)\}d\omega_1 d\omega_2 \tag{11}$$

where we have used the fact that $B(0,t)$ is real. Setting $\omega_1 = \omega + \varepsilon h$, $\omega_2 = \omega$, we get:

$$C^{\varepsilon}_{t,f}(\sigma) = \frac{1}{4\pi^2}\int\int e^{-i\omega\sigma}e^{ith}\overline{\hat{f}}(\omega+\varepsilon h)\hat{f}(\omega)\mathbb{E}\{\overline{\hat{b}^{\varepsilon}_1}(0,\omega+\varepsilon h)\hat{b}^{\varepsilon}_1(0,\omega)\}d\omega dh \tag{12}$$

A careful analysis of (7) at two nearby frequencies shows that $C^{\varepsilon}_{t,f}(\sigma)$ converges to:

$$C_{t,f}(\sigma) = \frac{1}{2\pi}\int e^{-i\omega\sigma}\Lambda(\omega,t)|\hat{f}(\omega)|^2 d\omega \tag{13}$$

which is the covariance of the limiting stationary gaussian process, $\Lambda(\omega,t)|\hat{f}(\omega)|^2$ being its spectral density. In the case of uniform background $\Lambda(\omega,t)$ can be computed explicitly:

$$\Lambda(\omega,t) = \frac{\alpha\omega^2}{(1+\alpha\omega^2 t)^2} \tag{14}$$

where $\alpha = \frac{1}{4}\int_0^{\infty}\mathbb{E}\{\eta(0)\eta(x)\}dx$.
The Fourier transform in σ of $B(0,t)B(0,t+\varepsilon\sigma)$ is an estimator of $\Lambda(\omega,t)|\hat{f}(\omega)|^2$ with an exponential asymptotic law.

3.2 Regime of constant amplitudes

Let us consider the case $\gamma = 0$. The representation (10) of the reflected signal becomes:

$$B(0,t+\varepsilon\sigma) = \frac{1}{2\pi}\int e^{-i\omega\sigma}e^{-i\frac{\omega t}{\varepsilon}}\hat{f}(\omega)\hat{b}^{\varepsilon}_1(0,\omega)d\omega \tag{15}$$

The time-reversal method consists in sending back into the medium a portion of the time-reversed reflected signal $B(0,-t)$.
This is done **physically as well as mathematically** by multiplying $B(0,t_0+t)$ by a cut-off function $G(t)$, reversing time and considering $B(0,t_0-t)G(-t)$ as a new pulse $g(\frac{t}{\varepsilon})$ in the $\varepsilon-$scale so that $g(\sigma) = B(0,t_0-\varepsilon\sigma)G(-\varepsilon\sigma)$; the function $G(t)$ can be anything like $\frac{1}{2M}\mathbb{I}(-M,M)(t)$ or $\frac{1}{\sqrt{2\pi}M}e^{-\frac{t^2}{2M^2}}$ where M is a fixed constant. A simple computation shows that:

$$\hat{g}(\omega) = \int e^{i\omega'\frac{t_0}{\varepsilon}}\overline{\hat{f}}(\omega')\overline{\hat{b}^{\varepsilon}_1}(0,\omega')\frac{1}{\varepsilon}\overline{\hat{G}}(\frac{\omega-\omega'}{\varepsilon})d\omega' \tag{16}$$

where, of course, $\hat{g}(\omega)$ is random and ε-dependent.
Sending back this pulse into the medium gives a new reflected signal denoted by $S^{\varepsilon}_{t_0}(\sigma)$ when observed around the same time t_0 and in the $\varepsilon-$scale. Using (16) and (10) with $\hat{f}$ replaced by

$\hat{g}$ one gets:

$$S^{\varepsilon}_{t_0}(\sigma) = \frac{1}{2\pi}\int\int e^{-i\omega\sigma} e^{-i\frac{(\omega-\omega')t_0}{\varepsilon}} \overline{\hat{f}}(\omega')\overline{\hat{b}^{\varepsilon}_1}(0,\omega')\hat{b}^{\varepsilon}_1(0,\omega)\frac{1}{\varepsilon}\overline{\hat{G}}(\frac{\omega-\omega'}{\varepsilon})d\omega' d\omega \tag{17}$$

Again the change of variables $\omega' - \omega = \varepsilon h$ gives:

$$S^{\varepsilon}_{t_0}(\sigma) = \frac{1}{2\pi}\int\int e^{-i\omega\sigma} e^{it_0 h}\overline{\hat{f}}(\omega+\varepsilon h)\overline{\hat{b}^{\varepsilon}_1}(0,\omega+\varepsilon h)\hat{b}^{\varepsilon}_1(0,\omega)\hat{G}(h)dh d\omega \tag{18}$$

Let us remark that if we observe the reflected signal $S^{\varepsilon}_{t_1}(\sigma)$ at a time $t_1 \neq t_0$ a term $e^{-i\frac{t_1-t_0}{\varepsilon}\omega}$ is introduced in the integral which will make $S^{\varepsilon}_{t_1}(\sigma)$ goes to 0 as $\varepsilon \to 0$. Taking the expectation in (18) gives:

$$\mathbb{E}\{S^{\varepsilon}_{t_0}(\sigma)\} = \frac{1}{2\pi}\int\int e^{-i\omega\sigma} e^{it_0 h}\overline{\hat{f}}(\omega+\varepsilon h)\mathbb{E}\{\overline{\hat{b}^{\varepsilon}_1}(0,\omega+\varepsilon h)\hat{b}^{\varepsilon}_1(0,\omega)\}\hat{G}(h)dh d\omega \tag{19}$$

The asymptotic analysis of $\mathbb{E}\{\overline{\hat{b}^{\varepsilon}_1}(0,\omega+\varepsilon h)\hat{b}^{\varepsilon}_1(0,\omega)\}$ and the smoothing in time by $\hat{G}(h)$ gives the convergence:

$$\lim_{\varepsilon\to 0}\mathbb{E}\{S^{\varepsilon}_{t_0}(\sigma)\} = \int e^{-i\omega\sigma}\overline{\hat{f}}(\omega)(\Lambda(\omega,t)*G(t))(t_0)d\omega = S_{t_0}(\sigma) \tag{20}$$

Moreover, when computing $\mathbb{E}\{S^{\varepsilon}_{t_0}(\sigma)^2\}$ one observes that we get terms such as:

$$\mathbb{E}\{\overline{\hat{b}^{\varepsilon}_1}(0,\omega+\varepsilon h)\hat{b}^{\varepsilon}_1(0,\omega)\overline{\hat{b}^{\varepsilon}_1}(0,\omega'+\varepsilon h')\hat{b}^{\varepsilon}_1(0,\omega')\}$$

which have been shown to have the same limit as:

$$\mathbb{E}\{\overline{\hat{b}^{\varepsilon}_1}(0,\omega+\varepsilon h)\hat{b}^{\varepsilon}_1(0,\omega)\}\mathbb{E}\{\overline{\hat{b}^{\varepsilon}_1}(0,\omega'+\varepsilon h')\hat{b}^{\varepsilon}_1(0,\omega')\}$$

for two distinct frequencies ω and ω' (cf [6]). This shows that $S^{\varepsilon}_{t_0}(\sigma)$ **converges to the deterministic limit** $S_{t_0}(\sigma)$. It has already been observed in [1] that the smoothing in time kills the fluctuations in the reflected process. In our model this smoothing in time is physically introduced in the time-reversal method. In other words, by Fourier inverse, $\hat{S^{\varepsilon}_{t_0}}(\omega)$ is a convergent estimator for $2\pi\overline{\hat{f}}(\omega)(\Lambda(\omega,t)*G(t))(t_0)$.

3.3 Low frequency information

Using (14) we see that $\lim_{\omega\to 0}\frac{\Lambda(\omega,t)}{\omega^2} = \alpha$ which means that $\frac{\hat{S}_{t_0}(\omega)}{\omega^2}$ for small ω^2 is a convergent (as $\varepsilon \to 0$) estimator for $2\pi\alpha\overline{\hat{f}}(0)\hat{G}(0)$. This remark is important in the inverse problem treated in the next paragraph; let us remark here that this low frequency estimate will be useful only for pulses such that $\hat{f}(0)\hat{G}(0) \neq 0$. $\hat{G}(0) = \int G(t)dt$ can always be chosen equal to 1 such as in the two choices suggested above. The condition $\hat{f}(0) = \int f(t)dt \neq 0$ is much more restrictive; in practice the device emitting the pulse contains a low frequency filter which imposes a frequency spectrum bounded away from 0 which implies $\hat{f}(0) = 0$. For the same reason when we are sending back the time reversed pulse $g(\sigma)$ we should include a cut-off function $K(\omega)$ in formula (16) which will give a factor $K(\omega)$ in the integral (18) defining $S^{\varepsilon}_{t_0}(\sigma)$ and then $K(\omega)\overline{\hat{f}}(\omega)$ in the integral (20) defining the limit $S_{t_0}(\sigma)$. Nevertheless, it will be possible to choose $K(\omega)\overline{\hat{f}}(\omega) = \overline{\hat{f}}(\omega)$ on the support of $\hat{f}$ which makes formula (20) correct.

3.4 Time-reversal in the high energy regime

We consider the case $\gamma = 1$ which makes the entering energy of order ε^{-1}. Replacing $G(t)$ by $G_\varepsilon = G(\frac{t}{\varepsilon})$, which means that we send back a portion of the reflected signal $B(0,t)$ of time duration ε, (16) becomes:

$$\hat{g}(\omega) = \int e^{i\omega' \frac{t_0}{\varepsilon}} \frac{1}{\varepsilon} \overline{\hat{f}}(\omega') \overline{\hat{b}^{\varepsilon}_1}(0,\omega') \overline{\hat{G}}(\omega - \omega') d\omega' \tag{21}$$

and $\hat{G}(h)$ has to be replaced by $\hat{G}(\varepsilon h)$ in (18); $\hat{G}(0)$ being again chosen equal to 1 we then get:

$$\lim_{\varepsilon \to 0} \mathbb{E}\{S^{\varepsilon}_{t_0}(\sigma)\} = \int e^{-i\omega\sigma} \overline{\hat{f}}(\omega) \Lambda(\omega, t_0) d\omega \tag{22}$$

$S^{\varepsilon}_{t_0}(\sigma)$ is then a possible estimator for $2\pi \overline{\hat{f}}(\omega)\Lambda(\omega, t_0)$. Its asymptotic law can be computed on its moments; this law is dependent upon G.

4 Nonuniform background

We are now in the situation of the model presented in section 2; the inverse problem consists in reconstructing $c_0(x)$ by analysing the reflected signal. The analysis presented in the previous paragraph can be extended by introducing the travel time τ defined by:

$$\tau(x) = \int_0^x \frac{dy}{c_0(y)} \tag{23}$$

As it is shown in [4], in the case $\gamma = \frac{1}{2}$, $C^{\varepsilon}_{t,f}(\sigma)$ given by (12) converges to $C_{t,f}(\sigma)$ given by (13). $\Lambda(\omega, t)$ is not explicitly computable anymore. It is given by an infinite hyperbolic system, the W-equations:

$$\frac{\partial W^{(N)}}{\partial x} - \frac{2N}{c_0(x)} \frac{\partial W^{(N)}}{\partial t} - \frac{2\omega^2 \alpha N^2}{c_0(x)^2} \{W^{(N+1)} - 2W^{(N)} + W^{(N-1)}\} = 0 \tag{24}$$

with boundary conditions:

$$W^{(N)}(L,t,\omega) = \delta(t)\delta_{N,0} \ , \ W^{(N)} = 0 \ for \ N < 0$$

The spectral density $\Lambda(\omega, t)$ is given by $\Lambda(\omega, t) = W^{(1)}(0, t, \omega)$.
We refer to [4] for the derivation of these equations from the Riccatti equations satisfied by

$\hat{b}_1^\varepsilon(x,\omega)$ and the corresponding asymptotics.
Here again $\lim_{\omega\to 0} \dfrac{\Lambda(\omega,t)}{\omega^2}$ gives a direct way to reconstruct $\frac{c_0(x)}{\alpha}$, since it gives $\frac{\alpha}{c_0(x)}$ at x such that $\tau(x) = \frac{t}{2}$.
As explained in the previous paragraph, $\Lambda(\omega,t)$ can be estimated only within the spectrum of f which, in practice, doesn't contain 0. Therefore the inverse problem goes through the resolution of the W-equations and matching of $c_0(x)$ to the observation of $W^{(1)}(0,t,\omega) = \Lambda(\omega,t)$; we refer to [7] for the inversion procedure. In any case the quality of the inversion relies strongly on the quality of the estimation of $\Lambda(\omega,t)$. Several methods, based on the reflected signal, have been proposed: a window Fourier transform is used in [1] and a wavelet transform method is proposed in [8]. These methods involve computations of covariances of the reflected signal. The time-reversal mirror method consists in having these computations done by the medium itself, in the time domain by nature; this enables us to stay as much as possible in the time domain and hopefully to get more robust estimates. Unfortunately it is difficult to test this method numerically because numerical simulations are usually done in the frequency domain. We are presently working with M. Fink on the possibility of testing this method experimentally.

4.1 Estimates in the regime of amplitudes of order 1 ($\gamma = 0$)

As shown in the previous paragraph, $S^\varepsilon_{t_0}(\sigma)$, given by (18) and obtained in the time domain by the time-reversal method with a function $G(t)$, gives a convergent estimator for $2\pi\overline{\hat{f}}(\omega)(\Lambda(\omega,t) * G(t))(t_0)$.
Let us remark that in the low frequency regime ($\omega \to 0$) the division by ω^2 translates into the estimation of a second derivative at $\omega = 0$ and therefore the estimation can be obtained directly in the time domain through the integral $-\int \sigma^2 S^\varepsilon_{t_0}(\sigma)d\sigma$ divided by $\int f(\sigma)d\sigma$. As already mentioned this estimate is useless in the case $\hat{f}(0) = 0$.
We have denoted by M the size of the support of $G(t)$; our estimation imposes $\varepsilon^2 \ll \varepsilon \ll M$. On the other hand the quality of the inversion of the W-equations depends on the choice of a time step $\Delta t \ll 1$. The convolution of the W-equations by $G(t)$ requires that $M \ll \Delta t$; putting these conditions together, we need that $\varepsilon^2 \ll \varepsilon \ll M \ll \Delta t \ll 1$. This is not the case in the numerical simulations done in [1] where a pointwise (in time) estimation of $\Lambda(\omega,t)$ is clearly needed.

4.2 Estimates in the high energy regime ($\gamma = 1$)

Choosing $\gamma = 1$, as in the last section of the previous paragraph, the time-reversal method with $G_\varepsilon(t) = G(\frac{t}{\varepsilon})$ gives a pointwise estimator for $\overline{\hat{f}}(\omega)\Lambda(\omega,t_0)$; the fact that this estimator is obtained in the time domain makes it very robust and it leaves the possibility to adjust the function G which enters explicitly in the law of $S_{t_0}(\sigma)$. Taking M, the support of G_ε, in the range of several wavelengths ($M = n\varepsilon$) should improve our estimation (experimental results show that $n = 4$ is enough).

4.3 Estimates of higher moments

In this section we show that it is possible to use back and forth into the medium to construct estimators for the higher moments $W^{(N)}(0,t,\omega), N>1$. This idea has been been very succesful in selecting targets as explained in [9].
Rewritting equation (18) in the general case of amplitudes of order $\varepsilon^{-\gamma}$ and using a cut-off function $G_\varepsilon(t)=G(\frac{t}{\varepsilon^\rho})$ with $\rho\geq 0$, we obtain the reflected signal:

$$S^\varepsilon_{t_0}(\sigma)=\frac{1}{2\pi}\int\int e^{-i\omega\sigma}e^{it_0h}\varepsilon^{-\gamma}\overline{\hat{f}}(\omega+\varepsilon h)\overline{\hat{b}^\varepsilon_1}(0,\omega+\varepsilon h)\hat{b}^\varepsilon_1(0,\omega)\varepsilon^\rho\hat{G}(\varepsilon^\rho h)dhd\omega \tag{25}$$

We now use this signal as a new pulse sent into the medium (without time-reversal). This will produce a reflected signal which we cut-off around another time t_1 with a new cut-off function $H_\varepsilon(t)=H(\frac{t}{\varepsilon^\delta})$ for some $\delta\geq 0$. After time-reversal this signal is sent back into the medium and gives a new reflected signal denoted by $S^\varepsilon_{t_0,t_1}(\sigma)$. With the same compution used to obtain (25), we get:

$$S^\varepsilon_{t_0,t_1}(\sigma)=\frac{1}{2\pi}\int\int e^{-i\omega\sigma}e^{it_1h}\overline{\hat{S^\varepsilon_{t_0}}}(\omega+\varepsilon h)\overline{\hat{b}^\varepsilon_1}(0,\omega+\varepsilon h)\hat{b}^\varepsilon_1(0,\omega)\varepsilon^\delta\hat{H}(\varepsilon^\delta h)dhd\omega \tag{26}$$

From (25) we deduce $\hat{S^\varepsilon_{t_0}}$ which, computed at $\omega+\varepsilon h$ and replaced in (26), gives:

$$S^\varepsilon_{t_0,t_1}(\sigma)=\frac{1}{2\pi}\int d\omega e^{-i\omega\sigma}\varepsilon^{\delta+\rho-\gamma}\int\int dhdh'e^{it_1h}e^{-it_0h'}\hat{f}(\omega+\varepsilon(h+h'))Q^\varepsilon(\omega,h,h')\overline{\hat{G}}(\varepsilon^\rho h')\hat{H}(\varepsilon^\delta h) \tag{27}$$

where $Q^\varepsilon(\omega,h,h')$ denotes the product $\{\hat{b}^\varepsilon_1(0,\omega)(\overline{\hat{b}^\varepsilon_1}(0,\omega+\varepsilon h))^2\hat{b}^\varepsilon_1(0,\omega+\varepsilon(h+h'))\}$ involving three frequencies.
Writting $e^{it_1h}e^{-it_0h'}=e^{-it_0(h+h')}e^{i(t_0+t_1)h}$, fixing $t_0+t_1=t>0$ and integrating in t_0 (from 0 up to t by causality), the integral with respect to t_0 reduces to a Dirac mass at $h=-h'$ and we get:

$$\int_0^t S^\varepsilon_{t_0,t-t_0}(\sigma)dt_0=\frac{1}{2\pi}\int d\omega e^{-i\omega\sigma}\varepsilon^{\delta+\rho-\gamma}\int dhe^{ith}\hat{f}(\omega)\{(\hat{b}^\varepsilon_1(0,\omega))^2(\overline{\hat{b}^\varepsilon_1}(0,\omega+\varepsilon h))^2\}\hat{G}(\varepsilon^\rho h)\hat{H}(\varepsilon^\delta h) \tag{28}$$

In order to get a convergent quantity with a nontrivial limit we must choose $\delta+\rho=\gamma$: for instance, taking $\delta=\gamma$ and $\rho=0$, we get a convergent estimator for $W^{(2)}(0,t)$ appearing in the W-equations (24).
Iterating this back and forth procedure we are able to construct robust estimators for $W^{(N)}(0,t)$ for N as large as needed. This can be used to improve greatly the inversion procedure for the coefficient $c_0(x)$.
It is noticeable that formulas such as (28), in the case $N=2$, correspond to multiple scattering of the wave.

5 Conclusion and comments

We have presented the time-reversal mirror method in the context of one-dimensional acoustic equations. Using an already existing mathematical analysis of this model in the diffusion-approximation limit, we have shown how this method may improve the estimates needed in the inverse problem. We have also described a method for obtaining estimates for higher order moments by using a back and forth procedure with time-reversals. We are presently working on experimental tests for this method.
In a forthcoming paper we shall give results on the time-reversal method in the case of three-dimensional layered media in the framework of [1].
The mathematical analysis relies strongly on the fact that 1-d or 3-d layered random media localize waves. This means, in particular, that the transmitted energy at frequency $\frac{\omega}{\varepsilon}$ decreases exponentially with L or equivalently that the reflected energy $|\hat{b_1^\varepsilon}(0,\omega)|^2$ is exponentially close to 1 as L goes to $+\infty$. 3-d random media are known to be localizing in the high frequency regime [10]. This is an indication that the behavior of the time-reversal method in a 3-d random medium should be qualitatively the same as in a 1-d random medium; this is supported by experimental observations. Unfortunately no mathematical analysis, at the present day, can describe quantitatively the relation between the reflected signals and the medium.

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