Abstract

We consider a one-dimensional linear wave equation with a small mean zero dissipative field and with the boundary condition imposed by the so-called Goursat problem. In order to observe the effect of the randomness on the solution we perform a space-time rescaling and we rewrite the problem in a diffusion approximation form for two parameter processes. We prove that the solution converges in distribution toward the solution of a two-parameter stochastic differential equation which we identify. The diffusion approximation results for one-parameter processes are well known and well understood. In fact, the solution of the one-parameter analog of the problem we consider here is immediate. Unfortunately, the situation is much more complicated for two-parameter processes and we believe that our result is the first one of its kind.

A DIFFUSION APPROXIMATION RESULT FOR TWO PARAMETER PROCESSES

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1 INTRODUCTION

Let us consider the Goursat problem for the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial \tau^2} - \frac{\partial^2 u}{\partial x^2} = \epsilon a(\tau, x) u$$

on the domain defined by $\tau > |x|$ and with the boundary conditions $u(\tau,\tau) = u(-\tau,\tau) = 1$ for $\tau \geq 0$.
We assume that the field of dissipation $a(\tau, x)$ is a mean zero random field. We are interested in the limiting regime of small dissipation. This is the reason for the presence of the small parameter $\epsilon > 0$. Setting $s' = \tau + x$, $t' = \tau - x$, $X(s', t') = u(\tau, x)$ and $F(s', t') = a(\tau, x)$ the problem becomes:

$$\frac{\partial^2 X}{\partial s' \partial t'} = \epsilon F(s', t') X \tag{1}$$

for (s', t') in the first quadrant $s', t' \geq 0$. The boundary conditions read $X_{s',0} = X_{0,t'} = 1$ for all $s', t' \geq 0$. We want to observe the fluctuations of the solution (remember that F is centered). To do so we change the scale and we work with the new variables $s' = s/\epsilon$ and $t' = t/\epsilon$. The partial differential equation can now be rewritten in the form:

$$\frac{\partial^2 X^\epsilon}{\partial s \partial t} = \frac{1}{\epsilon} F(\frac{s}{\epsilon}, \frac{t}{\epsilon}) X^\epsilon \tag{2}$$

with the same boundary conditions $X^\epsilon_{0,t} = X^\epsilon_{s,0} = 1$. We emphasize the dependence upon the *small parameter* ϵ by using the notation $X^\epsilon_{s,t}$ for the solution. Also, we shall use the notation:

$$F^\epsilon(s,t) = F(\frac{s}{\epsilon}, \frac{t}{\epsilon})$$

and we rewrite the partial differential equation (2) as the integral equation:

$$X^\epsilon_{s,t} = 1 + \frac{1}{\epsilon} \int_0^s \int_0^t F^\epsilon(u,v) X^\epsilon_{u,v} \, du \, dv. \tag{3}$$

We are interested in the limiting behavior of the solution X^ϵ in the limit $\epsilon \searrow 0$. We believe that it is possible to prove the existence and a characterization of a limiting process X under fairly general conditions on the field $F(s,t)$. In order to avoid technical difficulties we shall restrict ourselves to a particular case. Our goal is to prove that results of the diffusion approximation type can be proved for hyperbolic partial differential equations and two parameter processes. We do not aim at generality for the technicalities of the two parameter stochastic calculus get in the way and make the problem extremely difficult. We shall concentrate on a very simple model. We choose to work with a random field F of the form:

$$F(s,t) = \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} Z_{k,\ell}\mathbf{1}_{[k-1,k)\times[\ell-1,\ell)}(s,t)$$

where $\{Z_{k,\ell}, k \geq 1, \ell \geq 1\}$ is an independent family of centered identically distributed random variables. Again, for the sake of simplicity we shall assume that the support of the common distribution is bounded. In other words we assume the existence of a positive number M satisfying:

$$|Z_{k,\ell}| \leq M, \qquad\qquad k,\ell \geq 1$$

almost surely. We shall denote by σ^2 the common variance of the $Z_{k,\ell}$'s. The main result of the paper is the following:

Theorem 1.1 *For each $S > 0$ and $T > 0$ the distribution of $\{X^\epsilon_{s,t};\ 0 \leq s \leq S,\ 0 \leq t \leq T\}$ converges weakly as $\epsilon \searrow 0$ on the Banach space $C([0,S]\times[0,T])$ of real valued continuous functions on $[0,S]\times[0,T]$ toward the distribution of the unique solution $\{X_{s,t};\ 0 \leq s \leq S;\ 0 \leq t \leq T\}$ of the Stratonovich equation:*

$$dX_{s,t} = \sigma X_{s,t} \circ dW_{s,t} \tag{4}$$

with the boundary conditions $X_{0,t} = X_{s,0} = 1$. Here $W_{s,t}$ is a standard Brownian sheet over the positive quadrant.

The corresponding problem for one-parameter processes is a simple particular case of the classical results going under the name of diffusion-approximation:

$$\frac{dX^\epsilon}{dt} = \frac{1}{\sqrt{\epsilon}}F(t/\epsilon)X^\epsilon$$

with the initial condition $X^\epsilon_0 = 1$. Here one could assume that:

$$F(t) = \sum_{k=1}^{\infty} Z_k \mathbf{1}_{[k-1,k)}(t)$$

where $\{Z_k : k \geq 1\}$ is an independent family of centered identically distributed random variables. Again, we denote the common variance of the Z_k's by σ^2. This equation has a unique solution. It is obviously given by the formula:

$$X^\epsilon_t = e^{\frac{1}{\sqrt{\epsilon}}\int_0^t F(s/\epsilon)ds}.$$

The functional central limit theorem implies that the process

$$\{\epsilon^{-1/2}\int_0^t F(s/\epsilon)ds;\ t \geq 0\}$$

converges in distribution toward a process $\{\sigma B_t;\, t \geq 0\}$ where $\{B_t;\, t \geq 0\}$ is a standard Brownian motion. Since the exponential function is continuous, the solution process X^ϵ converges in distribution toward the process $X_t = e^{\sigma B_t}$ which is the unique solution of the Stratonovich equation:

$$dX_t = \sigma X_t \circ dB_t$$

with the initial condition $X_0 = 1$. In particular $\{X_t;\, t \geq 0\}$ will be a continuous functional of $\{B_t;\, t \geq 0\}$.

This is not the case any longer for two-parameter processes and a straightforward generalization of the one parameter case cannot be expected. Indeed, the usual characterizations of the limit process as a Markov process of the diffusion type or as the solution of a martingale problem are not possible in the case of two parameter processes.

The purpose of the present paper is to prove such a result of the diffusion approximation type for two parameter processes. We show that the solution $X^\epsilon = \{X^\epsilon_{s,t};\, s,t \geq 0\}$ converges in distribution to the unique solution $X = \{X_{s,t};\, s,t \geq 0\}$ of the Stratonovich equation:

$$dX_{s,t} = \sigma X_{s,t} \circ dW_{s,t} \tag{5}$$

with the boundary conditions $X(s,0) = X(0,t) = 1$, where $W_{s,t}$ is a standard Brownian sheet. The reader is referred to [2] for an account of two parameter stochastic calculus. The equation can be equivalently written as an equation in the Ito's sense:

$$dX_{s,t} = \sigma X_{s,t} \cdot dW_{s,t} + \frac{\sigma^2}{4} X_{s,t} ds\, dt \tag{6}$$

See [8] and [4] for properties of this equation. The solution of equation (6) has been called a two parameter diffusion in the literature and this is the reason why we use the terminology *diffusion approximation* for our result. This solution process has been studied in [5]. Its behavior is very different from the behavior of the solution $X(t)$ of the one parameter problem. For instance $X_{s,t}$ may take negative values while $X(t)$ is always positive. Nevertheless, the results of [3] imply that $X_{s,t}$ has a smooth density as long as $s > 0$ and $t > 0$.

A possible approach is to look for a solution in the form of an expansion:

$$\begin{aligned} X^\epsilon_{s,t} = 1 &+ \frac{1}{\epsilon}\int_0^s \int_0^t F(u_1, v_1) du_1 dv_1 \\ &+ \frac{1}{\epsilon^2}\int_0^s \int_0^t \int_0^{u_1} \int_0^{v_1} F(u_1, v_1) F(u_2, v_2) du_2 dv_2 du_1 dv_1 + \cdots\cdots \ . \end{aligned}$$

The second term converges in distribution to $\sigma W_{s,t}$ but the following terms involve multiple integrals and the latter are not continuous functionals on the two parameter

Wiener space! In fact the whole solution $\{X_{s,t};\ s,t \geq 0\}$ is not a continuous functional of the Brownian sheet $\{W_{s,t};\ s,t \geq 0\}$. The difficulty created by this lack of continuity is one of the pitfalls of the stochastic calculus with two parameter processes.

The importance of diffusion approximation results is two fold. At the conceptual level, it justifies the use of model involving stochastic partial differential equations driven by space time white noise. At the practical level, it gives approximations to quantities of interest such as the moments of the solutions, say $m_p^\epsilon(s,t) = \mathbb{E}\{(X_{s,t}^\epsilon)^p\}$ for $p \geq 1$ integer. The latter can be used to infer properties of the solution such as location and speed of the wave fronts. Convergence in distribution does not imply automatically convergence of the moments. A uniform integrability condition has to be satisfied for that to be true. This condition is satisfied in the present situation. The moments $m_p^\epsilon(s,t)$ are difficult to compute directly when $\epsilon > 0$. On the other hand, in the limiting regime $\epsilon = 0$, Ito's formula can be used to compute or estimate the moments $m_p(s,t)$. See [5]. For example, the first moment $m_1(s,t) = \mathbb{E}\{X_{s,t}\}$ is easily seen to be a solution of the (deterministic) integral equation:

$$m_1(s,t) = 1 + \frac{\sigma^2}{4}\int_0^s\int_0^t m_1(u,v)\,du\,dv,$$

the solution of which is given by $m_1(s,t) = f(\sigma^2 st/4)$ where the function f is defined by:

$$f(u) = \sum_{k=0}^{\infty}\frac{u^k}{(k!)^2}.$$

Our proof is based on the introduction of an intermediate scale $1/N$ and the approximation of equation (2) by a finite difference equation. We prove the convergence in distribution for this finite difference equation which itself will be an approximation of the equation for $X_{s,t}$. The key estimates will be found in Lemma 2.1 which says that we have the right limit and in Proposition 4.2 which gives the necessary tightness condition. These estimates are rather technical but we were not able to find a simpler proof! Our main result can certainly be extended (without too much difficulty) to a nonlinear equation of the type:

$$\frac{\partial^2 X^\epsilon}{\partial s\partial t} = \frac{1}{\epsilon}F(\frac{s}{\epsilon},\frac{t}{\epsilon})g(X_{s,t}^\epsilon)$$

and presumably to more general random fields F (i.e. more general ergodic random fields satisfying various mixing conditions).

Our proof is divided in two parts:

1. Proof of the tightness
2. Identification of the limit.

The identification of the limit is done via an approximation by the solution of a finite difference equation. The following section contains the various steps of this approximation. The last section of the paper is devoted to the proof of the tightness. The latter follows from the estimate:

$$\mathbb{E}\{(\Delta_{[z_1,z_2]}(X^\epsilon)^4\} \leq C(s_2-s_1)^2(t_2-t_1)^2 \tag{7}$$

for all $z_1 = (s_1,t_1)$, $z_2 = (s_2,t_2)$ such that $0 \leq s_1 \leq s_2 \leq S$ and $0 \leq t_1 \leq t_2 \leq T$. The constant C may depend upon S and T but is independent of ϵ. We use (here and throughout the paper) the notation $\Delta_{[z_1,z_2]}(X^\epsilon)$ for the increment of X^ϵ over the rectangle $[z_1,z_2]$, namely:

$$\Delta_{[z_1,z_2]}(X^\epsilon) = X^\epsilon_{s_2,t_2} - X^\epsilon_{s_1,t_2} - X^\epsilon_{s_2,t_1} + X^\epsilon_{s_1,t_1}.$$

Note that (7) implies that:

$$C(S,T) = \sup_{0\leq s_1\leq s_2\leq S,\ 0\leq t_1\leq t_2\leq T} \mathbb{E}\{|\Delta_{[(s_1,t_1),)s_2,t_2)]}(X^\epsilon)^2\} < \infty. \tag{8}$$

This uniform boundedness of the second moment plays a crucial role in the derivation of several estimates in Section 2 and Section 3 below. The actual tightness estimate (7) will be proved in Section 4. We refer to [1] for the fact that (7) implies the tightness of the laws of $\{X^\epsilon_{s,t}\ :\ s,t\geq 0\}$ on the Banach space $C([0,S]\times[0,T])$.

Most of the estimates in this paper contain constants. We use the same letter C for these constants, even when the actual numerical values of these constants change from line to line. We shall sometime emphasize the dependence of these constants upon parameters such as S and T, but the crucial fact is that the constants we use are independent of the small parameter ϵ and the integer N defining the intermediate scale.

2 Approximation by Finite Difference Equations

We fix an integer $N\geq 1$ and define:

$$p = \left[\frac{1}{N}[S/\epsilon]\right], \qquad \text{and} \qquad q = \left[\frac{1}{N}[T/\epsilon]\right]$$

Notice that:

$$p \leq \frac{S}{N\epsilon} \qquad \text{and} \qquad q \leq \frac{T}{N\epsilon} \tag{9}$$

and:

$$\lim_{\epsilon\to 0}\epsilon^2 pq = \frac{ST}{N^2} \tag{10}$$

We shall need these two facts in the sequel. We shall study $X^\epsilon_{s,t}$ at the points:

$$(kp\epsilon, \ell q\epsilon) \quad \text{for} \quad k, \ell = 0, 1, \cdots, N.$$

The problem contains 3 scales:

- order 1: rectangle $[0, S] \times [0, T]$
- order $\frac{1}{N}$: rectangles: $[((k-1)p\epsilon, (\ell-1)q\epsilon), (kp\epsilon, \ell q\epsilon)]$
- order ϵ: rectangles $[((n-1)\epsilon, (m-1)\epsilon), (n\epsilon, m\epsilon)]$.

We are only interested in the regime $\epsilon \ll \frac{1}{N} \ll 1$. We shall use the notations:

$$\Delta_{k,\ell} = [((k-1)p\epsilon, (\ell-1)q\epsilon), (kp\epsilon, \ell q\epsilon)]$$

and:

$$\Delta_{k,\ell}(X^\epsilon) = \Delta_{[((k-1)p\epsilon,(\ell-1)q\epsilon),(kp\epsilon,\ell q\epsilon)]}(X^\epsilon)$$

to shorten some of the formulas. Equation (2) implies that, for $1 \leq k \leq N$ and $1 \leq \ell \leq N$, one has:

$$\begin{aligned}
\Delta_{k,\ell}(X^\epsilon) &= \frac{1}{\epsilon} \int\int_{\Delta_{k,\ell}} F^\epsilon(u,v) X^\epsilon_{u,v}\, dudv \\
&= \left(\frac{1}{\epsilon} \int\int_{\Delta_{k\ell}} F^\epsilon(u,v) dudv \right) X^\epsilon_{(k-1)p\epsilon,(\ell-1)q\epsilon} \\
&\qquad + \frac{1}{\epsilon} \int\int_{\Delta_{k\ell}} F^\epsilon(u,v) (X^\epsilon_{u,v} - X^\epsilon_{(k-1)p\epsilon,(\ell-1)q\epsilon})\, dudv.
\end{aligned}$$

We shall also need the following notations:

$$\begin{aligned}
\Delta^1_{k,\ell}(u,v) &= [((k-1)p\epsilon, (\ell-1)q\epsilon, (u,v)] \\
\Delta^2_{k,\ell}(u,v) &= [((k-1)p\epsilon, 0), (u, (\ell-1)q\epsilon)] \\
\Delta^3_{k,\ell}(u,v) &= [(0, (\ell-1)q\epsilon), ((k-1)p\epsilon, v)].
\end{aligned}$$

These domains of the plane are illustrated in the following figure.

Figure 2.1 *Example of domains* $\Delta 1 = \Delta^1_{k,\ell}(u,v)$, $\Delta 2 = \Delta^2_{k,\ell}(u,v)$ *and* $\Delta 3 = \Delta^3_{k,\ell}(u,v)$ *of the plane.*

Then we have:

$$X^\epsilon_{u,v} - X^\epsilon_{(k-1)p\epsilon,(\ell-1)q\epsilon} = \Delta^1_{k,\ell}(u,v)(X^\epsilon) + \Delta^2_{k,\ell}(u,v)(X^\epsilon) + \Delta^3_{k,\ell}(u,v)(X^\epsilon)$$

where we used the obvious notation $\Delta^i_{k,\ell}(u,v)(X^\epsilon)$ for the increment of the process X^ϵ over the rectangle $\Delta^i_{k,\ell}(u,v)$. Consequently we get:

$$\begin{aligned}
\Delta_{k,\ell}(X^\epsilon) &= \left(\frac{1}{\epsilon}\int\int_{\Delta_{k,\ell}} F^\epsilon(u,v)dudv\right) X^\epsilon_{(k-1)p\epsilon,(\ell-1)q\epsilon} \\
&+ \left(\frac{1}{\epsilon^2}\int\int_{\Delta_{k,\ell}} F^\epsilon(u,v)\int\int_{\Delta^1_{k,\ell}(u,v)} F^\epsilon(\alpha,\beta)d\alpha d\beta dudv\right) X^\epsilon_{(k-1)p\epsilon,(\ell-1)q\epsilon} \\
&+ \frac{1}{\epsilon^2}\int\int_{\Delta_{k,\ell}} F^\epsilon(u,v)\int\int_{\Delta^1_{k,\ell}(u,v)} F^\epsilon(\alpha,\beta) \\
&\qquad (X^\epsilon(\alpha,\beta) - X^\epsilon_{(k-1)p\epsilon,(\ell-1)q\epsilon})\, d\alpha d\beta dudv \\
&+ \frac{1}{\epsilon}\int\int_{\Delta_{k,\ell}} F^\epsilon(u,v)(\Delta^2_{k,\ell}(u,v)(X^\epsilon) + \Delta^3_{k,\ell}(u,v)(X^\epsilon))dudv. \qquad (11)
\end{aligned}$$

Definition 2.1 *The random variables* $X^{\epsilon,N}_{k,\ell}$ *are defined for* $k,\ell = 0,1,\cdots,N$ *inductively by* $X^{\epsilon,N}_{k,\ell} = 1$ *if* $k=0$ *or* $\ell = 0$ *and*

$$
\begin{aligned}
\Delta_{k,\ell}(X^{\epsilon,N}) &= X^{\epsilon,N}_{k,\ell} - X^{\epsilon,N}_{k-1,\ell} - X^{\epsilon,N}_{k,\ell-1} + X^{\epsilon,N}_{k-1,\ell-1} \\
&= \left(\frac{1}{\epsilon}\int\int_{\Delta_{k\ell}} F^\epsilon(u,v)dudv\right) X^{\epsilon,N}_{k-1,\ell-1} + \frac{\epsilon^2 pq\sigma^2}{4} X^{\epsilon,N}_{k-1,\ell-1}
\end{aligned}
$$

for $k,\ell = 1,\cdots,N$.

We are in a position to present the crucial technical estimate of this section.

Lemma 2.1

$$\mathbb{E}\{|X^\epsilon_{kp\epsilon,\ell q\epsilon} - X^{\epsilon,N}_{k,\ell}|^2\} \le \frac{C}{N}$$

for $k,\ell = 0,\cdots,N$ *and for some positive constant* $C = C(S,T)$ *which depends only upon* S *and* T *and which is independent of* ϵ *and* N.

Proof:
Let us define the random vector $\{Y^{\epsilon,N}_{k,\ell}, 0 \le k \le N; 0 \le \ell \le N\}$ by:

$$Y^{\epsilon,N}_{k,\ell} = X^\epsilon_{kp\epsilon,\ell q\epsilon} - X^{\epsilon,N}_{k,\ell}.$$

Because of this definition we have $Y^{\epsilon,N}_{0,p} = Y^{\epsilon,N}_{k,0} = 0$ and:

$$\Delta_{k,\ell}(Y^{\epsilon,N}) = \Delta_{k,\ell}(X^\epsilon) - \Delta_{k,\ell}(X^{\epsilon,N})$$

for $1 \le k,\ell \le N$. Here $\Delta_{k,\ell}(Y^{\epsilon,N})$ represents the rectangular increment defined by:

$$\Delta_{k,\ell}(Y^{\epsilon,N}) = Y^{\epsilon,N}_{k,\ell} - Y^{\epsilon,N}_{k-1,\ell} - Y^{\epsilon,N}_{k,\ell-1} + Y^{\epsilon,N}_{k-1,\ell-1}$$

The latter can be rewritten using (11) and Definition 2.1 in the form:

$$
\begin{aligned}
\Delta_{k,\ell}(Y^{\epsilon,N}) &= \left(\frac{1}{\epsilon}\int\int_{\Delta_{k,\ell}} F^\epsilon(u,v)dudv\right) Y^{\epsilon,N}_{k-1,\ell-1} + \frac{1}{4}\epsilon^2 pq\sigma^2 Y^{\epsilon,N}_{k-1,\ell-1} \qquad (12)\\
&+ \left[\left(\frac{1}{\epsilon^2}\int\int_{\Delta_{k,\ell}} F^\epsilon(u,v)\int\int_{\Delta^1_{k,\ell}(u,v)} F^\epsilon(\alpha,\beta)d\alpha d\beta dudv\right) - \frac{\sigma^2}{4}\epsilon^2 pq\right] X^\epsilon_{(k-1)p\epsilon,(\ell-1)q\epsilon} \\
&+ \frac{1}{\epsilon^2}\int\int_{\Delta_{k,\ell}} F^\epsilon(u,v)\int\int_{\Delta^1_{k,\ell}(u,v)} F^\epsilon(\alpha,\beta) \\
&\qquad (X^\epsilon_{\alpha,\beta} - X^\epsilon_{(k-1)p\epsilon,(\ell-1)q\epsilon})d\alpha d\beta dudv \\
&+ \frac{1}{\epsilon}\int\int_{\Delta_{k\ell}} F^\epsilon(u,v)(\Delta^2_{k,\ell}(u,v)(X^\epsilon) + \Delta^3_{k,\ell}(u,v)(X^\epsilon))\, dudv.
\end{aligned}
$$

We shall use a Gronwall type lemma in order to control $\mathbb{E}\{|Y^{\epsilon,N}_{k,\ell}|^2\}$ (which is the whole purpose of Lemma 2.1). For $1 \le k, \ell \le N$ we have:

$$Y^{\epsilon,N}_{k,\ell} = \sum_{i=1}^{k} \sum_{j=1}^{\ell} \Delta_{k,\ell}(Y^{\epsilon,N}),$$

and consequently we get from (12):

$$\begin{aligned}
&\frac{1}{5}\mathbb{E}\{|Y^{\epsilon,N}_{k,\ell}|^2\} \le \\
&\mathbb{E}\{|\sum_{i=1}^{k}\sum_{j=1}^{\ell}\left(\frac{1}{\epsilon}\int\int_{\Delta_{ij}} F^{\epsilon}(u,v)dudv\right) Y^{\epsilon,N}_{i-1,j-1}|^2\} + \mathbb{E}\{|\frac{\epsilon^2 pq\sigma^2}{4}\sum_{i=1}^{k}\sum_{j=1}^{\ell} Y^{\epsilon,N}_{i-1,j-1}|^2\} \\
&+\mathbb{E}\{|\sum_{i=1}^{k}\sum_{j=1}^{\ell}\left[\left(\frac{1}{\epsilon^2}\int\int_{\Delta_{ij}} F^{\epsilon}(u,v)\int\int_{\Delta^1_{ij}(u,v)} F^{\epsilon}(\alpha,\beta)d\alpha d\beta dudv\right) - \frac{\sigma^2}{4}\epsilon^2 pq\right] \\
&\qquad X^{\epsilon}_{(i-1)p\epsilon,(j-1)q\epsilon}|^2\} \\
&+\mathbb{E}\{|\sum_{i=1}^{k}\sum_{j=1}^{\ell}\frac{1}{\epsilon^2}\int\int_{\Delta_{ij}} F^{\epsilon}_{(u,v)}\int\int_{\Delta^1_{ij}(u,v)} F^{\epsilon}(\alpha,\beta) \\
&\qquad (X^{\epsilon}_{\alpha,\beta} - X^{\epsilon}_{(i-1)p\epsilon,(j-1)q\epsilon})\, d\alpha d\beta dudv|^2\} \\
&+\mathbb{E}\{|\sum_{i=1}^{k}\sum_{j=1}^{\ell}\frac{1}{\epsilon}\int\int_{\Delta_{ij}} F^{\epsilon}(u,v)(\Delta^2_{ij}(u,v)(X^{\epsilon}) + \Delta^3_{ij}(u,v)(X^{\epsilon}))dudv|^2\}. \qquad (13)
\end{aligned}$$

We estimate the various terms of the right hand side separately.

First term. Since $Y^{\epsilon,N}_{i-1,j-1}$ is independent of the mean zero random variable $\left(\frac{1}{\epsilon}\int\int_{\Delta_{ij}} F^{\epsilon}(u,v)dudv\right)$, the first term is less than or equal to:

$$\begin{aligned}
&\sum_{i=1}^{k}\sum_{j=1}^{\ell}\mathbb{E}\{|(\frac{1}{\epsilon}\int\int_{\Delta_{ij}} F^{\epsilon}(u,v)dudv)Y^{\epsilon,N}_{i-1,j-1}|^2\} \\
&\quad = \sum_{i=1}^{k}\sum_{\ell=1}^{\ell}\mathbb{E}\{|\frac{1}{\epsilon}\int\int_{\Delta_{ij}} F^{\epsilon}(u,v)dudv|^2\}\mathbb{E}\{|Y^{\epsilon,N}_{i-1,j-1}|^2\} \\
&\quad = \sum_{i=1}^{k}\sum_{\ell=1}^{\ell}\frac{1}{\epsilon^2}\mathbb{E}\{\left(\sum_{m=(i-1)p+1}^{ip}\ \sum_{n=(j-1)q+1}^{jq}\epsilon^2 Z_{m,n}\right)^2\}\mathbb{E}\{|Y^{\epsilon,N}_{i-1,j-1}|^2\} \\
&\quad = \sum_{i=1}^{k}\sum_{j=1}^{\ell}\epsilon^2 pq\sigma^2\mathbb{E}\{|Y^{\epsilon,N}_{i-1,j-1}|^2\}
\end{aligned}$$

$$\le \frac{ST\sigma^2}{N^2}\sum_{i=1}^{k}\sum_{j=1}^{\ell}\mathbb{E}\{|Y^{\epsilon,N}_{i-1,j-1}|^2\}$$

by using (9).

Second term. Cauchy-Schwarz inequality implies that the second term is less than or equal to:

$$\frac{1}{16}\epsilon^4p^2q^2k\ell\sum_{i-1}^{k}\sum_{j=1}^{\ell}\mathbb{E}\{|Y^{\epsilon,N}_{i-1,j-1}|^2\} \le \frac{1}{16}\frac{S^2T^2}{N^2}\sum_{i=1}^{k}\sum_{\ell=1}^{\ell}\mathbb{E}\{|Y^{\epsilon,N}_{i-1,j-1}|^2.$$

Third term. Let us introduce the notation:

$$Q^\epsilon_{ij} = \frac{1}{\epsilon^2}\int\int_{\Delta_{ij}}F^\epsilon(u,v)\int\int_{\Delta^1_{ij}(u,v)}F^\epsilon(\alpha,\beta)d\alpha d\beta dudv.$$

The expectation of Q^ϵ_{ij} can be computed by decomposing the integral over Δ_{ij} in a sum of integrals over rectangles of size ϵ^2.

$$\begin{aligned}
\mathbb{E}\{Q^\epsilon_{ij}\} &= \mathbb{E}\{\left(\frac{1}{\epsilon^2}\int\int_{\Delta_{ij}}F^\epsilon(u,v)^2\int_{\epsilon[u/\epsilon]}^{u}\int_{\epsilon[v/\epsilon]}^{v}d\alpha d\beta dudv\right)\\
&= \frac{\sigma^2}{\epsilon^2}\int\int_{\Delta_{ij}}(u-\epsilon[u/\epsilon])(v_-\epsilon[v/\epsilon])dudv\\
&= \frac{\sigma^2}{4}\epsilon^2pq.
\end{aligned}$$

Using the independence of $X^\epsilon_{(i-1)p\epsilon,(j-1)q\epsilon}$ and $F^\epsilon(u,v)$ in the rectangle Δ_{ij} we get that this third term is less than or equal to:

$$\sum_{i=1}^{k}\sum_{j=1}^{\ell}\mathbb{E}\{|Q^\epsilon_{i,j}-\frac{\sigma^2}{4}\epsilon^2pq|^2\mathbb{E}\{|X^\epsilon_{(i-1)p\epsilon,(j-1)q\epsilon}|^2\}$$

and by using again (8), this is not greater than:

$$N^2C(S,T)\mathbb{E}\{|Q^\epsilon_{i,j}-\frac{\sigma^2}{4}\epsilon^2pq|^2\} = N^2C(S,T)\mathrm{Var}(Q^\epsilon_{1,1}) \le N^2C(S,T)\mathbb{E}\{|Q^\epsilon_{1,1}|^2\}.$$

Decomposing $\Delta_{1,1}$ and $\Delta^1_{1,1}$into rectangles of sizes ϵ^2 we get:

$$Q^\epsilon_{1,1} = \frac{\epsilon^2}{4}\sum_{n=1}^{p}\sum_{m=1}^{q}Z^2_{m,n} + \epsilon^2\sum_{n=1}^{p}\sum_{m=1}^{q}\sum_{i=1,\, i\ne m}^{n}\sum_{j=1,\, j\ne n}^{m}Z_{m,n}Z_{i,j}$$

which shows that:

$$\mathbb{E}\{|Q^\epsilon_{1,1}|^2\} \leq \frac{\epsilon^4}{8}\mathbb{E}\{|\sum_{n=1}^{p}\sum_{m=1}^{q} Z^2_{m,n}|^2\} + 2\epsilon^4\mathbb{E}\{|\sum_{n=1}^{p}\sum_{m=1}^{q}\sum_{i=1,\,i\neq m}^{n}\sum_{j=1,\,j\neq n}^{m} Z_{m,n}Z_{i,j}|^2\}.$$

Using the independence and the uniform boundedness of the $Z_{m,n}$'s the second term can easily be bound from above and we get:

$$\mathbb{E}\{|Q^\epsilon_{1,1}|^2\} \leq \epsilon^4 M^4 (pq)^2(\frac{1}{8}+2)$$

and hence, the third term is less than or equal to a quantity of the form $C(S,T)/N^2$

The fourth term. Using Cauchy-Schwarz once more one sees that this fourth term is less than or equal to:

$$\frac{k\ell}{\epsilon^4}\sum_{i=1}^{k}\sum_{j=1}^{\ell}\mathbb{E}\{|\int\int_{\Delta_{ij}} F^\epsilon(u,v)\int\int_{\Delta^1_{i,j}(u,v)} F^\epsilon(\alpha,\beta)(X^\epsilon_{\alpha,\beta}-X^\epsilon_{(i-1)p\epsilon,(j-1)q\epsilon})\, d\alpha d\beta\, dudv|^2\}$$

which is less that $A_1 + A_2$ with:

$$\begin{aligned} A_1 &= \frac{2N^2}{\epsilon^4}\sum_{i=1}^{k}\sum_{j=1}^{\ell}\mathbb{E}\{|\sum_{n=(i-1)p+1}^{ip}\sum_{m=(j-1)q+1}^{jq} \\ &\quad \int_{(n-1)\epsilon}^{n\epsilon}\int_{(m-1)\epsilon}^{m\epsilon} Z^2_{n,m}\int_{\epsilon[u/\epsilon]}^{u}\int_{\epsilon[v/\epsilon]}^{v}(X^\epsilon_{\alpha,\beta}-X^\epsilon_{(i-1)p\epsilon,(j-1)q\epsilon})\, d\alpha d\beta\, dudv|^2\} \end{aligned}$$

and

$$\begin{aligned} A_2 &= \frac{2N^2}{\epsilon^4}\sum_{i=1}^{k}\sum_{j=1}^{\ell}\mathbb{E}\{|\sum_{n=(i-1)p+1}^{ip}\sum_{m=(j-1)q+1}^{jq} \\ &\quad \int_{(n-1)\epsilon}^{n\epsilon}\int_{(m-1)\epsilon}^{m\epsilon} Z^2_{n,m}\int\int_{\Delta^{1*}_{i,j}(u,v)}(X^\epsilon_{\alpha,\beta}-X^\epsilon_{(i-1)p\epsilon,(j-1)q\epsilon})\, d\alpha d\beta\, dudv|^2\} \end{aligned}$$

where $\Delta^{1*}_{i,j}(u,v) = \Delta^1_{i,j}(u,v)\setminus[(\epsilon[u/\epsilon],\epsilon[v/\epsilon]),(u,v)]$. Using once more Cauchy-Schwarz inequality, both in the summation in (n,m) and in the integral and the fact that $|Z_{m,n}|^4 \leq M^4$ we get:

$$A_1 \leq \frac{2N^2}{\epsilon^4} \sum_{i=1}^{k} \sum_{j=1}^{\ell} pq \sum_n \sum_m \epsilon^4 M^4 \int_{(n-1)\epsilon}^{n\epsilon} \int_{(m-1)\epsilon}^{m\epsilon} \int_{\epsilon[u/\epsilon]}^{u} \int_{\epsilon[v/\epsilon]}^{v} \mathbb{E}\{|X^\epsilon_{\alpha,\beta} - X^\epsilon_{(i-1)p\epsilon,(j-1)q\epsilon}|^2\} \, d\alpha d\beta \, du dv.$$

Decomposing $X^\epsilon_{\alpha,\beta} - X^\epsilon_{(i-1)p\epsilon,(j-1)q\epsilon}$ into the sum of the increments of X^ϵ over the rectangles:

$$[((i-1)p\epsilon,(j-1)q\epsilon),(\alpha,\beta)], \quad [((i-1)p\epsilon,0),(\alpha,(j-1)q\epsilon)]$$

and

$$[(0,(j-1)q\epsilon),((i-1)p\epsilon,\beta)]$$

the areas of which are all less than $C(s,t)/N$, we get easily that:

$$\begin{aligned} A_1 &\leq \frac{2N^2}{\epsilon^4} \times N^2 \times (pq)^2 \epsilon^4 M^4 C(s,t) \frac{1}{N} \times \epsilon^4 \\ &\leq \frac{C(s,t)}{N}. \end{aligned} \tag{14}$$

Using independence in A_2 and $|Z|^2 \leq M$ we get that:

$$\begin{aligned} A_2 &\leq \frac{2N^2}{\epsilon^4} \sum_{ij,m,n} M^2 \mathbb{E}\{| \int_{(m-1)\epsilon}^{m\epsilon} \int_{(n-1)\epsilon}^{n\epsilon} \int \int_{\Delta^{1*}_{i,j}(u,v)} F^\epsilon(\alpha,\beta) \\ &\qquad (X^\epsilon_{\alpha,\beta} - X^\epsilon_{(i-1)p\epsilon,(j-1)q\epsilon}) \, d\alpha d\beta \, du dv|^2\} \\ &\leq \frac{2N^2}{\epsilon^4} \sum_{i,j,m,n} M^2 \epsilon^2 \int_{(n-1)\epsilon}^{n\epsilon} \int_{(m-1)\epsilon}^{m\epsilon} \mathbb{E}\{| \int \int_{\Delta^{1*}_{i,j}(u,v)} F^\epsilon(\alpha,\beta) \\ &\qquad (X^\epsilon_{\alpha,\beta} - X^\epsilon_{(i-1)p\epsilon,(j-1)q\epsilon}) \, d\alpha d\beta|^2\} du dv \end{aligned}$$

because of Cauchy-Schwarz inequality. One can write $X^\epsilon_{\alpha,\beta} - X^\epsilon_{(i-1)p\epsilon,(j-1)q\epsilon} = \Delta_1(X^\epsilon) + \Delta_2(X^\epsilon) + \Delta_3(x^\epsilon) - \Delta_4(X^\epsilon)$ where the $\Delta_j(X^\epsilon)$'s are the increments of X^ϵ over the rectangles R_j defined by:

$$\begin{aligned} R_1 &= [(\epsilon[\alpha/\epsilon], \epsilon[\beta/\epsilon), (\alpha,\beta)], \\ R_2 &= [((i-1)p\epsilon, 0), (\alpha, \epsilon[\beta/\epsilon])) \\ R_3 &= [(0,(j-1)q\epsilon, (\epsilon[\alpha/\epsilon], \beta)] \\ R_4 &= [((i-1)p\epsilon,(j-1)q\epsilon), (\epsilon[\alpha/\epsilon], \epsilon[\beta/\epsilon])]. \end{aligned}$$

See figure 2. We have:

$$
\begin{aligned}
&\mathbb{E}\{|\int\int_{\Delta^{1*}_{i,j}(u,v)} F^{\epsilon}(\alpha,\beta)(X^{\epsilon}_{\alpha,\beta} - X^{\epsilon}_{(i-1)p\epsilon,(j-1)q\epsilon}) d\alpha d\beta|^2\} \\
&\qquad \leq 4\sum_{r=1}^{4} \mathbb{E}\{|\int\int_{\Delta^{1*}_{i,j}(u,v)} F^{\epsilon}(\alpha,\beta)\Delta_r(X^{\epsilon})\, d\alpha d\beta|^2\}.
\end{aligned}
$$

Figure 2.2 *Illustration for the rectangular domains* $R1 = R_1$, $R2 = R_2$, $R3 = R_3$ *and* $R4 = R_4$.

For $r = 1$ we use Cauchy-Schwarz inequality and the fact that $|F^{\epsilon}(\alpha,\beta)|^2 \leq M^2$:

$$
\begin{aligned}
&\mathbb{E}\{|\int\int_{\Delta^{1*}_{i,j}(u,v)} F^{\epsilon}(\alpha,\beta)\Delta_1(X^{\epsilon}) d\alpha d\beta|^2 \\
&\qquad \leq M^2 \mathrm{Area}(\Delta^{1*}_{i,j}(u,v)) \int\int_{\Delta^{1*}_{i,j}(u,v)} \mathbb{E}\{|\Delta_1(X^{\epsilon})|^2 d\alpha d\beta \\
&\qquad \leq M^2(\mathrm{Area}\Delta^{1*}_{i,j}(u,v))^2 C(S,T)\epsilon^2 \\
&\qquad \leq M^2(pq\epsilon^2)^2 C(S,T)\epsilon^2 \\
&\qquad \leq C(S,T)\frac{\epsilon^2}{N^4}
\end{aligned}
$$

For $r = 2, 3$ and 4 we decompose $\Delta^{1,*}_{i,j}(u,v)$ into rectangles where $F^{\epsilon}(\alpha,\beta))$ is constant (i.e. independent of (α,β). Their areas are ϵ^2 except for the right most or upper most rectangles for which it is less than ϵ^2. The independence between $F^\epsilon(\alpha,\beta)$ and $\Delta_r(X^\epsilon)$ combined with the facts that the number of rectangles is at most pq and that $\mathrm{Area}(R_r)$ is not greater than $C(S,T)/N$ we get:

$$\begin{aligned}\mathbb{E}\{|\int\int_{\Delta^{1*}_{i,j}(u,v)} F^\epsilon(\alpha,\beta)\Delta_r(X^\epsilon)d\alpha d\beta|^2\} &\leq M^2C(S,T)\times\frac{1}{N}pq\epsilon^4\\ &\leq C(S,T)\frac{\epsilon^2}{N^3}.\end{aligned}$$

Putting these two estimates together we have:

$$\begin{aligned}A_2 &\leq \frac{8N^2}{\epsilon^4}\times N^2\times(pq)M^2\epsilon^2\times\epsilon^2\left(\frac{\epsilon^2}{N^4}+3\frac{\epsilon^2}{N^3}\right)\\ &\leq \frac{C(S,T)}{N}.\end{aligned}$$

Consequently the fourth term is less than $C(S,T)/N$.

The fifth and last term !!! The values of the function $F^\epsilon(u,v)$ on the rectangle Δ_{ij} are independent of $\Delta^2_{i,j}(u,v)(X^\epsilon)$ and $\Delta^3_{i,j}(u,v)(X^\epsilon)$. Consequently, the fifth term is less then:

$$\frac{1}{\epsilon^2}\sum_{i=1}^{k}\sum_{j=1}^{\ell}\mathbb{E}\{|\int\int_{\Delta_{ij}}F^\epsilon(u,v)(\Delta^2_{ij}(u,v)(X^\epsilon)+\Delta^3_{ij}(u,v)(X^\epsilon))dudv|^2\}.$$

Decomposing Δ_{ij} and using again the independence, this last quantity is shown to be less than:

$$\begin{aligned}&\frac{M^2}{\epsilon^2}\sum_{i=1}^{k}\sum_{j=1}^{\ell}\sum_{n=(i-1)p+1}^{ip}\sum_{m=(j-1)q+1}^{jq}\\ &\quad 2\epsilon^2\int_{(m-1)\epsilon}^{m\epsilon}\int_{(n-1)\epsilon}^{n\epsilon}(\mathbb{E}\{|\Delta^2_{i,j}(X^\epsilon)|^2\}+\mathbb{E}\{|(\Delta^3_{i,j}(X^\epsilon)|)^2\})dudv\\ &\quad\leq C(S,T)\frac{1}{\epsilon^2}N^2pq\epsilon^4\times\frac{1}{N}\\ \leq\quad & C(S,T)\times\frac{1}{N}.\end{aligned}$$

End of the proof of Lemma 2.1. The five estimates proved above give:

$$\mathbb{E}\{|Y^{\epsilon,N}_{k,\ell}|^2\leq C_1(S,T)\times\frac{1}{N^2}\sum_{i=1}^{k}\sum_{j=1}^{\ell}\mathbb{E}\{|Y^{\epsilon,N}_{i-1,j-1}|^2\}+\frac{C_2(S,T)}{N}$$

and given the fact that $Y^{\epsilon,N}_{k,0} = Y^{\epsilon,N}_{0,e} = 0$, Gronwall's inequality gives:

$$\max_{\substack{1\le k\le N\\ 1\le \ell\le N}} \mathbb{E}\{|Y^{\epsilon,N}_{k,\ell}|^2\} \le \frac{C(S,T)}{N}.$$

This completes the proof of Lemma 2.1. ∎

We now describe the limiting behavior in distribution, for N fixed and $\epsilon \searrow 0$, of the vector $\{X^{\epsilon,N}_{k,\ell};\ k,\ell = 0,\cdots,N\}$. The proof of the following result is elementary. We give it for the sake of completeness.

Lemma 2.2 *For each fixed $N \ge 1$ the random vector $\{X^{\epsilon,N}_{k,\ell};\ k,\ell = 0,\cdots,N\}$ converges in distribution when ϵ tends to 0 to the distribution of the random vector $\{\tilde X^N_{k,\ell};\ k,\ell = 0,\cdots,N\}$ defined by $\tilde X^N_{0,\ell} = \tilde X^N_{k,0} = 1$ and:*

$$\Delta_{k,\ell}(\tilde X^N) = \sigma\Delta_{k,\ell}(W)\tilde X^N_{k-1,\ell-1} + \frac{\sigma^2}{4}\frac{ST}{N^2}\tilde X^N_{k-1,\ell-1}$$

for $k,\ell = 1,2,\cdots,N$. Here $\{W_{s,t};\ 0\le s\le S,\ 0\le t\le T\}$ is a standard Brownian sheet. We use the notation $\Delta_{k,\ell}(\tilde X^N) = \tilde X^N_{k,\ell} - \tilde X^N_{k-1,\ell} - \tilde X^N_{k,\ell-1} + \tilde X^N_{k-1,\ell-1}$. Also, $\Delta_{k,l}(W)$ denotes the rectangular increment of the Brownian sheet over $[((k-1)S/N,(\ell-1)T/N),(kS/N,\ell T/N)]$.

Proof:
By Definition 2.1 one has $X^{\epsilon,N}_{0,\ell} = X^{\epsilon,N}_{k,0} = 1$ and:

$$\begin{aligned}&\{X^{\epsilon,N}_{0,\ell};\ 1\le k\le N,\ 1\le \ell\le N\}\\ &\qquad = \Phi\left(\{\frac{1}{\epsilon}\int\int_{\Delta_{k,\ell}} F^\epsilon(u,v)dudv;\ 1\le k\le N,\ 1\le\ell\le N\},\epsilon^2 pq\right)\end{aligned}$$

for some continuous function Φ from $\mathbb{R}^{N^2+1}$ to $\mathbb{R}^{N^2}$. Since:

$$\left(\{\frac{1}{\epsilon}\int\int_{\Delta_{k,\ell}} F^\epsilon(u,v)dudv;\ 1\le k\le N,\ 1\le\ell\le N\},\epsilon^2 pq\right)$$

converges in distribution, as ϵ tends to 0, to:

$$\left(\{\sigma\Delta_{k,\ell}(W);\ 1\le k\le N,\ 1\le \ell\le N\},\frac{ST}{N^2}\right)$$

one concludes that $\{X^{\epsilon,N}_{k,\ell};\ 1\le k\le N,\ 1\le\ell\le N\}$ converges in distribution to $\Phi\left(\{\sigma\Delta_{k,\ell}(W);\ 1\le k\le N,\ 1\le\ell\le N\},\frac{ST}{N^2}\right)$ with initial conditions 1. But this is nothing but the sequence $\tilde X^N_{k,\ell}$defined in the statement of the lemma. qed

Definition 2.2 *For every (s,t) in $[0,S]\times[0,T]$ we define $X^N_{s,t}$ by:*

$$X^N_{s,t} = \tilde{X}^N_{[\frac{sN}{S}],[\frac{tN}{T}]}.$$

Lemma 2.3 *If $X_{s,t}$ is defined by (5), then we have:*

$$\sup_{\substack{0\leq s\leq S\\ 0\leq t\leq T}} E\{|X^N_{s,t}-X_{s,t}|^2\} \leq \frac{C(S,T)}{N}.$$

Proof:
We first define the following rectangles:

$$\begin{aligned}
\Delta^N_{s,t} &= [(0,0),([\frac{sN}{S}]\frac{S}{N},[\frac{tN}{T}]\frac{T}{N})]\\
\Delta^{N,1}_{s,t} &= [([\frac{sN}{S}]\frac{S}{N},[\frac{tN}{T}]\frac{T}{N}),(s,t)]\\
\Delta^{N,2}_{s,t} &= [([\frac{sN}{S}]\frac{S}{N},0),(s,[\frac{tN}{T}]\frac{T}{N})]\\
\Delta^{N,3}_{s,t} &= [(0,[\frac{tN}{T}]\frac{T}{N}),([\frac{sN}{S}]\frac{S}{N},t)].
\end{aligned}$$

With these notations we have:

$$X^N_{s,t} = 1+\Delta^N_{s,t}(X^N)+\sum_{r=1}^{3}\Delta^{N,r}_{s,t}(X^N)$$

and

$$\begin{aligned}
\Delta^N_{s,t}(X^N) &= \sum_{k=1}^{[sN/S]}\sum_{\ell=1}^{[tN/T]}\Delta_{k,\ell}(\tilde{X}^N)\\
&= \sum_k\sum_\ell(\sigma\Delta_{k,\ell}(W)\tilde{X}^N_{k-1,\ell-1}+\frac{\sigma^2}{4}\frac{ST}{N^2}\tilde{X}^N_{k-1,\ell-1})\\
&= \sigma\int\int_{\Delta^N_{s,t}} X^N_{u,v}\,dW_{u,v}+\frac{\sigma^2}{4}\int\int_{\Delta^N_{s,t}} X^N_{u,v}\,dudv
\end{aligned}$$

For $r=1,2$ and 3, the increment $\Delta^{n,r}_{s,t}(X^N)$ is 0 by definition of X^N and $\Delta^{N,r}_{s,t}$. If we set $Y^N_{s,t}=X_{s,t}-X^N_{s,t}$, we have:

$$Y^N_{s,t} = \sigma\int\int_{\Delta^N_{s,t}} X^N_{u,v}\,dW_{u,v}+\frac{\sigma^2}{4}\int\int_{\Delta^N_{s,t}} X^N_{u,v}\,dudv$$

$$
\begin{aligned}
& -\sigma \int_0^s \int_0^t X_{u,v}\, dW_{u,v} + \frac{\sigma^2}{4} \int_0^r \int_0^t X_{u,v}\, dudv \\
= \quad & \sigma \int\int_{\Delta_{s,t}^N} Y_{u,v}^N\, dW_{u,v} + \frac{\sigma^2}{4} \int\int_{\Delta_{s,t}^N} Y_{u,v}^N\, dudv \\
& \sigma \int\int_{\Delta_{s,t}^{N,*}} X_{u,v}\, dW_{uv} - \frac{\sigma^2}{4} \int\int_{\Delta_{s,t}^{N,*}} X_{u,v}\, dudv
\end{aligned}
$$

with $\Delta_{s,t}^{N,*} = [(0,0),(s,t)] \setminus \Delta_{s,t}^N$. Consequently:

$$
\begin{aligned}
\mathbb{E}\{|Y_{s,t}^N|^2\} \quad \leq \quad & 4\sigma^2 \mathbb{E}\{|\int\int_{\Delta_{s,t}^N} Y_{u,v}^N\, dW_{u,v}|^2\} + \frac{\sigma^4}{14} \mathbb{E}\{|\int\int_{\Delta_{s,t}^N} Y_{u,v}^N\, dudv|^2\} \\
& + 4\sigma^2 \mathbb{E}\{|\int\int_{\Delta_{s,t}^{N,*}} X_{u,v} \cdot dW_{u,v}|^2\} + \frac{\sigma^4}{4} \mathbb{E}\{|\int\int_{\Delta_{s,t}^{N,*}} X_{u,v}\, dudv|^2\}.
\end{aligned}
$$

Using the martingale property of the stochastic integrals, Cauchy-Schwarz inequality in the two other integrals and the facts that $\mathbb{E}\{|X_{uv}|^2\} \leq C(S,T)$, $\text{Area}(\Delta_{s,t}^N) \leq ST$ and $\text{Area}(\Delta_{s,t}^{N,*}) \leq C(S,T)/N$, we get:

$$
\begin{aligned}
\mathbb{E}\{|Y_{s,t}^N|^2\} \quad & \leq \quad C(S,T) \int\int_{\Delta_{s,t}^N} \mathbb{E}\{|Y_{u,v}^N|^2\}\, dudv + \frac{C'(S,T)}{N} \\
& \leq \quad C(S,T) \int_0^s \int_0^t \mathbb{E}\{|Y_{u,v}^N|^2\}\, dudv + \frac{C'(S,T)}{N}.
\end{aligned}
$$

Finally, a Gronwall type lemma gives:

$$
\sup_{\substack{0 \leq s \leq S \\ 0 \leq t \leq T}} \mathbb{E}\{|Y_{s,t}^N|^2 \leq \frac{C(S,T)}{N}
$$

which is the desired result. ∎

3 Convergence of the Finite Marginals

Proposition 3.1 *The finite dimensional marginal distributions of* $\{X_{s,t}^\epsilon : \ 0 \leq s \leq S,\ 0 \leq t \leq T\}$ *converge to the corresponding marginals of the solution* $\{X_{s,t} : 0 \leq s \leq S,\ 0 \leq t \leq T\}$ *of equation (5).*

Proof:

We show that this result is a consequence of the technical lemmas we proved in the previous section. Let $d \geq 1$ be a fixed integer and let $\{z_n = (s_n, t_n);\ n = 1, \cdots, d\}$ be a finite set of points in $[0,S] \times [0,T]$. We denote by V^ϵ the d-dimensional random vector:

$$V^\epsilon = (X^\epsilon_{z_1}, X^\epsilon_{z_2}, \cdots, X^\epsilon_{z_d}).$$

For each $\epsilon > 0$ and for $n = 1, \cdots, d$, we denote by $z_n^{\epsilon,N}$ the point $z_n^{\epsilon,N} = ([s_n/(p\epsilon)]p\epsilon, [t_n/(q\epsilon)]q\epsilon)$ and we define the random vector V^ϵ_N by:

$$V^\epsilon_N = (X^\epsilon_{z_1^{\epsilon,N}}, \cdots, X^\epsilon_{z_d^{\epsilon,N}})$$

With these notations at hand we have:

$$X^\epsilon_{z_n} - X^\epsilon_{z_n^{\epsilon,N}} = \Delta_{[z_n^{\epsilon,N}, z_n]}(X^\epsilon) + \Delta_{[(s_n^{\epsilon,N},0),(s_n,t_n^{\epsilon,N})]}(X^\epsilon) + \Delta_{[(0,t_n^{\epsilon,N}),(s_s^{\epsilon,N},t_n)]}(X^\epsilon).$$

The area of the first rectangle is not greater than ST/N^2. The areas of the second and the third are not greater than ST/N. Combined with (8), this gives that:

$$\mathbb{E}\{|X^\epsilon_{z_n^{\epsilon,N}} - X^\epsilon_{z_n}|^2\} \leq \frac{C(S,T)}{N}$$

and:

$$\mathbb{E}\{\|V^\epsilon - V^\epsilon_N\|^2\} \leq \frac{C(S,T)}{N} \tag{15}$$

where the norm is the L^2-norm, i.e. $\|\cdot\| = \|\cdot\|_{L^2(\mathbf{R}^d)}$. Lemma 2.1 implies that:

$$\mathbb{E}\{|X^\epsilon_{z_n^{\epsilon,N}} - X^{\epsilon,N}_{[\frac{s_n}{p\epsilon}],[\frac{t_n}{q\epsilon}]}|^2\} \leq \frac{C(S,T)}{N}.$$

Consequently, if we denote the random vector:

$$(X^{\epsilon,N}_{[\frac{s_1}{p\epsilon}],[\frac{t_1}{q\epsilon}]}, \cdots, X^{\epsilon,N}_{[\frac{s_n}{p\epsilon}],[\frac{t_n}{q\epsilon}]})$$

by $V^{\epsilon,N}$ we can write:

$$\mathbb{E}\{\|V^\epsilon_N - V^{\epsilon,N}\|^2\} \leq \frac{C(S,T)}{N}. \tag{16}$$

For N and $(z_1, \cdots, z_d)$ fixed, one can find an $\epsilon_0 > 0$ such that for $\epsilon \leq \epsilon_0$ one has:

$$([\frac{s_1}{p\epsilon}] = [\frac{Ns_1}{S}],\ \ [\frac{t_1}{q\epsilon}] = [\frac{Nt_1}{T}]), \cdots, ([\frac{s_d}{p\epsilon}] = [\frac{Ns_d}{S}],\ \ [\frac{t_d}{q\epsilon}] = [\frac{Nt_d}{T}]).$$

These terms are all independent of ϵ because $p\epsilon$ and $q\epsilon$ converge (from below to) S/N and T/N respectively. This observation together with Lemma 2.2 imply that:

$$\lim_{\epsilon \searrow 0} V^{\epsilon,N} = V_N \tag{17}$$

in distribution where:

$$V_N = \{\tilde{X}^N_{[\frac{Ns_n}{S}],[\frac{Nt_n}{T}]};\ n = 1, \cdots, d\} = \{X^N_{z_n};\ n = 1, 2, \cdots, d\}.$$

if we use Definition 2.2. Finally Lemma 2.3 implies that:

$$\max_{n=1,\cdots,d} \mathbb{E}\{|X^N_{z_n} - X_{z_n}|^2\} \leq \frac{C(S,T)}{N}.$$

If we use the notation V for the random vector $\{X_{z_n};\ n = 1, \cdots, d\}$ we obtain:

$$\mathbb{E}\{\|V_N - V\|^2\} \leq \frac{C(S,T)}{N}. \tag{18}$$

Putting together the estimates (15), (16) and (18) and the convergence (17) gives:

$$\lim_{\epsilon \searrow 0} V^{\epsilon} = V$$

in distribution and this completes the proof. ∎

4 Tightness

As explained in the introduction, we need to prove first an estimate which will be needed in the proof of the tightness estimate (7).

Proposition 4.1 *There exists a finite constant $C = C(S,T)$ depending only upon S and T such that:*

$$\sup_{0 \leq s \leq S,\ 0 \leq t \leq T} \mathbb{E}\{|X^{\epsilon}_{s,t}|^4\} \leq C(S,T)$$

for every $0 < \epsilon \leq 1$.

Proof:
Let us use the notation $\underline{u}$ for $\epsilon[u/\epsilon]$ and let us also set:

$$H^{\epsilon}_{s,t} = \sup_{\substack{0 \leq u \leq s \\ 0 \leq v \leq t}} \mathbb{E}\{|X^{\epsilon}_{u,v}|^4\}.$$

We first observe that:

$$\mathbb{E}\{|X^{\epsilon}_{s,t}|^4\} = \mathbb{E}\{|1 + \Delta_{[0,z]}(X^{\epsilon})|^4\} \leq 4 + 4\mathbb{E}\{|\Delta_{[0,z]}(X^{\epsilon})|^4\} \tag{19}$$

But coming back to the equation X^{ϵ} is a solution of we get:

$$
\begin{aligned}
\Delta_{[0,z]}(X^\epsilon) &= \frac{1}{\epsilon}\int_0^s\int_0^t F^\epsilon(u,v)X^\epsilon_{u,v}\,dudv \\
&= \frac{1}{\epsilon}\int_0^t\int_0^t F^\epsilon(u,v)\Delta_{[(\underline{u},\underline{v}),(u,v)]}(X^\epsilon)\,dudv \\
&\quad +\int_0^s\int_0^t F^\epsilon(u,v)(X^\epsilon_{\underline{u},v}+X^\epsilon_{u,\underline{v}}-X^\epsilon_{\underline{u},\underline{v}})\,dudv
\end{aligned}
\tag{20}
$$

so that, if we use the notation $\underline{\Delta}=\Delta_{[(\underline{u},\underline{v}),(u,v)]}$ we get:

$$
\begin{aligned}
\mathbb{E}\{|\frac{1}{\epsilon}\int_0^s\int_0^t F^\epsilon(u,v)\underline{\Delta}(X^\epsilon)dudv|^4\} &\leq \frac{1}{\epsilon^4}\mathbb{E}\{(st\int_0^s\int_0^t |F^\epsilon(u,v)|^2|\underline{\Delta}(X^\epsilon)|^2\,dudv)^2\} \\
&\leq \frac{s^2t^2M^4}{\epsilon^4}\mathbb{E}\{|\int_0^s\int_0^t |\underline{\Delta}(X^\epsilon)|^2dudv|^2\} \\
&\leq \frac{s^3t^3}{\epsilon^4}M^4\int_0^s\int_0^t \mathbb{E}\{|\underline{\Delta}(X^\epsilon)^4\}\,dudv.
\end{aligned}
$$

On the other hand, we also have:

$$
\begin{aligned}
\mathbb{E}\{|\underline{\Delta}(X^\epsilon)|^4\} &= \frac{1}{\epsilon^4}\mathbb{E}\{|\int_{\underline{u}}^u\int_{\underline{v}}^v F^\epsilon(\alpha,\beta)X^\epsilon(\alpha,\beta)d\alpha d\beta|^4\} \\
&\leq \frac{1}{\epsilon^4}(u-\underline{u})^3(v-\underline{v})^3M^4\int_{\underline{u}}^u\int_{\underline{v}}^v \mathbb{E}\{|X^\epsilon_{\alpha,\beta}|^4\}d\alpha d\beta \\
&\leq \epsilon^2M^4\int_{\underline{u}}^u\int_{\underline{v}}^v H^\epsilon_{u,v}d\alpha d\beta \\
&\leq \epsilon^4M^4H^\epsilon_{u,v}
\end{aligned}
$$

so that:

$$
\begin{aligned}
\mathbb{E}\{|\frac{1}{\epsilon}\int_0^s\int_0^t F^\epsilon(u,v)\underline{\Delta}(X^\epsilon)\,dudv|^4\} &\leq \frac{s^3t^3}{\epsilon^4}\times\epsilon^4M^4\int_0^s\int_0^t H^\epsilon_{u,v}\,dudv \\
&\leq S^3T^3M^8\int_0^s\int_0^t H^\epsilon_{u,,v}\,dudv
\end{aligned}
\tag{21}
$$

which takes care of the first term in (20). For the second term we remark that $F^\epsilon(u,v)$ is independent of $Y^\epsilon_{u,v}=X^\epsilon_{\underline{u},v}+X^\epsilon_{u,\underline{v}}-X^\epsilon_{\underline{u},\underline{v}}$. In order to use martingale inequalities we decompose the rectangle $[0,z]$ in the form:

$$[0,z]=[0,\underline{z}]+[\underline{z},z]+[(0,\underline{t}),(\underline{s},t)]+[(\underline{s},0),(s,\underline{t}]$$

as shown in Figure 3.

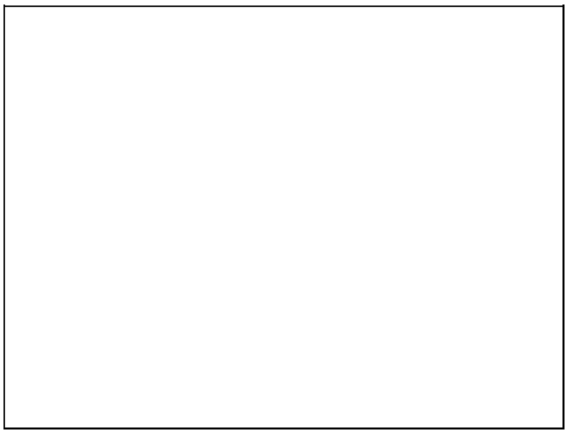

Figure 4.1 *Example of decomposition of the rectangle* $[0,z]$.

Then:

$$\frac{1}{\epsilon^4}\mathbb{E}\{|\int_0^{\underline{s}}\int_0^{\underline{t}} F^\epsilon(u,v)Y^\epsilon_{u,v}\,dudv|^4\}=\frac{1}{\epsilon^4}\mathbb{E}\{|\sum_{k=1}^{[s/\epsilon]}\sum_{\ell=1}^{[t/\epsilon]} Z_{k,\ell}\int_{\epsilon(k-1)}^{\epsilon k}\int_{\epsilon(\ell-1)}^{\epsilon\ell} Y^\epsilon_{u,v}\,dudv|^4\}.$$

Using Burkhölder inequality for discrete two-parameter martingales, this last expectation is bounded from above by:

$$\begin{aligned}
&\frac{1}{\epsilon^4}\mathbb{E}\{\left(\sum_k\sum_\ell |Z_{k,\ell}|^2\int_{\epsilon(k-1)}^{\epsilon k}\int_{\epsilon(\ell-1)}^{\epsilon\ell} Y^\epsilon_{u,v}\,dudv|^2\right)^2\}\\
&\le \frac{M^4}{\epsilon^4}[\frac{s}{\epsilon}][\frac{t}{\epsilon}]\sum_k\sum_\ell \mathbb{E}\{|\int_{\epsilon(k-1)}^{\epsilon k}\int_{\epsilon(\ell-1)}^{\epsilon\ell} Y^\epsilon_{u,v}\,dudv|^4\}\\
&\le \frac{stM^4}{\epsilon^6}\sum_k\sum_\ell \epsilon^6\int_{\epsilon(k-1)}^{\epsilon k}\int_{\epsilon(\ell-1)}^{\epsilon\ell}\mathbb{E}\{|Y^\epsilon_{u,v}|^4\}\,dudv\\
&\le stM^4\sum_k\sum_\ell\int_{\epsilon(k-1)}^{\epsilon k}\int_{\epsilon(\ell-1)}^{\epsilon\ell} 4\left(\mathbb{E}\{|X^\epsilon_{\underline{u},v}|^4\}+\mathbb{E}\{|X^\epsilon_{u,\underline{v}}|^4\}+\mathbb{E}\{|X^\epsilon_{\underline{u},\underline{v}}|^4\}\right)\,dudv
\end{aligned}$$

$$
\begin{aligned}
&\leq 12stM^4 \sum_k \sum_\ell \int_{\epsilon(k-1)}^{\epsilon k} \int_{\epsilon(\ell-1)}^{\epsilon\ell} H^\epsilon_{u,v}\, dudw \\
&\leq 12stM^4 \int_0^{\underline{s}} \int_0^{\underline{t}} H^\epsilon_{u,v}\, dudv \\
&\leq 12STM^4 \int_0^s \int_0^t H^\epsilon_{u,v}\, dudv. \qquad (22)
\end{aligned}
$$

For the second rectangle $[\underline{z}, z]$ we have:

$$
\begin{aligned}
\frac{1}{\epsilon^4}\mathbb{E}\{|\int_{\underline{s}}^s \int_{\underline{t}}^t F^\epsilon(u,v) Y^\epsilon_{u,v}\, dudv|^4\} &\leq \frac{M^4}{\epsilon^4}(s-\underline{s})^3(t-\underline{t})^3 \int_{\underline{s}}^s \int_{\underline{t}}^t \mathbb{E}\{|Y^\epsilon_{u,v}|^4\}\, dudv \\
&\leq 12M^4\epsilon^2 \int_0^s \int_0^t H^\epsilon_{u,v}\, dudv \\
&\leq 12M^4 \int_0^s \int_0^t H^\epsilon_{u,v}\, dudv \qquad (23)
\end{aligned}
$$

as long as $\epsilon \leq 1$. For the third rectangle $[(0,\underline{t}),(\underline{s},t)]$ we have:

$$
\frac{1}{\epsilon^4}\mathbb{E}\{|\int_0^{\underline{s}} \int_{\underline{t}}^t F^\epsilon(u,v) Y^\epsilon_{u,v}\, dudv|^4\} = \frac{1}{\epsilon^4}\mathbb{E}\{|\sum_{k=1}^{[s/\epsilon]} Z_{k,[t/\epsilon]+1} \int_{\epsilon(k-1)}^{\epsilon k} \int_{\underline{t}}^t Y^\epsilon_{u,v}\, dudv|^4\}.
$$

Using Burkhölder inequality for discrete martingales, this last quantity is not greater than:

$$
\begin{aligned}
&\frac{1}{\epsilon^4}\mathbb{E}\{\left(\sum_k |Z_{k,[t/\epsilon]+1}|^2 | \int_{\epsilon(k-1)}^{\epsilon k} \int_{\underline{t}}^t Y^\epsilon_{u,v}\, dudv|^2\right)^2\} \\
&\leq \frac{M^4}{\epsilon^4}[\frac{s}{\epsilon}] \sum_k \mathbb{E}\{|\int_{\epsilon(k-1)}^{\epsilon k} \int_{\underline{t}}^t Y^\epsilon_{u,v}\, dudv|^4\} \\
&\leq \frac{M^4}{\epsilon^4}\frac{S}{\epsilon} \sum_k \epsilon^3(t-\underline{t})^3 \int_{\epsilon(k-1)}^{\epsilon k} \int_{\underline{t}}^t \mathbb{E}\{|Y^\epsilon_{u,v}|^4\}\, dudv \\
&\leq 12\epsilon M^4 S \int_0^{\underline{s}} \int_{\underline{t}}^t H^\epsilon_{u,v}\, dudv \\
&\leq 12M^4 S \int_0^s \int_0^t H^\epsilon_{u,v}\, dudv \qquad (24)
\end{aligned}
$$

as long as $\epsilon \leq 1$. The fourth rectangle is handled in the same way and we get:

$$\frac{1}{\epsilon^4}\mathbb{E}\{|\int_{\underline{s}}^{s}\int_0^{\underline{t}} F^\epsilon_{(u,v)} Y^\epsilon_{u,v}\, dudv|^4\} \le 12M^4T\int_0^S\int_0^T H^\epsilon_{u,v}\, dudv. \tag{25}$$

Putting together (22), (23), (24) and (25) we get:

$$\mathbb{E}\{|\frac{1}{\epsilon}\int_0^s\int_0^t F^\epsilon(u,v)(X^\epsilon_{\underline{u},v}+X^\epsilon_{u,\underline{v}}-X^\epsilon_{\underline{u},\underline{v}})\, dudv|^4\} \le C(S,T)\int_0^s\int_0^t H^\epsilon_{u,v}\, dudv$$

which together with (19), (20) and (21) implies:

$$\mathbb{E}\{|X^\epsilon_{s,t}|^4\} \le 4 + C(S,T)\int_0^s\int_0^t H^\epsilon_{u,v}\, dudv.$$

and consequently:

$$H^\epsilon_{s,t} \le 4 + C(S,T)\int_0^s\int_0^t H^\epsilon_{u,v}\, dudv$$

and a Gronwall type lemma can be used to conclude the proof. ∎

We are now in a position to prove the tightness estimate as we announced it in the introduction.

Proposition 4.2 *There exists a finite constant $C(S,T)$ depending only upon S and T such that*

$$\mathbb{E}\{|\Delta_{[z,z']}(X^\epsilon)|^4\} \le C(S,T)|\Delta_{[z,z']}|^2$$

for every $z=(s,t)$ and $z'=(s',t')$ such that $0\le s<s'\le S$ and $0\le t<t'\le T$ and for every ϵ in $(0,1]$ and where we used the notation:

$$|\Delta_{[z,z']}| = (s'-s)(t'-t).$$

Proof:
As before $\underline{u}=\epsilon[u/\epsilon]$ and we shall also denote $\epsilon([u/\epsilon]+1)$ by $\bar{u}$. Generalizing (20) we get:

$$\begin{aligned}\Delta_{[z,z']}(X^\epsilon) &= \frac{1}{\epsilon}\int_s^{s'}\int_t^{t'} F^\epsilon(u,v)\Delta_{[(\underline{u},\underline{v}),(u,v)]}(X^\epsilon)\, dudv \\ &+\frac{1}{\epsilon}\int_s^{s'}\int_t^{t'} F^\epsilon(u,v)(X^\epsilon_{\underline{u},v}+X^\epsilon_{u,\underline{v}}-X^\epsilon_{\underline{u},\underline{v}})\, dudv.\end{aligned} \tag{26}$$

As for (21) we obtain:

$$\frac{1}{\epsilon^4}\mathbb{E}\{|\int_s^{s'}\int_t^{t'} F^\epsilon(u,v)\underline{\Delta}(X^\epsilon)dudv|^4\} \le \frac{M^4}{\epsilon^4}|\Delta_{[z,z']}|^3\int_s^{s'}\int_t^{t'} \mathbb{E}\{|\underline{\Delta}(X^\epsilon)|^4\}\,dudv.$$

Using Proposition 3.1 we have:

$$\mathbb{E}\{|\underline{\Delta}(X^\epsilon)|^4\} \le \epsilon^4 M^4 H^\epsilon_{u,v} \le \epsilon^4 M^4 C(S,T)$$

which combined with $|\Delta_{[z,z']}|^2 \le C(S,T)$ gives:

$$\mathbb{E}\{|\frac{1}{\epsilon}\int_s^{s'}\int_t^{t'} F^\epsilon(u,v)\underline{\Delta}(X^\epsilon)dudv|^4\} \le C(S,T)|\Delta_{[z,z']}|^2. \tag{27}$$

For the second term is (26) we decompose $\Delta_{[z,z']}$ as in Figure 4.

Figure 4.2 *Decomposition of the rectangle $[z,z']$ for small ϵ.*

Some of the 9 rectangles may be absent for some values of ϵ. But we have to consider them all because they are all present when ϵ is small enough. The increments over the rectangles Δ_2, Δ_3 and Δ_4 are treated as in the case of Δ_1 which we consider now. Using again the notation $Y^\epsilon_{u,v} = X^\epsilon_{\underline{u},v} + X^\epsilon_{u,\underline{v}} - X^\epsilon_{\underline{u},\underline{v}}$ we have, like in (23):

$$\begin{aligned}
\mathbb{E}\{|\frac{1}{\epsilon}\int\int_{\Delta_1} F^\epsilon(u,v)Y^\epsilon_{u,v}\,dudv|^4\} &\le \frac{1}{\epsilon^4}|\Delta_1|^3 M^4\int\int_{\Delta_1}\mathbb{E}\{|Y^\epsilon_{u,v}|^4\}\,dudv \\
&\le 12|\Delta_1|M^4\int\int_{\Delta_1} H^\epsilon_{u,v}\,dudv
\end{aligned}$$

$$
\begin{aligned}
&\leq C(S,T)|\Delta_1|^2 \\
&\leq C(S,T)|\Delta_{[z,z']}|^2 \qquad (28)
\end{aligned}
$$

for $\epsilon \leq 1$ because $\Delta_1 \subset \Delta_{[z,z']}$. The terms corresponding to the rectangles Δ_5, Δ_6 and Δ_7 are treated as in the case of Δ_8 which we consider now. As in (24) using Burkhölder inequality for discrete (one parameter) martingales we get:

$$
\begin{aligned}
&\mathbb{E}\{|\frac{1}{\epsilon}\int\int_{\Delta_8} F^{\epsilon}_{(u,v)} Y^{\epsilon}_{u,v}\, dudv|^4\} \\
&\quad = \frac{1}{\epsilon^4}\mathbb{E}\{|\sum_{\ell=[t/\epsilon]+2}^{[t'/\epsilon]} Z_{\underline{s},\ell}\int_s^{\bar{s}}\int_{\epsilon(\ell-1)}^{\epsilon\ell} Y^{\epsilon}_{u,v}\, dudv|^4\} \\
&\quad \leq \frac{M^4}{\epsilon^4}\mathbb{E}\{\left(\sum_\ell |\int_s^{\bar{s}}\int_{\epsilon(\ell-1)}^{\epsilon\ell} Y^{\epsilon}_{u,v}\, dudv|^2\right)^2\} \\
&\quad \leq \frac{M^4}{\epsilon^4}([t'/\epsilon]-[t/\epsilon]-2)\sum_\ell(\bar{s}-s)^3\epsilon^3\int_s^{\bar{s}}\int_{\epsilon(\ell-1)}^{\epsilon\ell}\mathbb{E}\{|Y^{\epsilon}_{u,v}|^4\}\, dudv \\
&\quad \leq C(S,T)\frac{1}{\epsilon^4}([t'/\epsilon]-[t/\epsilon]-2)^2(\bar{s}-s)^4\epsilon^4 \\
&\quad \leq C(S,T)(\epsilon([t'/\epsilon]-[t/\epsilon]-2))^2(\bar{s}-s)^2 \\
&\quad \leq C(S,T)(t'-t)^2(\bar{s}-s)^2 \\
&\quad \leq C(S,T)|\Delta_{[z,z']}|^2. \qquad (29)
\end{aligned}
$$

Finally case of Δ_9 is treated as in (22) by using Burkhölder inequality for discrete two-parameter martingales:

$$
\begin{aligned}
&\mathbb{E}\{|\frac{1}{\epsilon}\int\int_{\Delta_9} F^{\epsilon}(u,v)\, dudv|^4\} \\
&\quad = \frac{1}{\epsilon^4}\mathbb{E}\{|\sum_{k=[s/\epsilon]+2}^{[s'/\epsilon]}\sum_{\ell=[t/\epsilon]+2}^{[t'/\epsilon]} Z_{k,\ell}\int_{\epsilon(k-1)}^{\epsilon k}\int_{\epsilon(\ell-1)}^{\epsilon\ell} Y^{\epsilon}_{u,v}\, dudv|^4\} \\
&\quad \leq \frac{M^4}{\epsilon^4}\frac{(s'-s)(t'-t)}{\epsilon^2}\sum_k\sum_\ell(\epsilon^2)^3\int_{\epsilon(k-1)}^{\epsilon k}\int_{\epsilon(\ell-1)}^{\epsilon\ell}\mathbb{E}\{|Y^{\epsilon}_{u,v}|^4\}\, dudv \\
&\quad \leq C(S,T)|\Delta_{[z,z']}|^2 \qquad (30)
\end{aligned}
$$

Putting (26) through (30) gives the desired result. ∎

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