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Wave Propagation and Time Reversal in Randomly Layered Media

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To our families

Preface

Our motivation for writing this book is twofold: First, the theory of waves propagating in randomly layered media has been studied extensively during the last thirty years but the results are scattered in many different papers. This theory is now in a mature state, especially in the very interesting regime of separation of scales as introduced by G. Papanicolaou and his coauthors and described in [8], which is a building block for this book. Second, we were motivated by the time-reversal experiments of M. Fink and his group in Paris. They were done with ultrasonic waves and have attracted considerable attention because of the surprising effects of enhanced spatial focusing and time compression in random media. An exposition of this work and its applications is presented in [56]. Time reversal experiments were also carried out with sonar arrays in shallow water by W. Kuperman [113] and his group in San Diego. The enhanced spatial focusing and time compression of signals in time reversal in random media have many diverse applications in detection and in focused energy delivery on small targets as, for example, in the destruction of kidney stones. Enhanced spatial focusing is also useful in sonar and wireless communications for reducing interference. Time reversal ideas have played an important role in the development of new methods for array imaging in random media as presented in [19]. A quantitative mathematical analysis is crucial in the understanding of these phenomena and for the development of new applications. In a series of recent papers by the authors and their coauthors, starting with [40] in the one-dimensional case and [16] in the multidimensional case, a complete analysis of time reversal in random media has been proposed in the two extreme cases of strongly scattering layered media, and weak fluctuations in the parabolic approximation regime. These results are important in the understanding of the intermediate situations and will contribute to future applications of time reversal.

Wave propagation in three-dimensional random media has been studied mostly by perturbation techniques when the random inhomogeneities are small. The main results are that the amplitude of the mean waves decreases with distance traveled, because coherent wave energy is converted into incoherent fluctuations, while the mean energy propagates diffusively or by radiative transport. These phenomena are analyzed extensively from a physical and engineering point of view in the book of Ishimaru [90]. It was first noted by Anderson [5] that for electronic waves in strongly disordered materials there is wave localization. This means that wave energy does not propagate, because the random inhomogeneities trap it in finite regions. What is different and special in one-dimensional random media is that wave localization always occurs, even when the inhomogeneities are weak. This means that there is never a diffusive or transport regime in one-dimensional random media. This was first proved by Goldsheid, Molchanov, and Pastur in [79]. It is therefore natural that the analysis of waves in one-dimensional or strongly anisotropic layered media presented in this book should rely on methods and techniques that are different from those used in general, multidimensional random media.

The content of this book is multidisciplinary and presents many new physically interesting results about waves propagating in randomly layered media as well as applications in time reversal. It uses mathematical tools from probability and stochastic processes, partial differential equations, and asymptotic analysis, combined with the physics of wave propagation and modeling of time-reversal experiments. It addresses an interdisciplinary audience of students and researchers interested in the intriguing phenomena related to waves propagating in random media. We have tried to gradually bring together ideas and tools from all these areas so that no special background is required. The book can also be used as a textbook for advanced topics courses in which random media and related homogenization, averaging, and diffusion approximation methods are involved. The analytical results discussed here are proved in detail, but we have chosen to present them with a series of explanatory and motivating steps instead of a "theorem-proof" format. Most of the results in the book are illustrated with numerical simulations that are carefully calibrated to be in the regimes of the corresponding asymptotic analysis. At the end of each chapter we give references and additional comments related to the various results that are presented.

Acknowledgments

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Contents

1

Introduction and Overview of the Book

We begin by describing the organization of the book as shown in the diagram in Figure 1.1.

The basic theory of wave propagation in one-dimensional random media is contained in Chapters 2–9. Background for waves in deterministic, layered media is given in Chapters 2 and 3. In Chapters 4 and 5 we introduce the modeling of random media and describe in detail the scaling regimes that we consider in this book. In Chapter 6 we give a self-contained presentation of the asymptotic theory of random differential equations in a form that can be applied directly to the analysis of waves in random media in the following chapters. The asymptotic theory of reflection and transmission of waves in one-dimensional random media is presented in Chapters 7–9. Monochromatic reflection and transmission is analyzed in Chapter 7, which contains the well-known results of exponential decay of transmitted energy as the size of the random medium increases. In Chapter 8 we analyze the propagation of wave fronts and in Chapter 9 we characterize the statistical properties of wave fluctuations in the time domain.

The theory of time reversal in one-dimensional random media, both for reflected and for transmitted waves, along with applications to detection and communications, is presented in Chapters 10–13.

The extension of the theory of Chapters 8 and 9 to wave propagation in three-dimensional randomly layered media is given in Chapter 14. Time reversal in such media is analyzed in Chapter 15, where we derive analytical formulas that characterize the enhanced spatial focusing. An application to echo-mode energy refocusing on a passive scatterer is presented in Chapter 16.

Chapters 17–19 contain special topics and various generalizations to other asymptotic regimes and other types of waves. In Chapter 20 we analyze in detail wave propagation in randomly perturbed waveguides. This chapter is self-contained and could be read right after Chapter 6.

We now describe in more detail the contents of the chapters.

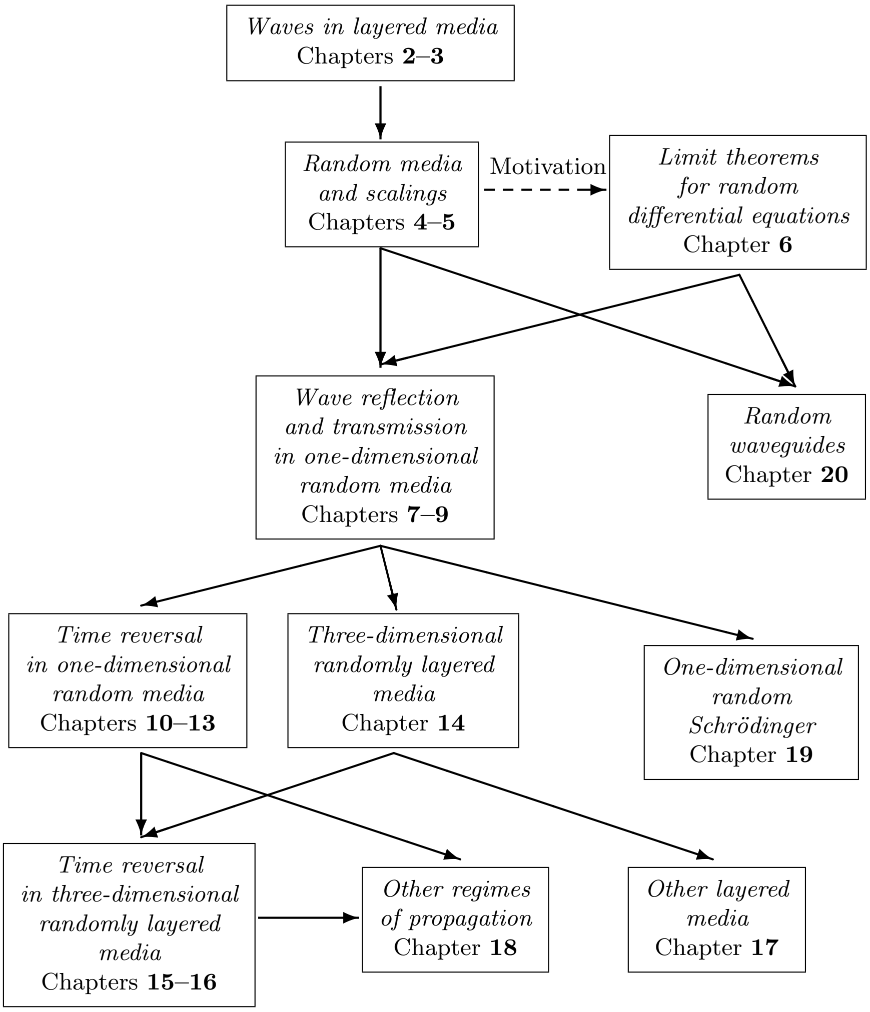

Fig. 1.1. Interdependence of the chapters.

Basic facts about wave propagation in **homogeneous media** are presented in **Chapter 2**.

In **Chapter 3** we consider one-dimensional **piecewise constant layered media**, and we introduce the usual formulation of reflection and transmission in terms of products of matrices.

Starting with **Chapter 4** we consider **randomly layered media**. We introduce the linear system of acoustic equations for waves propagating in one dimension, and then carefully describe the sequence of transformations that will be carried out throughout the rest of the book. We pay particular attention to boundary conditions and their interpretation, and to the reflected and transmitted waves in both the frequency and time domains. The concepts of random media and correlation lengths are introduced in this chapter. Our point of view is that randomness is closely associated with small-scale inhomogeneities leading naturally to the regime of homogenization and the notion of effective medium. This is done with an application of the law of large numbers, in the context of differential equations with random coefficients. This regime corresponds to waves propagating over distances of a few wavelengths, which are, however, much larger than the correlation length of the inhomogeneities.

We go a step further in **Chapter 5** by considering waves propagating over distances much larger than wavelengths. The fluctuations due to the multiple scattering by the random inhomogeneities accumulate and create "*noisy*" reflected and transmitted waves. We introduce important scaling regimes in which **diffusion approximations** are valid, leading to differential equations with random coefficients that are white noise. Even though the equations are linear, the probability distribution of the "noisy" wave field is a highly nonlinear function of the distribution of the random coefficients that model the random inhomogeneities. For a given frequency the random differential equations that enter are finite-dimensional, but in the time domain the problems become infinite-dimensional. Asymptotic approximations greatly simplify the analysis in the scaling regimes, and enable us to obtain useful information about the statistics of the reflected and transmitted waves.

In **Chapter 6** we present concepts and results about **stochastic processes** needed in the modeling of one-dimensional wave propagation and its asymptotic analysis. It is important to note that distance along the one-dimensional direction of propagation plays the role of the usual time parameter for these stochastic processes. The physical time is transformed by going into the frequency domain. In this chapter we present briefly the elements of the theory of Markov processes used for modeling randomly layered media and for describing the limit processes arising in the regime of diffusion approximations. A summary of the **stochastic calculus** is given at the end of the chapter, including Itô's formula, stochastic differential equations, the link with parabolic partial differential equations through the Feynman–Kac formula, and applications to the study of Lyapunov exponents of linear random differential equations.

A detailed analysis of the reflection and transmission of monochromatic waves in a one-dimensional random medium is given in **Chapter 7**. In one-dimensional random media all the wave energy is eventually converted into fluctuations, giving rise to the phenomenon of wave **localization**. This means that the energy is trapped by the random medium. It is entirely reflected back in the case of a random half-space. We show that the **exponential decay** of the transmitted energy through a random slab of random medium is closely related to the stability of the random harmonic oscillator, studied in this chapter. We also compute the moments of the transmitted energy, quantifying the exponential decay, as well as the almost-sure exponential decay that is related to the usual localization theory.

In **Chapter 8** we study the transmitted **wave front** in one-dimensional random media, in the regimes of the diffusion approximation introduced in the previous chapters. A pulse is sent from one end of a one-dimensional random medium and it is observed at the other end (see Figure 1.2). When the pulse exits the slab it looks like a smeared and faded version of the original one, followed by a noisy, incoherent coda. It is quite remarkable that in these asymptotic regimes, the front of the transmitted pulse has a simple description: (i) its deterministic shape is given by the convolution of the original pulse with a deterministic kernel that depends only on the second-order statistics of the random medium, and (ii) the transmitted wave front is centered at a random arrival time whose probability distribution is explicitly given in terms of a single Brownian motion. In this chapter we also describe the wave front reflected from a strong interface in a random medium.

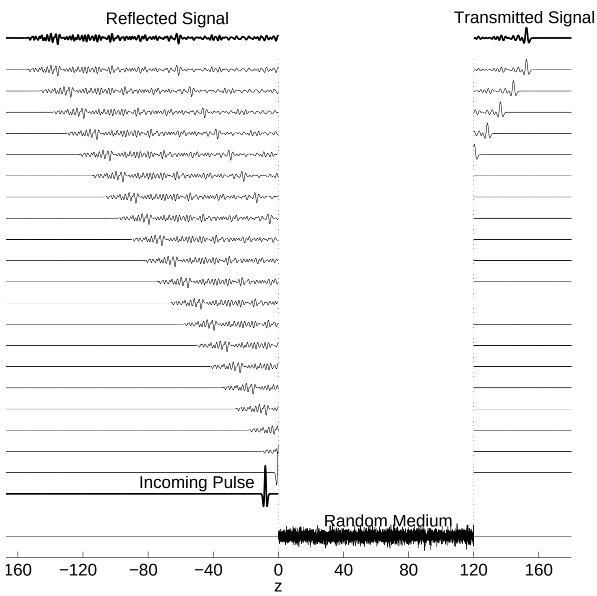

Fig. 1.2. Propagation of a pulse through a slab of random medium $(0, L)$. A right-going wave is incoming from the left. Snapshots of the wave profile (here the pressure) at different times are plotted from bottom to top. The reflected and transmitted signals at the last time of the numerical simulation are plotted at the top.

In **Chapter 9** we characterize the **statistics of the reflected and transmitted waves**, including the coda, in both the frequency and time domains (see the wave signals plotted at the top of Figure 1.2). This is done by a careful asymptotic analysis of the moments of the reflection and transmission coefficients. They satisfy a system of differential equations with random coefficients and are scaled so that the diffusion approximation can be applied. The limiting moments are obtained as solutions of systems of transport equations, which play a central role in the analysis of time reversal with incoherent waves, discussed in the following chapters. The solutions of these deterministic transport equations admit a probabilistic representation in terms of jump Markov processes, which is particularly convenient for Monte Carlo simulations and, in some cases, for deriving explicit formulas.

In **Chapter 10** we analyze **time reversal in reflection** where the incoherent reflected waves are recorded and sent time-reversed back into the medium. We show that stable refocusing takes place at the original source point. This is observed in physical experiments and illustrated in numerical simulations in Figures 1.3 and 1.4. Time-reversal refocusing can be used to estimate power spectral densities of reflected waves. They contain information about the medium. In this chapter we also compare, with a detailed analysis of **signal-to-noise ratios**, the **spectral estimation** method using time reversal with a direct estimation of cross-correlations of the reflected signal.

In **Chapter 11** we present two applications of time reversal to detection. In the first application, we use time reversal to detect the presence of a **weak reflector** buried in the many random layers. In this case the refocusing kernel of the time-reversal process has a jump that is related to the depth and strength of the reflector, and we exploit this to identify the reflector. In the second application, we introduce **absorption** in the one-dimensional model and show that refocusing still takes place after time reversal. We apply this to the **detection** and characterization of a dissipative region embedded in the random medium. In the presence of a dissipative region the refocusing kernel is modified and has a jump in its derivative. The time of this jump is related to the depth of the dissipative region, and its amplitude to the strength of absorption.

In **Chapter 12** we study time reversal of waves in randomly layered media described in the previous chapters. In this chapter we analyze **time reversal in transmission**, which means that a pulse is emitted at one end of a random slab, recorded at a time-reversal mirror at the other end, and then sent back. The wave refocuses at the original source point and the quality of the refocusing depends on how much of the transmitted wave has been recorded. In particular, it is shown that recording some part of the incoherent coda wave improves refocusing.

Applications to **communications** are presented in **Chapter 13**, where we analyze **signal-to-interference ratios** with and without using time reversal for communications through a one-dimensional random channel.

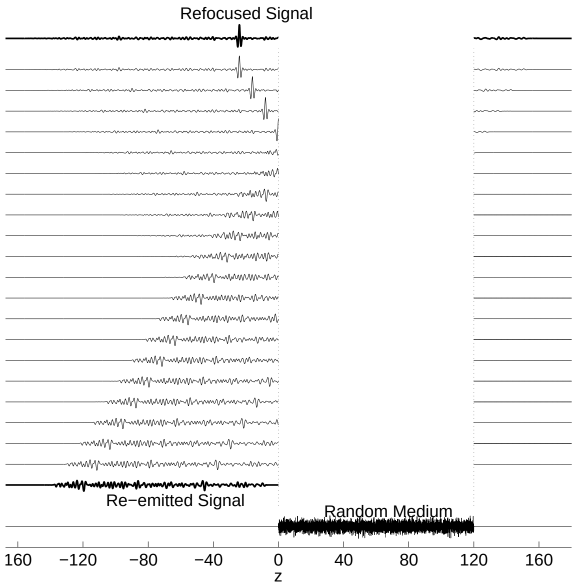

Fig. 1.3. We use the same random medium as in Figure 1.2 and send back, to the right, the time-reversed reflected signal (the one plotted at the top left corner of Figure 1.2). Snapshots of the wave profile (here the pressure) at different times are plotted from bottom to top. The refocused pulse is seen emerging from the random medium at the top.

Starting with **Chapter 14** we analyze waves propagating in a randomly layered **three-dimensional medium**. By taking Fourier transforms with respect to time and along the layers, the problem can be formulated as infinitely many one-dimensional problems. We model a physical source located at the surface of the random medium. Using a stationary phase analysis, we show that in the regime of diffusion approximations, and because of the separation of scales as in previous chapters, the stable wave front can again be described with an explicit formula that we derive.

Time reversal of waves propagating in three-dimensional randomly layered media is discussed in **Chapter 15**, where we consider a time-reversal mirror that records the signals generated by a source embedded in the random layers. We show that the time-reversed waves refocus around the original source point. We give a detailed analytical description of the refocused pulse in time and space. We compare this refocusing with **diffraction-limited** refocusing in homogeneous media and show that there is **superresolution** from **multi-**

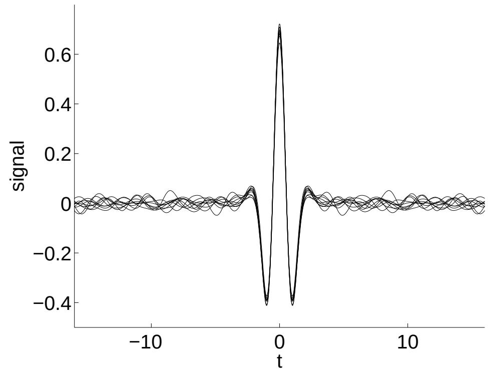

Fig. 1.4. We plot the refocused pulses generated by 10 independent simulations of time reversal (we follow the same procedure as in Figures 1.2–1.3, and we magnify the refocused pulse seen at the top line of Figure 1.3). The initial pulse is the second derivative of a Gaussian. We see here the remarkable statistical stability of the refocused pulse. Its shape and center do not depend on the realization of the medium, in contrast to the small-amplitude random wave fluctuations before and after the refocusing time.

pathing. This means that the focusing is much tighter, as well as stable, in the random medium.

In **Chapter 16** we present an application of time reversal in three-dimensional randomly layered media to **echo-mode energy refocusing** on a passive scatterer. This means that when the reflected signals received at the time-reversal mirror from a scatterer in a randomly layered medium are time-reversed and suitably reemitted, they tend to focus on the scatterer.

In **Chapter 17** we present an extension of the theory of wave propagation and time reversal to more **general randomly layered media**. We analyze models in which the effective parameters of the random medium do not match those of the adjacent homogeneous medium. We also analyze the case in which the effective parameters of the random medium vary smoothly at the macroscopic scale. The case in which both the bulk modulus and the density of the medium are randomly fluctuating is analyzed in Section 17.3.

Chapter 18 is devoted to several extensions and generalizations including the following ones.

- We reconsider the analysis for a **different regime** of scale separation, in which the amplitude of the fluctuations of the medium parameters is small and the typical wavelength is comparable to the small correlation length of the random medium.
- We extend the analysis to **dispersive or weakly nonlinear** random media. In the dispersive case, time reversal succeeds in recompressing the

dispersive oscillatory tail as well as the incoherent part of the waves. We analyze the combined effect of randomness and weak nonlinearity on the front of a propagating pulse. We show that randomness helps in preventing shock formation, so that time reversal in transmission can be done for longer propagation distances.

- We study the effect of **changes in the medium** parameters before and after time reversal. Although refocusing is affected by these changes, we still have partial refocusing. We also quantify the partial loss of statistical stability.

In **Chapter 19** we discuss the robustness of wave localization in a randomly layered medium when there is also nonlinearity, in the context of the nonlinear Schrödinger (NLS) equation. Using a perturbed inverse scattering transform, we show in this chapter that a **soliton** can overcome the exponential decay experienced by linear waves propagating through a slab in random medium.

Wave propagation in **waveguides** is analyzed in **Chapter 20**. We consider the case in which the waveguide supports a finite number of propagating modes and the random fluctuations of the medium are three-dimensional. We analyze only transmitted waves through a randomly perturbed waveguide, in the forward-scattering approximation, and the space-time refocusing of these waves after time reversal. We show that stable refocusing does occur, especially when the number of modes is large. This chapter may be considered as a link with the theory of wave propagation in three-dimensional random media.